A Full System of Invariants for Third-Order Linear Partial Differential Operators (LPDOs) in General Form

Ekaterina Shemyakova and Franz Winkler

RISC

Research Institute for Symbolic Computation

Johannes Kepler Universität · Linz · Austria

Outline

1. Ring of LPDOs
2. Invariants of LPDOs
3. Obstacles to Factorizations
4. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
5. Discussions

Outline

Ring of LPDOs

Differential field

a field K with a set $\Delta = \{\partial_x, \partial_y\}$ of commuting derivations acting on it.

Corresponding ring of linear differential operators

$$K[D] = K[D_x, D_y] \ ,$$

where D_x, D_y correspond to the derivations ∂_x, ∂_y, respectively.

Symbol of an LPDO:

Any operator $L \in K[D]$ is of the form

$$L = \sum_{i+j=0}^{d} a_{ij} D_x^i D_y^j \ , \tag{1}$$

where $a_{ij} \in K$. Then the (principal) symbol is the formal polynomial

$$\mathrm{Sym}_L = \sum_{i+j=d} a_{ij} X^i Y^j \ .$$

LPDO is hyperbolic

if its symbol is completely factorable (all factors are of first order) and each factor has multiplicity one.

Outline

Invariants of LPDOs

The gauge transformations (G.T.) of LPDOs:

$$L \rightarrow g^{-1} \circ L \circ g \quad , \; g \in K^* \; ,$$

where K^* denotes the set of invertible elements in K.

Remark

The symbol of an LPDO is unaltered under G.T. Thus, in particular, hyperbolic LPDOs in the normalized form admit G.T. (the form of such LPDOs is preserved by G.T.)

Invariant of certain class of LPDOs:

an algebraic expression of coefficients of an LPDO and their derivatives which is unaltered (under G.T. in our case).

Invariants for Hyperbolic Bivariate LPDOs of Second-Order

The normalized form of such operators is

$$L = D_{xy} + a(x,y)D_x + b(x,y)D_y + c(x,y) \ . \qquad (2)$$

The invariants (w.r.t. G.T.)

$$h = c - a_x - ab \ , \quad k = c - b_y - ab$$

were found by Laplace, Euler (maybe), and are called the Laplace invariants.

h, k form a generating set of differential invariants of (2) w.r.t. G.T.

What is the use of those h and k?

1. Classification of PDEs:

For ex., equation of the form $z_{xy} + a(x,y)z_x + b(x,y)z_y + c(x,y)z = 0$ is equivalent to the wave equation

$$z_{xy} = 0$$

whenever $h = k = 0$.

2. Invariant description of invariant properties

For ex., operator of the form $L = D_{xy} + a(x,y)D_x + b(x,y)D_y + c(x,y)$ has a factorization if and only if $h = 0$ or $k = 0$.

Moreover,

the Laplace Transformation Method is based on invariants. Thus, instead of an operator L one considers h and k, and instead of a sequence of transformed operators – a sequence of invariants.

Invariants for Hyperbolic Bivariate LPDOs of Third-Order

- Symbol with Constant Coefficients: 4 invariants were determined, but they are not sufficient to form a generating set of invariants [Kartaschova].
- Arbitrary Symbol: an idea to get some invariants, but again insufficient to form a generating set of invariants [Tsarev].
- Arbitrary Symbol: 5 independent invariants are found which form a generating set of invariants [Shemyakova, Winkler - this talk].

Outline

Obstacles to Factorizations

Motivation: Laplace's incomplete factorizations for $\mathrm{ord} = 2$

$$L = D_x \circ D_y + aD_x + bD_y + c$$

can be rewritten in the following ways:

$$L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k,$$

where h, k are the Laplace invariants.

Definition (Generalization to $\mathrm{ord} = n$)

For $\mathrm{Sym}_L = S_1 \dots S_k$, we can always find a partial factorization:

$$L = L_1 \circ \dots \circ L_k + R,$$

where $\mathrm{Sym}(L_i) = S_i$, $i = 1, \dots, k$, and R is of the smallest possible order.
R is a common obstacle to factorizations of the type $(S_1)(S_2)\dots(S_k)$.

Theorem

For a hyperbolic $L \in K[D_x, D_y]$ of order d and its factorizations into first-order factors

1. *the order of common obstacles $\leq d-2$;*
2. *a common obstacle is unique for each factorization type;*
3. *there are $d!$ common obstacles;*
4. *the symbol of a common obstacle is an invariant.*

Corollary for $L = (pD_x + qD_y)D_xD_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00}$

(all the coefficients belong to K, and $p, q \neq 0$, complete factorizations).

1. The order of common obstacles is ≤ 1;
2. a common obstacle is unique for each factorization type;
3. there are 6 common obstacles to factorizations into exactly three factors;
4. the symbol of a common obstacle is an invariant.

Example of Computing of an Invariant

p and q are invariants.

Assume for a while $p = 1$.

Compute the symbol of the common obstacle to factorization of the type $(X)(Y)(X + qY)$

Do it by means of Grigoriev-Schwarz method (differential version of Hensel descent). The result will be of the form

$$Coeff_1 X + Coeff_2 Y \ .$$

$Coeff_2 =$

$$(a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_x q a_{11} - a_{11x}q^2 + qa_{02x} + 2q_x^2 - q_{xx})/q^2 =$$

$$(I_4 + 2q_x^2 - q_{xx})/q^2 \ .$$

At long last one gets

Theorem

$$
\begin{aligned}
q^2\mathrm{Sym}_{XY(X+qY)} &= (q^2 I_3 + I_2 - q_{xy}q + q_{yy}q^2 + q_x q_y)X \\
&+ (I_4 + 2q_x^2 - q_{xx})Y \ , \\
q^2\mathrm{Sym}_{X(X+qY)Y} &= (i_2 + I_2)X + (I_4 + 2q_x^2 - q_{xx})Y \ , \\
q^2\mathrm{Sym}_{YX(X+qY)} &= (q^2 I_3 + q^2 q_{yy})X + i_3 Y \ , \\
q^2\mathrm{Sym}_{Y(X+qY)X} &= (q^2 I_3 + q^2 q_{yy})X + i_1 Y \ , \\
q^2\mathrm{Sym}_{(X+qY)XY} &= (i_2 + I_2)X + (i_1 + I_2 q)Y \ , \\
q^2\mathrm{Sym}_{(X+qY)YX} &= i_2 X + i_1 Y \ ,
\end{aligned}
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where

$$
\begin{aligned}
i_1 &= I_4 - 2\partial_x(I_1)q + 4q_x I_1 - 2I_2 q \ , \\
i_2 &= q^2 I_3 - 2\partial_y(I_1)q + 2I_1 q_y + I_2 \ , \\
i_3 &= I_4 - I_2 q - q_x q_y q + q_{xy}q^2 + 2q_x^2 - q_{xx}q \ .
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Four independent invariants were found,

At long last one gets

Theorem

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q^2 \mathrm{Sym}_{XY(X+qY)} &= (q^2 I_3 + I_2 - q_{xy} q + q_{yy} q^2 + q_x q_y) X \\
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q^2 \mathrm{Sym}_{X(X+qY)Y} &= (i_2 + I_2) X + (I_4 + 2q_x^2 - q_{xx}) Y \ , \\
q^2 \mathrm{Sym}_{YX(X+qY)} &= (q^2 I_3 + q^2 q_{yy}) X + i_3 Y \ , \\
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i_1 &= I_4 - 2\partial_x(I_1) q + 4 q_x I_1 - 2 I_2 q \ , \\
i_2 &= q^2 I_3 - 2\partial_y(I_1) q + 2 I_1 q_y + I_2 \ , \\
i_3 &= I_4 - I_2 q - q_x q_y q + q_{xy} q^2 + 2q_x^2 - q_{xx} q \ .
\end{aligned}
$$

Four independent invariants were found,
a fifth one was found by scientific guessing :-)

For $p \neq 1$

substitute a_{ij}/p for a_{ij} into expressions of the invariants.

Outline

Generating Set of Invariants for $L = (pD_x + qD_y)D_xD_y + \dots$

Theorem

The following 7 invariants form a generating set of invariants:

$$
\begin{aligned}
I_p &= p\ ,\\
I_q &= q\ ,\\
I_1 &= 2a_{20}q^2 - a_{11}pq + 2a_{02}p^2\ ,\\
I_2 &= a_{20x}pq^2 - a_{02y}p^2q + a_{02}p^2q_y - a_{20}q^2p_x\ ,\\
I_3 &= a_{10}p^2 - a_{11}a_{20}p + a_{20}(2q_yp - 3qp_y) + a_{20}^2q - a_{11,y}p^2 + a_{11}p_yp + a_{20}\\
I_4 &= a_{01}q^2 - a_{11}a_{02}q + a_{02}(2qp_x - 3pq_x) + a_{02}^2p - a_{11,x}q^2 + a_{11}q_xq + a_{02}\\
I_5 &= a_{00}p^3q - p^3a_{02}a_{10} - p^2qa_{20}a_{01} + p^2a_{02}a_{20}a_{11} + pqp_xa_{20}a_{11} +\\
&\quad (pI_1 - pq^2p_y + qp^2q_y)a_{20x} + (qq_xp^2 - q^2p_xp)a_{20y} + (\frac{1}{2}p_{xy}p^2q - p_x\\
&\quad +(4q^2p_xp_y - 2qp_xq_yp + qq_{xy}p^2 - q^2p_{xy}p - 2qq_xpp_y)a_{20} - \frac{1}{2}p^3qa_{11}\\
&\quad \frac{1}{2} \qquad \frac{1}{2}
\end{aligned}
$$

Proof

Prove that the operators $L = (pD_x + qD_y)D_xD_y + \sum_{i+j=0}^{2} a_{ij}D_x^iD_y^j$ and $L' = (p'D_x + q'D_y)D_xD_y + \sum_{i+j=0}^{2} a'_{ij}D_x^iD_y^j$ are equivalent if the value of their corresponding invariants are equal, that is

$$I_i = I'_i,\ i = p, q, 1, 2, 3, 4, 5\ .$$

We will be looking for some $g = g(x, y) = e^{f(x,y)} = e^f$,

such that

$$g^{-1}Lg = L'\ . \tag{3}$$

Equate the coefficients of D_{xx}, D_{yy} on both sides of (3), and get

$$\begin{aligned} \partial_y(f) &= b_{20} - a_{20} \,, & (4) \\ \partial_x(f) &= (b_{02} - a_{02})/q \,. & (5) \end{aligned}$$

In addition, the assumption $I_2 = I_2'$ implies

$$(b_{20} - a_{20})_x = ((b_{02} - a_{02})/q)_y .$$

Therefore, there is only one (up to a multiplicative constant) function f, which satisfies the conditions (4) and (5).

Check that then $\exp^f$ connects L and L'.

Outline

Discussions

Generalization to arbitrary order hyperbolic bivariate LPDOs

Discussions

Generalization to arbitrary order hyperbolic bivariate LPDOs

`done!!`

Discussions

Generalization to arbitrary order hyperbolic bivariate LPDOs

done!!

Generalization to arbitrary order NON-hyperbolic bivariate LPDOs

Discussions

Generalization to arbitrary order hyperbolic bivariate LPDOs

done!!

Generalization to arbitrary order NON-hyperbolic bivariate LPDOs

done!!