Exact solutions of completely integrable systems and linear ODE's having elliptic function coefficients

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Abstract

We present an algorithm for finding closed form solutions in elliptic functions of completely integrable systems. First we solve the linear differential equations in spectral parameter of Hermite-Halphen type. The integrability condition of the pair of equations of Hermite-Halphen type gives the large family of completely integrable systems of Lax-Novikov type. This algorithm is implemented on the basis of the computer algebra system MAPLE. Many examples, such as vector nonlinear Schödinger equation, optical cascaded equations and restricted three wave system are considered. New solutions for optical cascaded equations are presented. The algorithm for linear ODE's with elliptic functions coefficients is generalized to 2×2 matrix equations with elliptic coefficients.

1 Intordution

We consider a linear differential equation in spectral parameter λ

$$L\Psi = \left(p_0 \frac{d^m}{dx^m} + \sum_{j=1}^{m-1} p_{m-j}(x) \frac{d^{m-j}}{dx^{m-j}} \right) \Psi = \lambda \Psi, \tag{1}$$

where p_j are expressed in terms of *doubly periodic* functions having the same periods. We will also require that our coefficients are in fact *elliptic functions*. New important development of algorithms for ODE's with elliptic coefficients is paper [BLH]. They are implemented in Maple 9 as DEsolve function. The geometry of ODE's with elliptic function is considered in [GK] using Hermite-Halphen algorithm. The aim of present paper is to present applications of algorithms for ODE's with elliptic coefficients to obtain periodic solutions of completely integrable and near to completely integrable systems.

Let $z(x)$ be a solution of

$$y'''(x) - 4r(x)y'(x) - 2r'(x)y(x) = 0, \tag{2}$$

and set

$$y_1(x) = \sqrt{z(x)} \exp\left(-\frac{C}{2} \int \frac{dx'}{z(x')} \right), \tag{3}$$

and

$$y_1(x) = \sqrt{z(x)} \exp\left(\frac{C}{2} \int \frac{dx'}{z(x')} \right), \tag{4}$$

where C is a constant given by

$$C^2 = z'(x)^2 - 2z(x)z''(x) + 4r(x)z(x)^2, \tag{5}$$

if $C \neq 0$, the $y_1(x), y_2(x)$ are linearly independent and form a basis of

$$y''(x) - r(x)y(x) = 0, \tag{6}$$

If $C = 0$, the the basis for solution space of (6) is given by $y_1(x)$ and $y_2(x) = \sqrt{z(x)} \int \frac{1}{z(x)} dx$.

For proof see for example [BLH].

Introduce Hermite polynomial in spectral parameter λ as solution of the following nonlinear differential equation

$$\frac{1}{2}FF_{xx} - \frac{1}{4}F_x^2 - (u(x)+\lambda)F^2 + \frac{1}{4}R(\lambda) = 0, \tag{7}$$

where $r(x) = (u(x)+\lambda)$, $C^2 = R(\lambda)$, $F(x) = z(x)$. In modern literature (see for example references in [CEEK]) functions (3) and (4) are called Baker-Akhiezer (BA) functions.

1.1 Vector nonlinear Schrödinger equation

We consider the system of coupled nonlinear Schrödinger equations

$$i\frac{\partial}{\partial t}Q_j + s\frac{\partial^2}{\partial x^2}Q_j + \sigma\left(\sum_{k=1}^{n}|Q_k|^2\right)Q_j = 0, \quad j = 1,\ldots,n, \tag{8}$$

where $s = \pm 1$, $\sigma = \pm 1$. We seek solution of (8) in the following form [EEK])

$$Q_j = q_j(z)\,\mathrm{e}^{i\Theta_j}, \quad j = 1,\ldots n, \tag{9}$$

where $z = x - ct$, $\Theta_j = \Theta_j(z,t)$, with q_j, Θ_j real. Substituting (9) into (8) and separating real and imaginary parts by supposing that the functions $\Theta_j, j = 1,\ldots n$ behave as

$$\Theta_j = \frac{1}{2}scx + (a_j - \frac{1}{4}sc^2)t - s\,\mathcal{C}_j\int_0^z \frac{\mathrm{d}z'}{q_j(z')^2} + \Theta_{j0},$$

we obtain the system ($\sigma = s = \pm 1$)

$$\frac{d^2}{dz^2}q_j + \left(\sum_{k=1}^{n}\frac{\sigma}{s}q_k^2 - \frac{a_j}{s}\right)q_j - \frac{\mathcal{C}_j^2}{q_j^3} = 0, \quad k,j = 1,\ldots n, \tag{10}$$

where C_j, $j = 1,\ldots n$ are free parameters and Θ_{j0} are constants. These equations describe the integrable case of motion of a particle in a quartic potential perturbed with inverse squared potential, which is separable in ellipsoidal coordinates. The solutions of the system (10) are then given as

$$q_i^2(z) = 2\frac{\mathcal{F}(z, a_i - \Delta)}{\prod_{k\neq i}^{n}(a_i - a_k)}, \; i = 1,\ldots,n, \tag{11}$$

where $\mathcal{F}(z,\lambda)$ is Hermite polynomial associated with Lamé potential and is defined as solution of (7). The final formula for the solutions of the system (8) then reads

$$Q_i(x,t) = \sqrt{2\frac{\mathcal{F}(z,a_i-\Delta)}{\prod_{k\neq i}^{n}(a_i-a_k)}}\exp(\Theta_i), \tag{12}$$

where

$$\Theta_j = \left\{\frac{1}{2}icx + i(a_j - \frac{1}{4}c^2)t - \frac{1}{2}\nu(a_j-\Delta)\int\limits_0^z \frac{\mathrm{d}z'}{\mathcal{F}(z',a_j-\Delta)}\right\},$$

and $i = 1,\ldots,n$ and we have made use of (11) and (9). To obtain the special class of periodic solution of (10) we introduce the following ansatses

$$q_i(\zeta) = \sqrt{A_i\wp(\zeta+\omega')+B_i}, \quad i=1,2,3, \text{ or } i=1,\ldots,4. \tag{13}$$

As a result we obtain:

$$\sum_{k=1}^{m} A_k = -2, \quad a_i = \sum_{k=1}^{m} B_k - \frac{B_i}{A_i},\ m=3 \text{ or } 4, \tag{14}$$

$$-\frac{4\mathcal{C}_i^2}{A_i^2} = (4\lambda^3 - \lambda g_2 - g_3)_{|\lambda=-B_i/A_i}, \quad i=1,\ldots,3 \text{ or } 4 \tag{15}$$

and using the well known relations

$$\int_0^z \frac{dz'}{\wp(z')-\wp(\tilde{a}_j)} = \frac{1}{\wp'(\tilde{a}_j)}\left(2z\zeta(\tilde{a}_j) + \ln\frac{\sigma(z-\tilde{a}_j)}{\sigma(z+\tilde{a}_j)}\right), \tag{16}$$

and

$$\wp(z+\omega') - \wp(\tilde{a}_j) = -\frac{\sigma(z+\omega'+\tilde{a}_j)\sigma(z+\omega'-\tilde{a}_j)}{\sigma(z+\omega')^2\sigma(\tilde{a}_j)^2}. \tag{17}$$

We derive the following result

$$\begin{aligned} Q_j = \quad & \sqrt{-A_j}\frac{\sigma(z+\omega'+\tilde{a}_j)}{\sigma(z+\omega')\sigma(\tilde{a}_j)} \times \\ & \exp\left(\frac{i}{2}cx + i(a_j - \frac{1}{4}c^2)t - (z+\omega')\zeta(\tilde{a}_j)\right), \end{aligned} \tag{18}$$

where

$$\sum_{j=1}^{\epsilon_1} A_j = -2, \quad a_j = \sum_{k=1}^{\epsilon_1} B_k - \frac{B_j}{A_j},$$
$$\frac{\mathcal{C}_j}{A_j} = \frac{i}{2}\sqrt{4\lambda^3 - \lambda g_2 - g_3}_{|\lambda = -\frac{B_j}{A_j}}$$
$$\wp(\tilde{a}_j) = -\frac{B_j}{A_j} = \hat{a}_j, \quad j = 1 \dots \epsilon_1,\, \epsilon_1 = 3, 4 \tag{19}$$

To obtain the class of periodic solutions of system (10) for $n = 3, 4$ we introduce the following two ansatses in terms of the Weierstrass function $\wp(\zeta + \omega')$

$$q_i(\zeta) = \sqrt{A_i\wp(\zeta+\omega')^3 + B_i\wp(\zeta+\omega')^2 + C_i\wp(\zeta+\omega') + D_i}, \tag{20}$$

where $i = 1, \dots 3$. Next for conciseness we denote $\wp = \wp(\zeta + \omega')$, then the second ansatz have the form

$$q_i(\zeta) = \sqrt{A_i\wp^4 + B_i\wp^3 + C_i\wp^2 + D_i\wp + E_i}, \qquad i = 1, \dots 4 \tag{21}$$

with the constants A_i, B_i, C_i, D_i, E_i defined from the compatibility condition of the ansatz with the equations of motion (10). Inserting (20) and (21) into Eqs. (10), using the basic equations for Weierstrass $\wp$ function [WW]

$$\left(\frac{d}{d\zeta}\wp(\zeta)\right)^2 = 4\wp(\zeta)^3 - g_2\wp(\zeta) - g_3, \quad \frac{d^2}{d\zeta^2}\wp(\zeta) = 6\wp(\zeta) - \frac{g_2}{2}, \tag{22}$$

and equating to zero the coefficients at different powers of $\wp$ we obtain the following algebraic equations for the parameters of the solutions $A_i, B_i, C_i, D_i, i = 1, 2, 3$ for $n = 3$

$$A_1 + A_2 + A_3 = 0, \qquad B_1 + B_2 + B_3 = 0, \tag{23}$$
$$C_1 + C_2 + C_3 = -12, \quad C_i = \frac{2}{3}\frac{B_i^2}{A_i} - \frac{1}{4}A_i g_2, \tag{24}$$
$$a_i = \sum_{i=1}^{3} D_i - 5\frac{B_i}{A_i}, \quad D_i = \frac{5}{9}\frac{B_i^3}{A_i^2} - \frac{1}{3}B_i g_2 - \frac{1}{4}A_i g_3. \tag{25}$$

The analogical algebraic system for $n=4$ is as follows

$$A_1+A_2+A_3+A_4=0, \qquad B_1+B_2+B_3+B_4=0, \tag{26}$$
$$C_1+C_2+C_3+C_4=0, \quad D_1+D_2+D_3+D_4=-20, \tag{27}$$
$$C_i=\frac{3}{5}\frac{B_i^2}{A_i}-\frac{3}{10}A_i g_2, \quad D_i=\frac{14}{45}\frac{B_i^3}{A_i^2}-\frac{53}{180}B_i g_2-\frac{2}{9}A_1 g_3 \tag{28}$$
$$E_i=\frac{49}{225}\frac{B_i^4}{A_i^3}-\frac{113}{450}\frac{B_i^2}{A_i}g_2-\frac{11}{36}B_i g_3+\frac{9}{400}A_i g_2^2. \tag{29}$$
$$a_i=\sum_{i=1}^{4}E_i-7\frac{B_i}{A_i},$$

Another result from the algebraic systems is the expression for constants $\mathcal{C}_i$ which parametrise our solutions. For them we obtain

$$\mathcal{C}_i^2=-\frac{\nu(a_i-\Delta)^2}{\prod_{k\neq i}(a_i-a_k)},$$

where $i,k=3\,\mathrm{or}\,4$ and parameters ν are defined by (for $n=3$)

$$\begin{aligned}\nu^2= \quad & \lambda^7-\frac{63}{2}g_2\lambda^5+\frac{297}{2}g_3\lambda^4+\frac{4185}{16}g_2^2\lambda^3- \\ & \frac{18225}{8}g_2g_3\lambda^2+\frac{91125}{16}g_3^2\lambda-\frac{3375}{16}g_2^2\lambda,\end{aligned} \tag{30}$$

and (for $n=4$)

$$\begin{aligned}\nu^2 \;=\; & \lambda^9-\frac{231}{2}\lambda^7 g_2+\frac{2145}{2}g_3\lambda^6+\frac{63129}{16}\lambda^5 g_2^2-\frac{518505}{8}g_2g_3\lambda^4 \\ & +\left(-\frac{563227}{16}g_2^3+\frac{4549125}{16}g_3^2\right)\lambda^3+\frac{991515}{2}g_3g_2^2\lambda^2+ \\ & \left(\frac{361179}{4}g_2^4-\frac{5273625}{4}g_2g_3^2\right)\lambda \\ & -972405g_3g_2^3-1500625g_3^3.\end{aligned} \tag{31}$$

Using the general formulae, we will consider below the physically important cases of $n=3,4$ [EEK] which are associated with the three-gap $12\wp(\zeta+\omega')$, and four-gap elliptic potentials $20\wp(\zeta+\omega')$.

The Hermite polynomial $\mathcal{F}(\wp(x),\lambda)$ associated to the Lamé potential $12\wp(\zeta)$ has the form

$$\begin{aligned}\mathcal{F}(\wp(\zeta),\lambda) = \quad & \lambda^3 - 6\wp(\zeta+\omega')\lambda^2 - 3\cdot 5(-3\wp(\zeta+\omega')^2 + g_2)\lambda \\ & -\frac{3^2\cdot 5^2}{4}(4\wp(\zeta+\omega')^3 - g_2\wp(\zeta+\omega') - g_3). \end{aligned} \tag{32}$$

The solution is real under the choice of the arbitrary constants $a_i, i = 1,\dots,n$ in such way, that the constants $a_i - \Delta, i = 1,\dots,n$ lie in *different* lacunae. Comparing (20) and (32) and using (11) the solutions of polynomial equations (23),(24),(25) can be given by

$$A_i = \frac{2\cdot 5^2\cdot 3^2}{\prod_{k\neq i}^3 (a_i - a_k)}, \tag{33}$$

$$B_i = -\frac{2\cdot 3^2\cdot 5(a_i - \Delta)}{\prod_{k\neq i}^n (a_i - a_k)}, \tag{34}$$

$$\Delta = \frac{2}{5}\sum_{i=1}^3 a_i. \tag{35}$$

The Hermite polynomial $\mathcal{F}(\wp(\zeta),\lambda)$ associated to the Lamé potential $20\wp(\zeta)$ can be written as

$$\begin{aligned}\mathcal{F}(\wp(\zeta),\lambda) \quad = \quad & 11025\wp(\zeta+\omega')^4 - 1575\wp(\zeta+\omega')^3\lambda + \\ & (135\lambda^2 - \frac{6615}{2}g_2)\wp(\zeta+\omega')^2 + \\ & (-10\lambda^3 + \frac{1855}{4}\lambda g_2 - 2450 g_3)\wp(\zeta+\omega') + \\ & \lambda^4 - \frac{113}{2}\lambda^2 g_2 + \frac{3969}{16}g_2^2 + \frac{195}{4}\lambda g_3. \end{aligned} \tag{36}$$

Comparing (21) and (36) and using (11) the solutions of polynomial equations (26-29) can be given by

$$\begin{aligned} A_i &= \frac{11025\cdot 2}{\prod_{k\neq i}(a_i - a_k)}, \\ B_i &= -\frac{1575\cdot 2(a_i - \Delta)}{\prod_{k\neq i}(a_i - a_k)} \\ \Delta &= \frac{2}{7}\sum_{i=1}^4 a_i. \end{aligned} \tag{37}$$

Next solution of system (10,$n=3$) we obtain using the following ansatz

$$q_i(\zeta) = \sqrt{A_i\wp(\zeta+\omega')^2 + B_i\wp(\zeta+\omega') + C_i}, \quad i = 1,2,3, \tag{38}$$

then we have

$$\sum_{i=1}^{3} A_i = 0, \quad \sum_{i=1}^{3} B_i = -6, \tag{39}$$

$$a_i = \sum_{k=1}^{3} C_k - 3\frac{B_i}{A_i}, \quad C_i = \frac{B_i^2}{A_i} - \frac{1}{4}A_i g_2, \tag{40}$$

$$\frac{\mathcal{C}_i^2 \cdot 3^3 \cdot 4}{A_i^2} = (4\lambda^5 + 27\lambda^2 g_3 + 27\lambda g_2^2 - 21\lambda^3 g_2 - 81 g_2 g_3), \tag{41}$$

where $\lambda = -3B_i/A_i$.

1.2 Optical cascading equations

Let us consider the system of two ordinary differential equations,

$$q_{1\xi\xi} + A_0 q_1 + B_0 q_1 q_2 = 0, \tag{42}$$

$$q_{2\xi\xi} + C_0 q_2 + D_0 q_2^2 = 0, \tag{43}$$

where we have A_0, B_0, C_0, D_0 are constants.

Introducing new variable

$$q_1^2 = \frac{4F}{B_0 D_0}, \quad F = \lambda^2 - 3\wp\lambda + 9\wp^2 - \frac{9}{4}g_2 \tag{44}$$

where F is Hermite polynomial [WW], g_2, g_3 are elliptic invariants defined in [WW]. $\wp = \wp(\xi+\omega')$ is Weierstrass function shifted by half period ω' is related to sn Jacobian elliptic function with modulus k

$$\wp(\xi+\omega'; g_2, g_3) = \alpha^2 k^2 \mathrm{sn}^2(\alpha\xi, k) - (1+k^2), \tag{45}$$

where $\alpha = \sqrt{e_1 - e_3}$ and $e_i, i = 1,2,3,\ e_3 \le e_2 \le e_1$ are the real roots of the cubic equation

$$4\lambda^3 - g_2\lambda - g_3 = 0. \tag{46}$$

Using wave height α and modulus $k = \sqrt{(e_2 - e_3)/(e_1 - e_2)}$ we have the following relations

$$
\begin{aligned}
&e_1 = \frac{1}{3}(2 - k^2)\alpha^2, \quad e_2 = \frac{1}{3}(2k^2 - 1)\alpha^2, \quad e_3 = -\frac{1}{3}(1 + k^2)\alpha^2, \\
&g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3) = \frac{4}{3}\alpha^2(1 - k^2 + k^4), \\
&g_3 = 4 e_1 e_2 e_3 = \frac{4}{27}\alpha^6 (k^2 + 1)(2 - k^2)(1 - 2k^2). \qquad (47)
\end{aligned}
$$

Inserting this expression in (42) we have the following nonlinear differential equation with spectral parameter $\lambda = -C_0/2$

$$
\frac{1}{2} F F_{\xi\xi} - \frac{1}{4} F_\xi^2 - (u(\xi) + \lambda) F^2 + \frac{1}{4} R(\lambda) = 0, \qquad (48)
$$

with eigenvalue equations

$$
\begin{aligned}
&R(\lambda) = 4\lambda^5 - 21\lambda^3 g_2 + 27\lambda g_2^2 + 27\lambda^2 g_3 - 81 g_2 g_3 = 0, \\
&u(\xi) = -(B_0 q_2 + \lambda + A_0) = 6\wp(\xi + \omega'), \qquad (49)
\end{aligned}
$$

or in factorized form

$$
\begin{aligned}
&R(\lambda) = 4\prod(\lambda - \lambda_i) = 0, \quad \lambda_1 = -\sqrt{3g_2}, \quad \lambda_2 = 3e_3 \\
&\lambda_3 = 3e_2, \quad \lambda_4 = 3e_1, \quad \lambda_5 = \sqrt{3g_2}. \qquad (50)
\end{aligned}
$$

It is well known that equation (48) is reduced to linear periodic spectral problem of one dimensional Schrödinger equation with two gap potential $u(x) = 6\wp(\xi + \omega')$ and with five normalized eigenfunctions $q_1^{(i)}$, $(i) = 1, \ldots 5$:

$$
\frac{d^2 q_1^{(i)}}{d^2 \xi^2} - u(\xi) q_1^{(i)} = \lambda_i q_1^{(i)}, \quad (i) = 1, \ldots, 5. \qquad (51)
$$

Under these conditions the second equation (43) is automatically satisfied. Second equation can be considered as "self-consistent" equation for potential $u(\xi)$. Finally the five spectral families of periodic solutions can be written in the following Table 1

Table 1: Five spectral families of periodic solutions

(I).	$q_1 = \frac{6}{\sqrt{B_0 D_0}} \alpha^2 k^2 \,\mathrm{E}_2^{(u-)}$	$q_2 = -\frac{1}{B_0}(u(\xi) + \frac{3g_2}{\lambda_1} - 2\lambda_1)$	(i)=1
(II)	$q_1 = \frac{6}{\sqrt{B_0 D_0}} \alpha^2 k \,\mathrm{E}_2^{(cd)}$	$q_2 = -\frac{1}{B_0}(u(\xi) + \frac{3g_2}{\lambda_2} - 2\lambda_2)$	(i)=2
(III)	$q_1 = \frac{6}{\sqrt{B_0 D_0}} \alpha^2 k \,\mathrm{E}_2^{(sd)}$	$q_2 = -\frac{1}{B_0}(u(\xi) + \frac{3g_2}{\lambda_3} - 2\lambda_3)$	(i)=3
(IV)	$q_1 = \frac{6}{\sqrt{B_0 D_0}} \alpha^2 k^2 \,\mathrm{E}_2^{(sc)}$	$q_2 = -\frac{1}{B_0}(u(\xi) + \frac{3g_2}{\lambda_4} - 2\lambda_4)$	(i)=4
(V)	$q_1 = \frac{6}{\sqrt{B_0 D_0}} \alpha^2 k^2 \,\mathrm{E}_2^{(u+)}$	$q_2 = -\frac{1}{B_0}(u(\xi) + \frac{3g_2}{\lambda_5} - 2\lambda_5)$	(i)=5

where

$$
\begin{aligned}
\mathrm{E}_2^{(sc)} &= \mathrm{sn}(\alpha\xi, k)\mathrm{cn}(\alpha\xi, k),\\
\mathrm{E}_2^{(sd)} &= \mathrm{sn}(\alpha\xi, k)\mathrm{dn}(\alpha\xi, k),\\
\mathrm{E}_2^{(cd)} &= \mathrm{cn}(\alpha\xi, k)\mathrm{dn}(\alpha\xi, k),\\
\mathrm{E}_2^{(u\pm)} &= \mathrm{sn}^2(\alpha\xi, k) - \frac{1 + k^2 \pm \sqrt{1 - k^2 + k^4}}{3k^2},
\end{aligned}
\tag{52}
$$

are normalized two-gap Lamé functions [WW], cn, dn are Jacobian elliptic functions and potential $u(\xi)$ have the form

$$
u(\xi) = 6\alpha^2 k^2 \mathrm{sn}^2(\alpha\xi, k) - 2(1 + k^2)\alpha^2. \tag{53}
$$

2 2×2 matrix spectral problems and integrable systems

2.1 Baker-Akhiezer function

Let us start with two linear systems

$$
\frac{d\Psi_{1j}}{dx} + F\Psi_{1j} + G\Psi_{2j} = 0, \quad \frac{d\Psi_{1j}}{dt} + \tilde{A}\Psi_{1j} + \tilde{B}\Psi_{2j} = 0, \tag{54}
$$

$$
\frac{d\Psi_{2j}}{dx} + H\Psi_{1j} - F\Psi_{2j} = 0, \quad \frac{d\Psi_{2j}}{dt} + \tilde{C}\Psi_{1j} - \tilde{A}\Psi_{2j} = 0, \tag{55}
$$

which constitute [AKNS] scheme in particular case $F = -i\lambda, G = iu(x,t), H = \pm iu(x,t)$, where coefficients depend on an arbitrary spectral parameter λ.

The compatibility conditions $\Psi_{j,xt} = \Psi_{j,tx}, j = 1,2$ yield to the following nonlinear system of equations:

$$\begin{aligned}
&F_t - \tilde{A}_x + \tilde{C}G - \tilde{B}H = 0,\\
&G_t - \tilde{B}_x + 2(\tilde{B}F - \tilde{A}G) = 0,\\
&H_t - \tilde{C}_x + 2(\tilde{A}H - \tilde{C}F) = 0.
\end{aligned} \tag{56}$$

The general system (56) is equivalent also to zero curvature representation

$$U_t - V_x + [U, V] = 0, \tag{57}$$

where

$$U = \begin{pmatrix} F & G \\ H & -F \end{pmatrix}, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{58}$$

The periodic solutions in elliptic functions are generated through special matrices L whose representations are polynomials in the spectral parameter λ and L obey the following set of equations:

$$L_x = [U, L], \qquad L_t = [V, L], \qquad L\Psi = \nu\Psi, \tag{59}$$

where

$$V = \sum_{k=0}^{N} \begin{pmatrix} V_k^{11} & V_k^{12} \\ V_k^{21} & V_k^{22} \end{pmatrix} \lambda^{N-k} = \begin{pmatrix} \tilde{V}^{11} & \tilde{V}^{12} \\ \tilde{V}^{21} & \tilde{V}^{22} \end{pmatrix}. \tag{60}$$

or in explicit form

$$\frac{dA}{dx} = -HB + GC, \quad A(x,\lambda) = \sum_{j=0}^{n+1} A_{n+1-j}(x,t)\lambda^j, \tag{61}$$

$$\frac{dB}{dx} = 2FB - 2GA, \quad B(x,\lambda) = \sum_{j=0}^{n} B_{n-j}(x,t)\lambda^j, \tag{62}$$

$$\frac{dC}{dx} = -2FC + 2HA, \quad C(x,\lambda) = \sum_{j=0}^{n} C_{n-j}(x,t)\lambda^j, \tag{63}$$

and for $N = 1$

$$\frac{dA}{dt} = \tilde{V}^{12}C - \tilde{V}^{12}B, \quad A(\xi,\lambda) = \sum_{j=0}^{n+1} A_{n+1-j}(x,t)\lambda^j, \tag{64}$$

$$\frac{dB}{dt} = 2\tilde{V}^{11}B - 2\tilde{V}^{12}A, \quad B(\xi,\lambda) = \sum_{j=0}^{n} B_{n-j}(x,t)\lambda^j, \tag{65}$$

$$\frac{dC}{dt} = -2\tilde{V}^{11}C + 2\tilde{V}^{21}A, \quad C(\xi,\lambda) = \sum_{j=0}^{n} C_{n-j}(x,t)\lambda^j. \tag{66}$$

The equations (61),(62),(63) yield that

$$(A(x,\lambda)^2 - B(x,\lambda)C(x,\lambda))_x = 0 \tag{67}$$

and hence

$$\nu^2 = A(x,\lambda)^2 - B(x,\lambda)C(x,\lambda) = R_{2n+2}(\lambda), \tag{68}$$

where the integration constant R_{2n+2} is a polynomial in λ of degree $2n+2$. After a chain of simple transformations we obtain

$$\Psi_{1,x} = \frac{1}{2B}\left(B_x - 2R_{2n+2}G\right)\Psi_1, \tag{69}$$

$$\Psi_{2,x} = \frac{1}{2C}\left(C_x - 2R_{2n+2}H\right)\Psi_2. \tag{70}$$

After integration as result we have

$$\Psi_1 = \sqrt{B}\exp\left(\pm\sqrt{R_{2n+2}}\int^x \frac{G}{B}dx\right), \tag{71}$$

$$\Psi_2 = \sqrt{-C}\exp\left(\pm\sqrt{R_{2n+2}}\int^x \frac{H}{C}dx\right). \tag{72}$$

2.2 Restricted multiple three wave interaction system

Let us consider coupled quadratic nonlinear oscillators

$$\imath\frac{db_j}{d\xi} + uc_j - \frac{1}{2}\epsilon_j b_j = 0, \tag{73}$$

$$\imath\frac{dc_j}{d\xi} + u^* b_j + \frac{1}{2}\epsilon_j c_j = 0, \tag{74}$$

$$\imath\frac{du}{d\xi} + \sum_{j=1}^{n} b_j c_j^* = 0, \tag{75}$$

where ξ is the evolution coordinate and ϵ_j are constants. The equations (73-75) can be written as Lax representation

$$\frac{dL}{d\xi} = [M, L], \tag{76}$$

of the following linear system:

$$\frac{d\psi}{d\xi} = M(\xi,\lambda)\psi(\xi,\lambda) \quad L(\xi,\lambda)\psi(\xi,\lambda) = 0, \tag{77}$$

where L, M are 2×2 matrices and have the form

$$L(\xi,\lambda) = \begin{pmatrix} A(\xi,\lambda) & B(\xi,\lambda) \\ C(\xi,\lambda) & D(\xi,\lambda) \end{pmatrix}, \tag{78}$$

$$M(\xi,\lambda) = \begin{pmatrix} -\imath\lambda/2 & iu \\ u^* & \imath\lambda/2 \end{pmatrix}. \tag{79}$$

where

$$A(\xi,\lambda) = a(\lambda)\left(-\imath\frac{\lambda}{2} + \frac{\imath}{2}\sum_{j=1}^{n}\frac{(c_j c_j^* - b_j b_j^*)}{\lambda - \epsilon_j}\right), \tag{80}$$

$$B(\xi,\lambda) = a(\lambda)\left(\imath u - \imath\sum_{j=1}^{n}\frac{b_j c_j^*}{\lambda - \epsilon_j}\right), \tag{81}$$

$$C(\xi,\lambda) = a(\lambda)\left(\imath u^* - \imath\sum_{j=1}^{n}\frac{c_j b_j^*}{\lambda - \epsilon_j}\right), \tag{82}$$

where $D(\xi,\lambda) = -A(\xi,\lambda)$ and $a(\lambda) = \prod_{i=1}^{n}(\lambda - \epsilon_i)$. The Lax representation yields the hyperelliptic curve $K = (\nu, \lambda)$

$$\det(L(\lambda) - \frac{1}{2}\nu\mathbf{1}_2) = 0, \tag{83}$$

where $\mathbf{1}_2$ is the 2×2 unit matrix. The curve (83) can be written in canonical form as

$$\nu^2 = 4\prod_{j=1}^{2n+2}(\lambda - \lambda_j) = R(\lambda), \tag{84}$$

where $\lambda_j \neq \lambda_k$ are branching points. Next we develop a method which allows to construct periodic solutions of system (73-75). The method is based on the application of spectral theory for self-adjoint one dimensional Dirac equation with periodic finite gap potential $\mathcal{U} = -u$ cf. Eqs. (73,74)

$$\imath \frac{d\Psi_{1j}}{d\xi} - \mathcal{U}\Psi_{2j} - i\lambda_j \Psi_{1j} = 0, \tag{85}$$

$$\imath \frac{d\Psi_{2j}}{d\xi} - \mathcal{U}^* \Psi_{1j} + i\lambda_j \Psi_{1j} = 0, \tag{86}$$

with spectral parameter λ and eigenvalues $\lambda_j = i\epsilon_j/2$. The equation (76) is equivalently written as

$$\frac{dA}{d\xi} = iuC - iu^*B, \quad A(\xi,\lambda) = \sum_{j=0}^{n+1} A_{n+1-j}(\xi)\lambda^j, \tag{87}$$

$$\frac{dB}{d\xi} = -i\lambda B - 2iuA, \quad B(\xi,\lambda) = \sum_{j=0}^{n} B_{n-j}(\xi)\lambda^j, \tag{88}$$

$$\frac{dC}{d\xi} = i\lambda C + 2iu^*A, \quad C(\xi,\lambda) = \sum_{j=0}^{n} C_{n-j}(\xi)\lambda^j, \tag{89}$$

or in different form we have

$$A_{j+1,\xi} = iuC_j - iu^*B_j,\ A_0 = 1, A_1 = c_1, \tag{90}$$

$$iB_{j+1} = -B_{j,\xi} - 2iuA_{j+1}, \quad B_0 = -2u, \tag{91}$$

$$iC_{j+1} = C_{j,\xi} - 2iu^*A_{j+1} \quad C_0 = -2u^*, \tag{92}$$

where c_1 is the constant of integration. Differenciating Eq. (87) and using (83) we can obtain

$$BB_{\xi\xi} - \frac{u_\xi}{u}BB_\xi - \frac{1}{2}B_\xi^2 + \left(\frac{\lambda^2}{2} - i\lambda\frac{u_\xi}{u} + |u|^2\right)B^2 = 2u^2\nu. \tag{93}$$

Using (69) the eigenfunction Ψ_1 for finite-gap potential $\mathcal{U}$ have the form

$$\Psi_1(\xi,\lambda) = \left[\frac{\mathcal{U}(\xi)}{\mathcal{U}(0)}\frac{B(\xi,\lambda)}{B(0,\lambda)}\right]^{1/2} \exp\left\{-i\int_0^\xi \frac{\sqrt{R(\lambda)}}{B(\xi',\lambda)}d\xi'\right\}.$$

Analogously we can write expression for $\Psi_2(\xi, \lambda)$ and finally elliptic solutions of initial system of restricted three interaction system take the form

$$b_j(\xi) = b_j^0\, \Psi_1(\xi, \lambda_j), \qquad c_j(\xi) = c_j^0\, \Psi_2(\xi, \lambda_j), \qquad j = 1 \dots n, \tag{94}$$

where b_j^0, c_j^0 are constants fixed by initial conditions.

3 Implementation

In [GK] H-H (Hermite-Halphen) algorithm is presented and implemented in computer algebra REDUCE. Geometric interpretation of solutions found in [GK] is discussed in [EK]. Recently [BLH] Maple 9 [CGGMW] implementation of algorithm for solving linear ODE's having elliptic function coefficients is reported. New algorithm is found. This implementation is very important for deriving new solutions of integrable and nonintegrable dynamical systems with elliptic solutions [EK, GR, GH, B]. Important problem in deriving elliptic solutions is factorization of algebraic curves [CGHKW]. Algorithm for deriving elliptic solutions presented above is implemented on computer algebra Maple 10. The source code is available under request. Using this implementation new solutions of Manakov system in external potential are derived in [KEGKS]

References

[BLH] Burger, R., Labahn, G., van Hoeji, M: Closed form solutions of linear odes having elliptic function coefficients. ISSAC'04 Proceedings (2004) 58–64

[GK] Gerdt, V., Kostov, N: Computer algebra in the theory of ordinary differential equations of Halphen type, Computers and mathematics,Proc.of Int. Conf. Cambridge, USA, Springer, New-York, (1989) 178–188.

[WW] Whittaker, E., Watson, G.: A Course of Modern Analysis, (Cambridge University Press, Cambridge, 1986).

[EK] Enolskii, V., Kostov N.: On the geometry of elliptic solitons, Acta Applicandae Math., **36** (1994) 57–86

[CEEK] Christiansen, P., Eilbeck, J., Enolskii, V., Kostov, N.: Quasi-periodic and periodic solutions for Manakov type systems of coupled nonlinear Schrodinger equations, Proc. Royal Soc. London A. **456** (2000), 2263–2281

[EEK] Eilbeck, J., Enolskii, V., Kostov, N.: Quasi-periodic and periodic solutions for vector nonlinear Schrodinger equations, Journ. Math. Phys. **41** (2000) 8236–8248

[K] Kostov, N.: Quasi-periodic and periodic solutions for dynamical systems related to Korteweg-de Vries equation, The European Physical Journal B. **29** (2002), 255–260

[AKNS] Ablowitz, M., Kaup, D., Newell, A., Segur, H.: The inverse scattering transform–Fourier analysis for nonlinear problems, *Studies in Appl. Math.* **53** (1974) 249–315

[KEGKS] Kostov, N., Enolskii, V., Gerdjikov, V., Konotop, V.,Salerno, M., Two-component Bose-Einstein condensates in periodic potential, Phys. Rev E. **70** (2004), 056617

[S] Smirnov, A.: Elliptic soliton and Heun's equation, The Kowalevski property, CRM Proc. Lecture Notes. **32** (2002) 287–305

[CGGMW] Maple Reference Manual, B.W. Char, K.O. Geddes, G.H. Gonnet, M.B. Monagan and S.M. Watt, Watcom Publications, Waterloo, (1988)

[CGHKW] Corless, R., Giesbrecht, M., van Hoeij, M., Kotsireas, I., Watt, S.:Towards factoring bivariate approximate polynomials, International Conference on Symbolic and Algebraic Computation , Proceedings of ISSAC'01, ACM Press New York, NY, USA, (2001)

[GH] Gesztesy, F., Holden, H.: Soliton Equations and Their Algebro-Geometric Solutions. Vol. I: (1+1)-Dimensional Continuous Models, Cambridge University Press, Cambridge, (2003).

[GR] Gesztesy, F.,Ratneseelan, R.: An alternative approach to algebro-geometric solutions of AKNS hierarchy, Rev. Math. Phys., **10** (1998) 345–391

[B] Brezhnev, Yu.: Elliptic solitons and Gröbner bases, Journal of Mathematical Physics, **45** (2004) 696–712