On the weight spectra of Conway matrices related to the non-transitive head-or-tail game

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We consider non-transitive head-or-tail game:

Two players agree on some integer n. Then both of them select a binary n-word ("head"=0, "tail"=1) and begin flipping a coin. The winner is the player whose word appears first as a block of n consecutive outcomes.

e.g. $n=5$, $w_1=00000$, $w_2=10000$

For example:

B \ A	00	01	10	11
00		1/2	1/4	1/2
01	1/2		1/2	3/4
10	3/4	1/2		1/2
11	1/2	1/4	1/2	

Given w_1 and w_2, who has better chances to win?

Given w_1, how to choose w_2?

The solution of the problem that was suggested by Conway requires construction of a special matrix.

Martin Gardner. **Time Travel and Other Mathematical Bewilderements.** NY, 1988)

Guibas L.J., Odlyzko. **String Overlaps, Pattern Matching, and Nontransitive Games**. Journal of Combinatorial Theory, March 1981, 30, 183-208 .

Conway $2^n \times 2^n$ matrix $C_n = (c_{ij})$ $\quad 0 \le i\ ,\ j \le 2^n - 1$

$i,j \rightarrow$ HTTHT $\Leftrightarrow 01101 \Leftrightarrow 13$

The probability that the word *i* will win over the word *j*

$$P_{ij} = \frac{c_{jj} - c_{ji}}{(c_{ii} + c_{jj}) - (c_{ij} + c_{ji})}$$

$$P_{ii} = 0$$

Notations

E_r - identity matrix

$O_{r\times s}$ - zero matrix

$u \cap v$ - componentwise product (intersection) of binary vectors u and v

$|u|$ - Hamming weight (number of components equal to 1)

e_{00} - matrix unit (matrix with upper-left element 1, other elements are 0s)

Definition 1

Let V_n be the space of the binary n-words, i.e. the n-dimensional vector space over the field $Z_2 = \{0, 1\}$ and k is an integer, $0 \le k \le n$. We say that binary vector $u = (\varepsilon_0 , \varepsilon_1 \ldots \varepsilon_{n-1})$ k-overlaps vector $v = (\eta_0 , \eta_1 \ldots \eta_{n-1})$ if $(\varepsilon_k , \varepsilon_{k+1} \ldots \varepsilon_{n-1}) = (\eta_0 , \eta_1 \ldots \eta_{n-k-1})$ and write this as $u =_k = v$.

Definition 2

The Conway number of the pair (u, v) is $c(u,v) = \sum_{k=0}^{n-1} 2^{n-1-k} \| u =_k = v \|$, where $\| u =_k = v \|$ equals to 1 (resp. 0) if $u =_k = v$ is “true” (resp. “false”).

n=3

$$C_3 = \begin{pmatrix} 7 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 1 & 3 & 3 \\ 3 & 3 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 7 \end{pmatrix}$$

$$P_{ij} = \frac{c_{jj} - c_{ji}}{(c_{ii} + c_{jj}) - (c_{ij} + c_{ji})}$$

For example, the probability that the word i=100 = 4 will win over the word j=000 = 0 is P_{40}=7/8.

	000	001	010	011	100	101	110	111
000	0	1/2	2/5	2/5	1/8	5/12	3/10	1/2
001	1/2	0	2/3	2/3	1/4	5/8	1/2	7/10
010	3/5	1/3	0	1/2	1/2	1/2	3/8	7/12
011	3/5	1/3	1/2	0	1/2	1/2	3/4	7/8
100	7/8	3/4	1/2	1/2	0	1/2	1/3	3/5
101	7/12	3/8	1/2	1/2	1/2	0	1/3	3/5
110	7/10	1/2	5/8	1/4	2/3	2/3	0	1/2
111	1/2	3/10	5/12	1/8	2/5	2/5	1/2	0

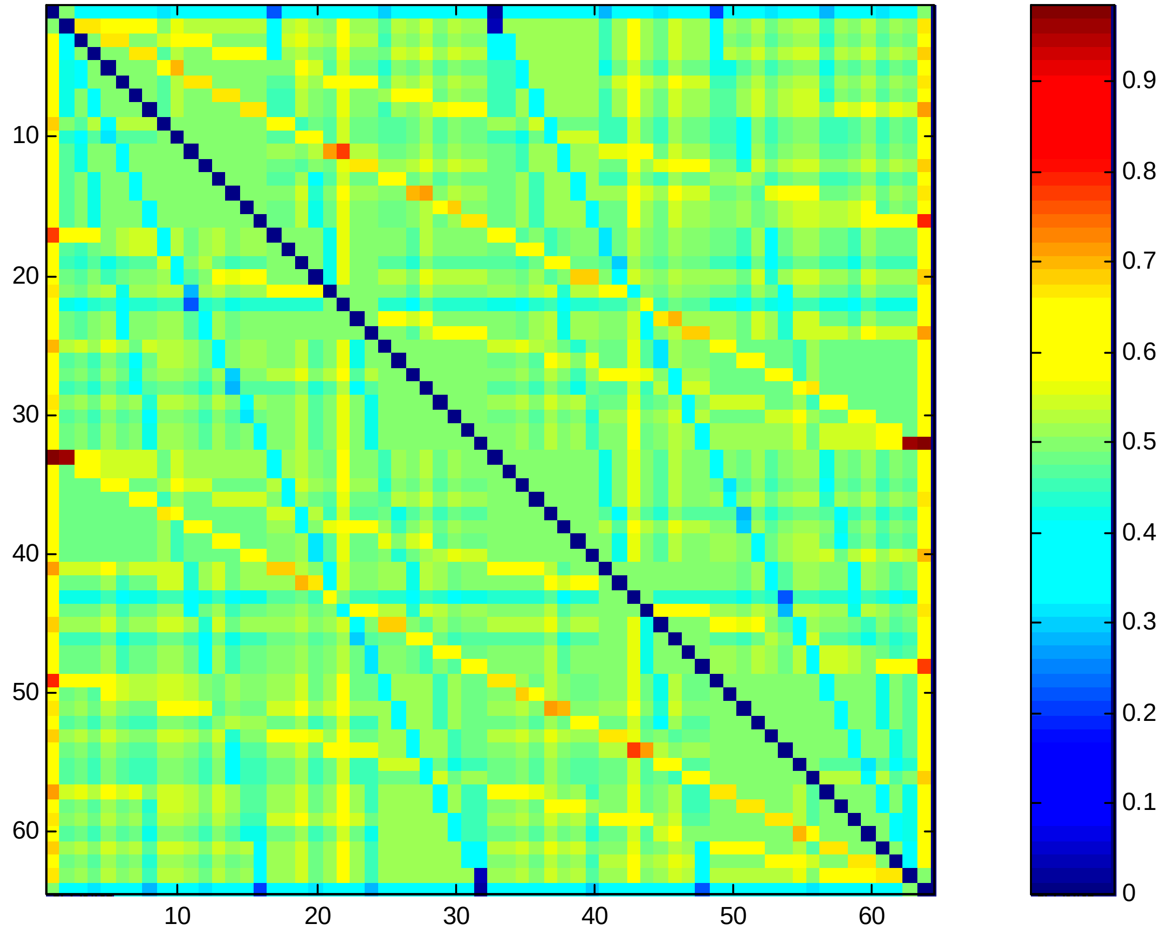
We developed a recursive algorithm for the fast construction of the Conway matrix. Results are visualized as a square, where numbers of rows and columns code binary words in a usual way; color depends on the probability of the event “row wins”.

$$C_n = \begin{pmatrix} C_{n-1} & O \\ O & C_{n-1} \end{pmatrix} \begin{pmatrix} E + 2^{-1}U_{n-1} & E - 2^{-1}U_{n-1} \\ E + 2^{-1}U_{n-1} & E - 2^{-1}U_{n-1} \end{pmatrix}$$

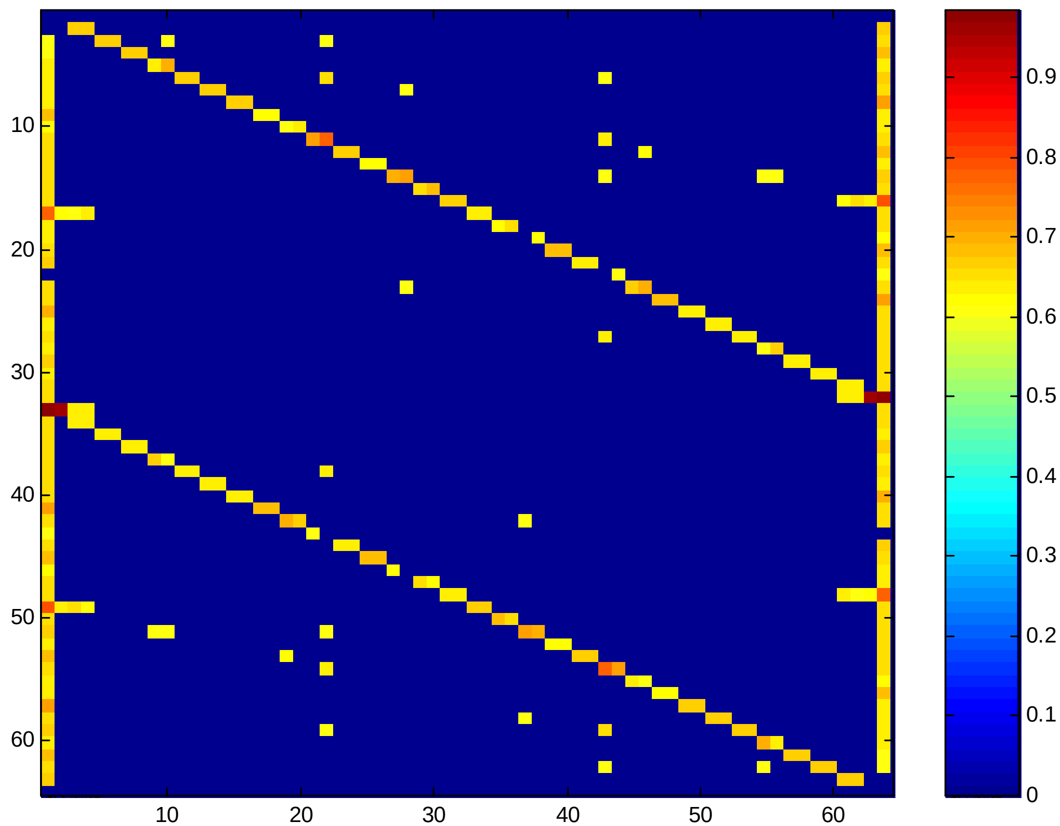
$$+ 2^{n-2} \begin{pmatrix} E & -E \\ -E & E \end{pmatrix} + 2^{-1} J_n \qquad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where J_n is $2^n \times 2^n$ matrix consisting of 1’s,

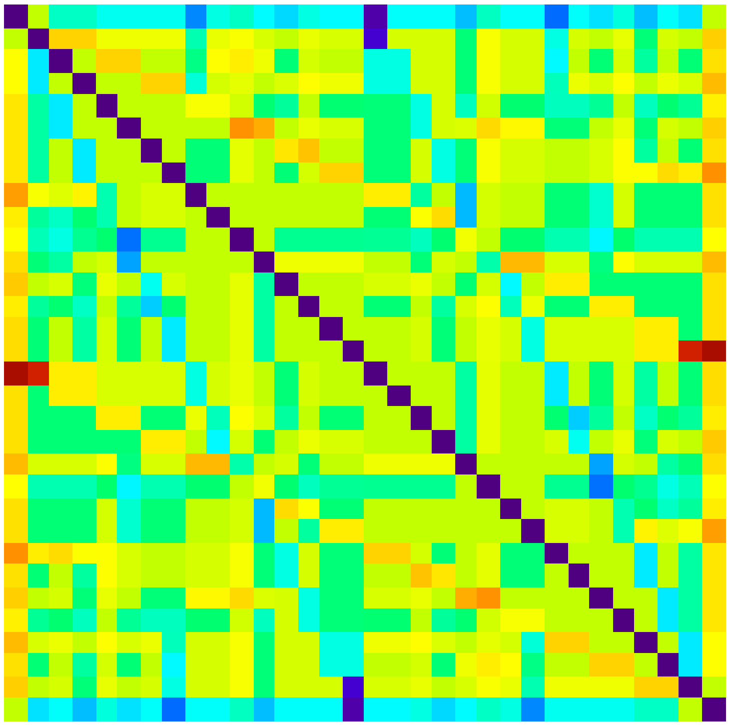
$$U_n = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes E_{2^{n-1}} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}$$



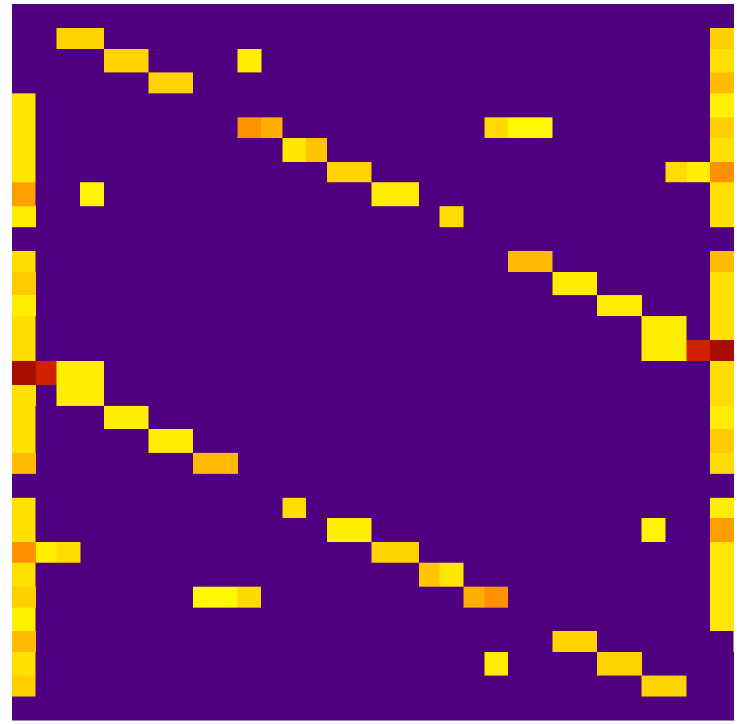
CONWAY[6,0] – recursive construction of the probabilities matrix P for n=6.



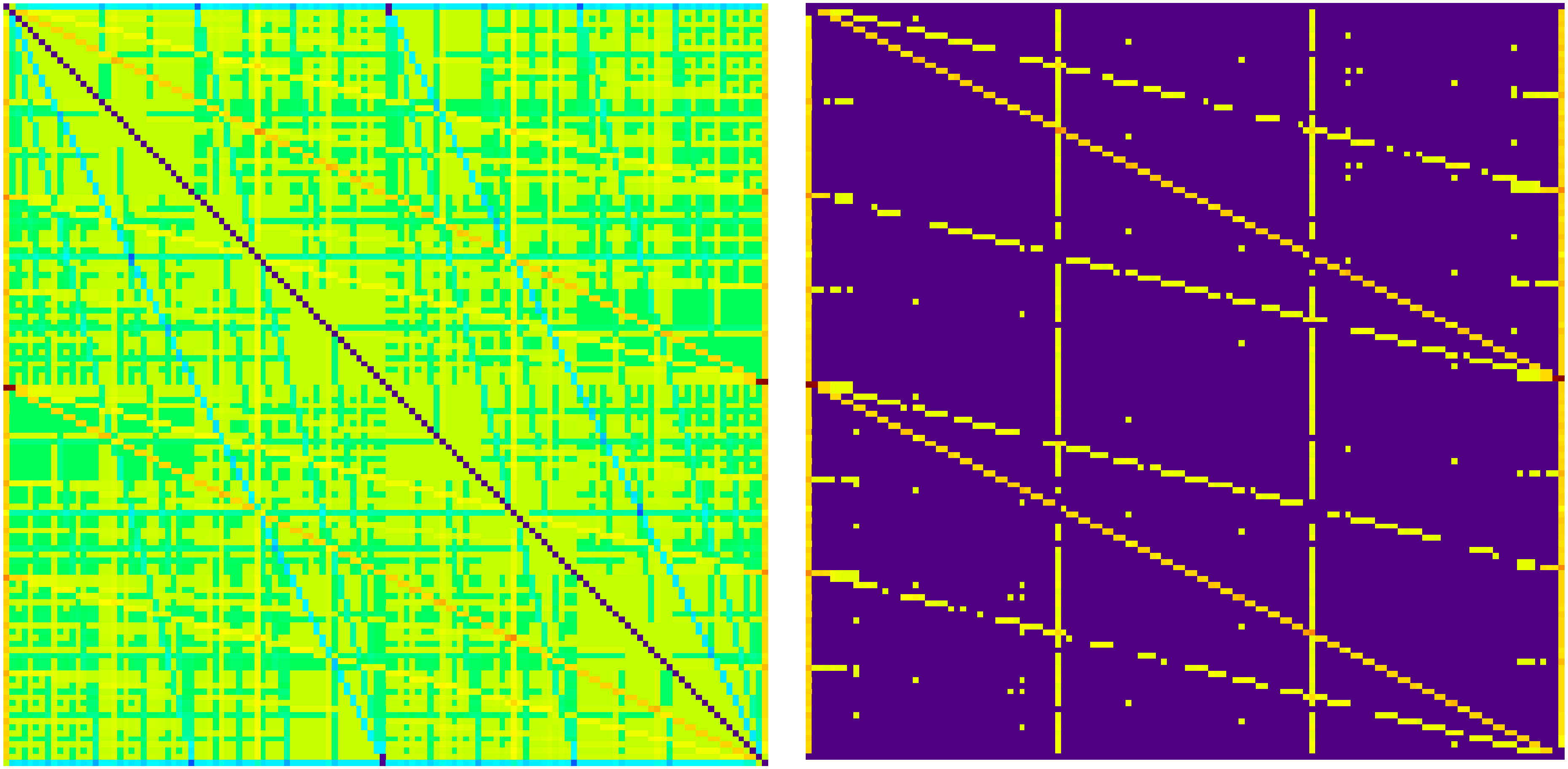
P≥0.6, i.e. the probability that “raw wins the column" is not less than 60% (blue background – probabilities less than 60%). As can be seen, ***whatever*** word selected by the “column”, the “raw” ***always*** has an adequate answer (with probability *p*≥0.6), but the set of these answers is “or measure zero”.



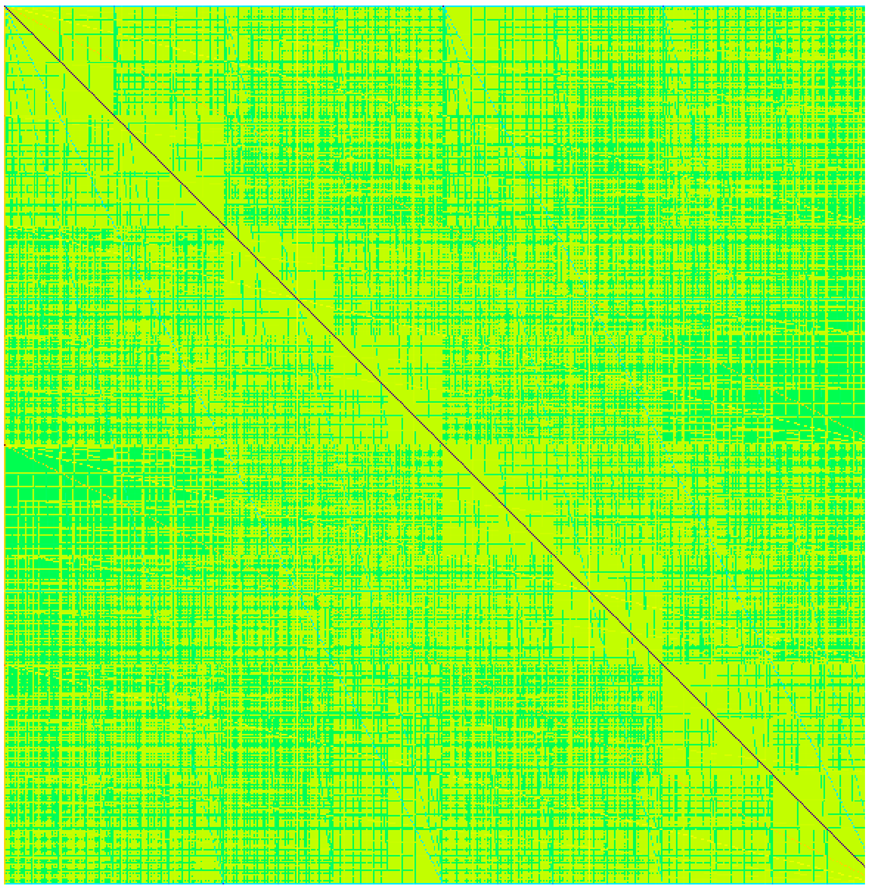
CONWAY[5, 0]



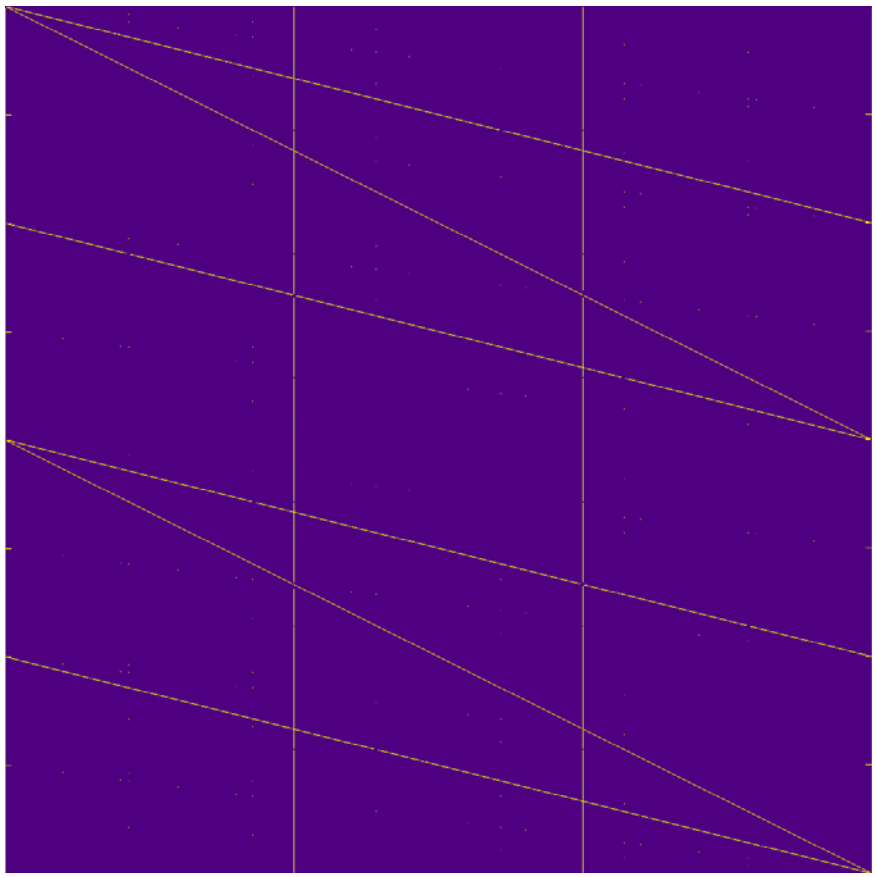
CONWAY[5, 0.6]



CONWAY[7, 0]   CONWAY[7, 0.55]



CONWAY[10, 0.]



CONWAY[10, 0.55]

Conway matrix is a map

$$C_n : \quad W \to \{0, 1 \ldots, 2^n - 1\}$$

$$W = V \oplus V$$

$$C_n(w) = C_n(u, v) = c_{uv}$$

In other words, we have an integer-valued function on the Abelian group. Our aim is to find the weight-spectrum of this function, i.e. the *(n+1)*x*(n+1)* integer matrix $S(C_n)$

$$S_{p,q} = \sum_{wt(u)=p, wt(v)=q} c(u, v)$$

$$W = V \oplus V \qquad C_n : \quad W \to \{0, 1 \ldots, 2^n - 1\}$$

$$C_n(w) = C_n(u, v) = c_{uv}$$

To find the weight-spectrum let us first consider the Fourier transform.

Since W is a direct sum of two Abelian (commutative) groups, it is an Abelian (commutative) group, and we can apply the Fourier transform. In this case it is usually called the Hadamard transform. Since we prefer to deal with integers, we will consider non-normalized transform.

$$\widehat{C}_n = H_n C_n H_n$$

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where H_n is the $2^n \times 2^n$ Hadamard matrix defined by the well-known recurrent formula

$$H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \qquad n \geq 2, \quad H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The reason for this move from C_n to its Hadamard transform $\widehat{C}_n$ is that it has been shown by us that the latter has the very simple matrix structure:

$$(\widehat{C}_n)_{ij} = 2^{2n-1} \begin{cases} n \quad , \quad if \;\; i = j = 0 \\ 1 \quad , \quad if \;\; j = 2^k i \, , \; for \; some \; k \, , 0 \le k \le n-1 \end{cases}$$

$$0 \le i \, , j \le 2^n - 1$$

For example:

***n*=3**

$$C_3 = \begin{pmatrix} 7 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 1 & 3 & 3 \\ 3 & 3 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 7 \end{pmatrix}$$

$$\widehat{C}_3 = 2^{2\times 3-1} \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Given the spectrum $S(\widehat{C}_n)$, we can easily find the spectrum $S(C_n)$ by the MacWilliams formula for the dual spectrums

$$2^{2n} S_{k_1, k_2}(C_n) = \sum_{m_1, m_2 = 0}^{n} S_{m_1, m_2}(\widehat{C}_n) P_{k_1}(m_1) P_{k_2}(m_2)$$

$$0 \le k_1, k_2 \le n$$

where $P_k(m)$ are the Krawtchouk polynomials (of order n) defined by the generating function

$$(1+t)^{n-m}(1-t)^m = \sum_{k=0}^{n} P_k(m) t^k$$

$$P_k(m) = \sum_{l=0}^{k} (-1)^l \binom{n-m}{k-l} \binom{m}{l}$$

After some calculations we find:

$$S_{m_1,m_2}(\widehat{C}_n) = 2^{2n-1} \cdot \begin{cases} n, & m_1 = m_2 = 0 \\ \binom{n+1}{m+1} \cdot \delta_{m_1,m_2} & m_1 = m_2 = m\,(\neq 0) \end{cases}$$

And, applying the MacWilliams formula we find after some transformations

$$S_{k_1,k_2}(C_n) = \frac{1}{2}\left(\left(\sum_{p \le \min(k_1,k_2)} \binom{n+1}{p}\right) \cdot \left(\sum_{p > \max(k_1,k_2)} \binom{n+1}{p}\right) - \binom{n}{k_1}\binom{n}{k_2}\right)$$

$$S_{k_1,k_2}(C_n)=\frac{1}{2}\left(\left(\sum_{p\le\min(k_1,k_2)}\binom{n+1}{p}\right)\cdot\left(\sum_{p>\max(k_1,k_2)}\binom{n+1}{p}\right)-\binom{n}{k_1}\binom{n}{k_2}\right)$$

Let

$$S(C_n)=\frac{1}{2}\left(\mathrm{B}_n-Y_n\otimes Y_n{}^{\mathrm{T}}\right)$$

where

$$\mathrm{B}_n=(\beta_{k_1,k_2})_{0\le k_1,k_2\le n}$$

$$Y_n=\left(\binom{n}{s}\right)_{0\le s\le n}$$

$$S(C_n) = \frac{1}{2}\left(\mathrm{B}_n - Y_n \otimes Y_n^{\mathrm{T}}\right)$$

Since Y_n is a row of the binomial coefficients, everywhere in what follows, we shall refer to the Kronecker product $Y_n \otimes Y_n^{\mathrm{T}}$ as a "trivial term" of the matrix $S(C_n)$ and our attention will be mainly devoted to the matrix B_n (ignoring the multiplier 1/2) as a nontrivial number-theoretical, combinatorial and computational object.

For example:

$$S(C_3) = \begin{pmatrix} 7 & 4 & 1 & 0 \\ 4 & 23 & 8 & 1 \\ 1 & 8 & 23 & 4 \\ 0 & 1 & 4 & 7 \end{pmatrix}$$

$$\mathrm{B}_3 = \begin{pmatrix} 15 & 11 & 5 & 1 \\ 11 & 55 & 25 & 5 \\ 5 & 25 & 55 & 11 \\ 1 & 5 & 11 & 15 \end{pmatrix} \qquad Y_3 \otimes Y_3^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

The recurrent formula and fast computational algorithm for matrices B_n

Let Δ_{n+1} be the diagonal matrix: $\Delta_{n+1} = 2^n \cdot diag\left(\binom{n}{r}\right)_{0\le r\le n}$

Furthermore, for any matrix A let $|\overline{A}$ stands for the zero-padding of A from its left and upper sides and the notations $\overline{A}|$, $|\underline{A}$ and $\underline{A}|$ have the analogous meaning.

Then it can be demonstrated that

$$\mathrm{B}_{n+1} = \left| \overline{\mathrm{B}}_n + \overline{\mathrm{B}}_n \right| + \left| \underline{\mathrm{B}_n} + \underline{\mathrm{B}_n} \right| + \Delta_{n+1}$$

$$\Delta_{n+1} = 2 \cdot \left(\left| \overline{\Delta_n} + \underline{\Delta_n} \right| \right) \qquad n \geq 1$$

$$\mathrm{B}_0 = \Delta_0 = (1)$$

These formulas provide us the recursive construction of matrices B_n "without multiplications" (ignoring the multiplication by 2), so we refer to them as a "fast algorithm" for B_n

For example:

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 7 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 7 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 4 & 16 & 4 \\ 0 & 1 & 4 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 7 & 4 & 1 & 0 \\ 4 & 16 & 4 & 0 \\ 1 & 4 & 7 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 7 & 4 & 1 \\ 0 & 4 & 16 & 4 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 7 & 4 & 1 & 0 \\ 4 & 16 & 4 & 0 \\ 1 & 4 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ 2\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 15 & 11 & 5 & 1 \\ 11 & 55 & 25 & 5 \\ 5 & 25 & 55 & 11 \\ 1 & 5 & 11 & 15 \end{pmatrix}$$

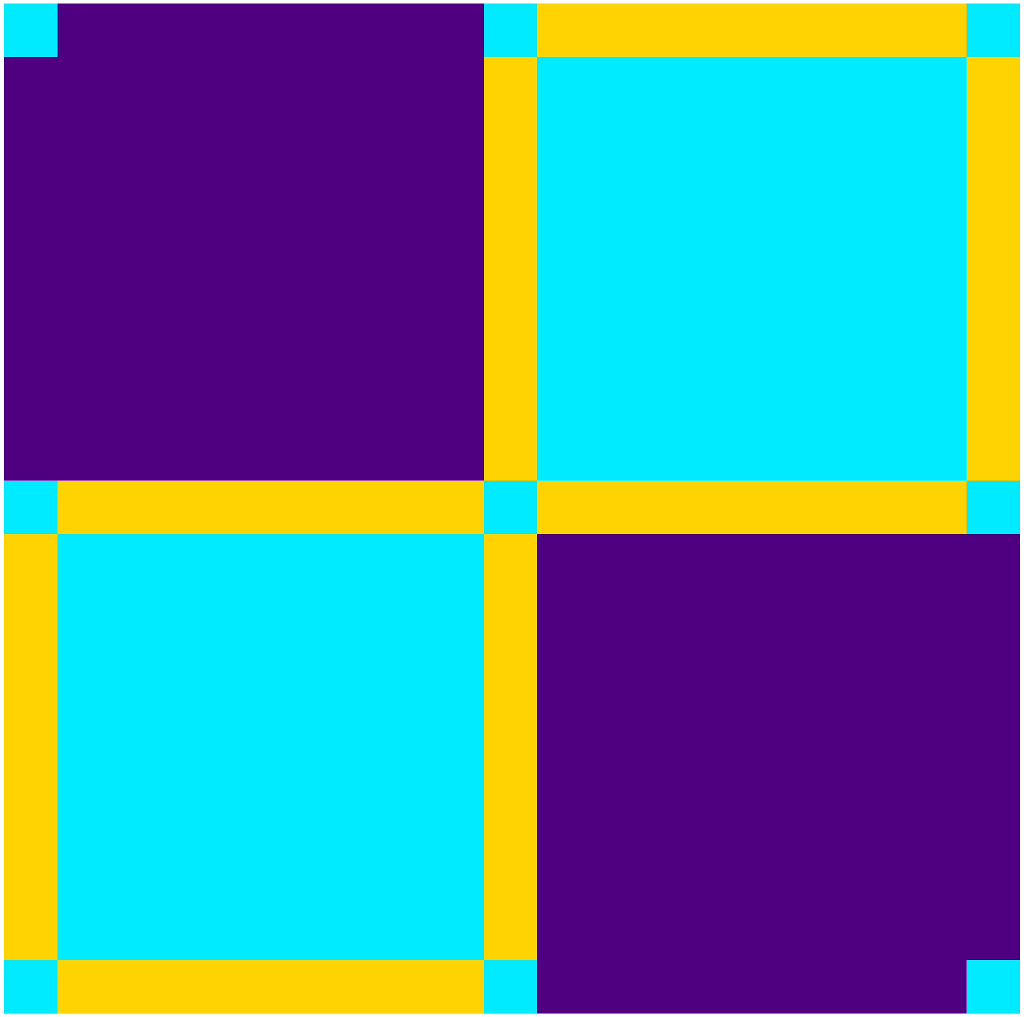
$$\mathrm{B}_{n+1} = |\overline{\mathrm{B}}_n + \overline{\mathrm{B}}_n| + |\underline{\mathrm{B}_n} + \underline{\mathrm{B}_n}| + \Delta_{n+1}$$

$$\Delta_{n+1} = 2 \cdot \left(|\overline{\Delta_n} + \underline{\Delta_n}| \right)$$

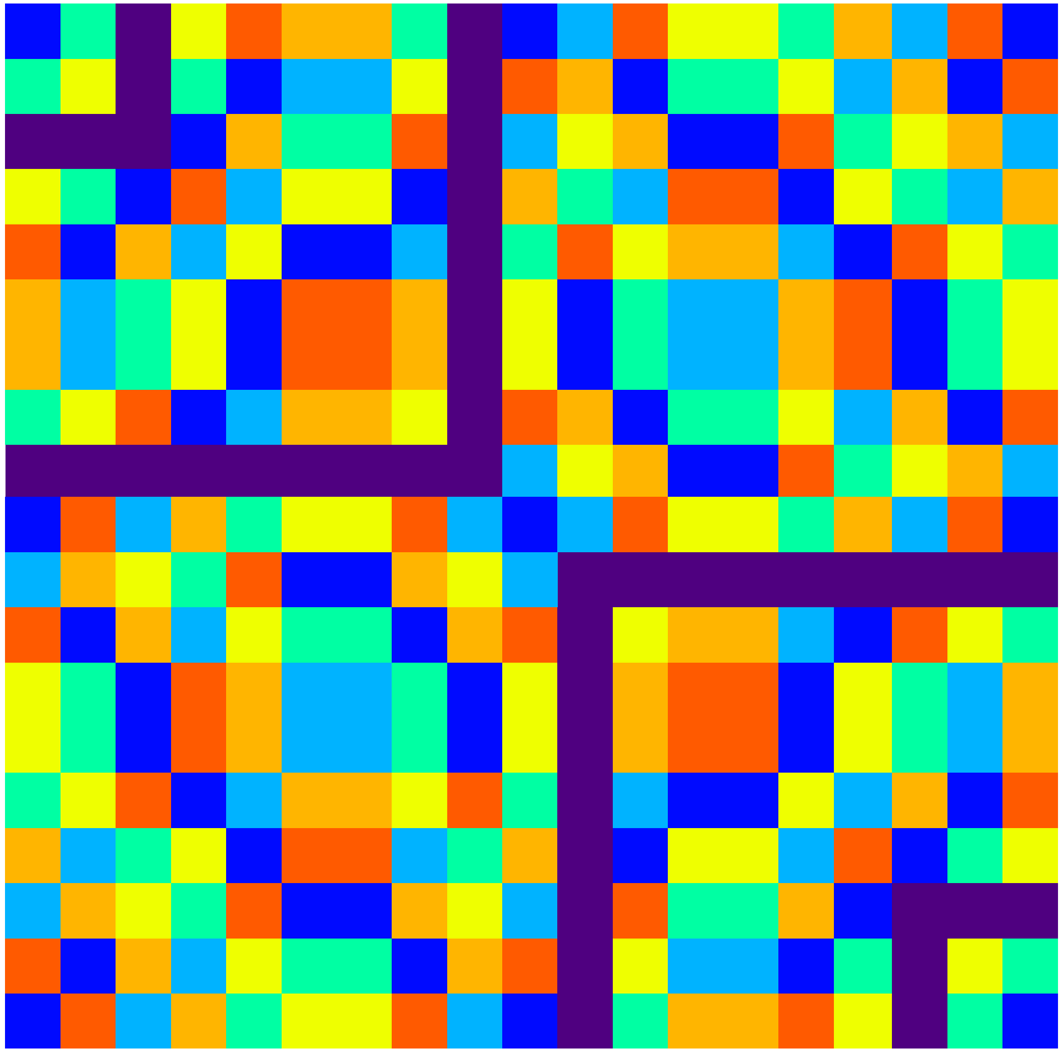
These recursive formulae show that in fact B_n is the sum of the results of the *n*-th steps of two cellular automata, where B_n is 2D cellular automata and its diagonal Δ_n is one-dimensional.

We found most interesting the behavior of B_n modulo some integer m that we observed with the help of WSCONW.nb, a MATHEMATICA program for visualizing formulae that was made by the authors of this paper. The structure of the resulting images reflects the difficulties related to studying these partial binomial sums, difficulties which have been mentioned by many authors.

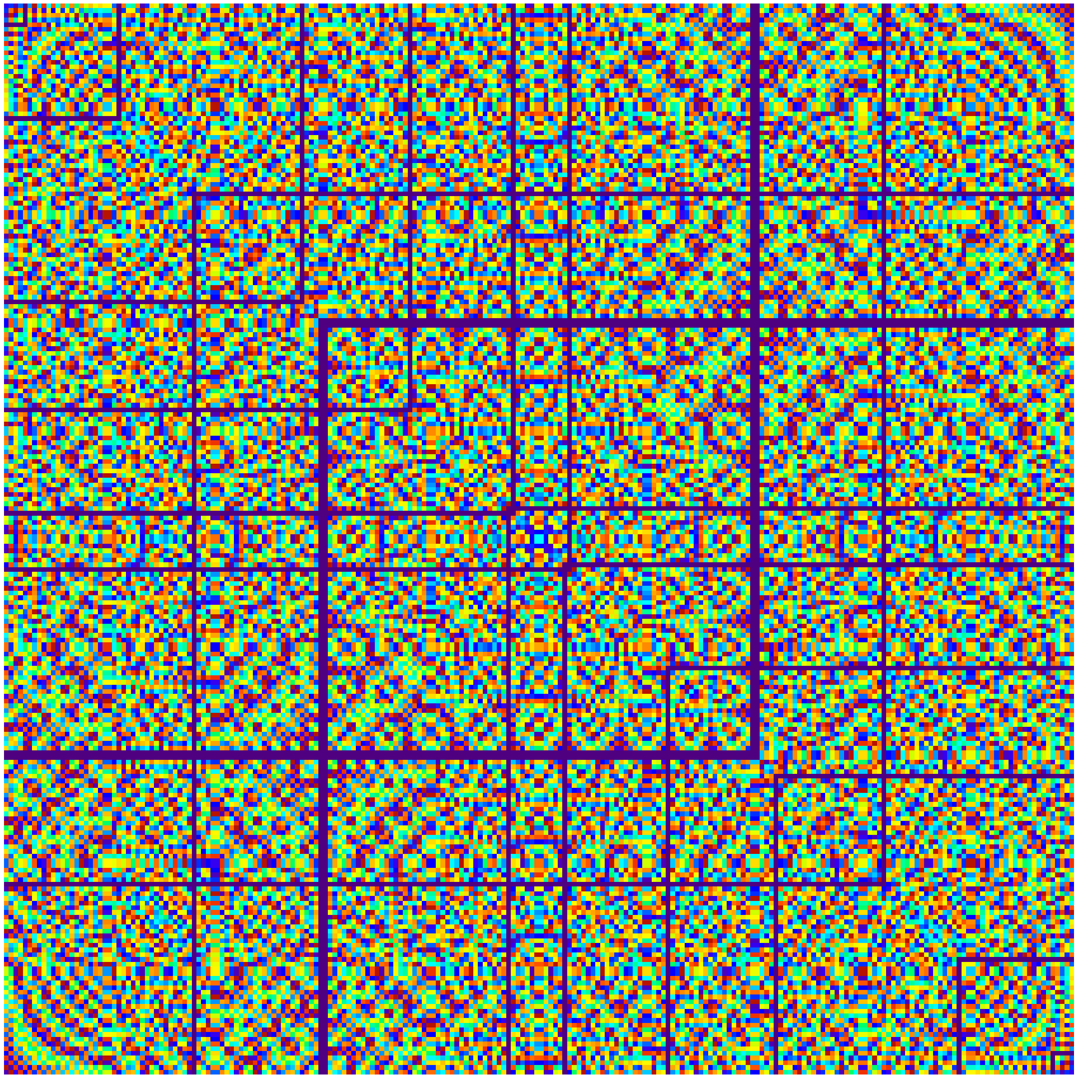
Some examples are given below. Notice the effect of the optical illusions of the “looking through the frosted glass” in the case n=361, m=121 and askew deformations in the case n=196, m=49.



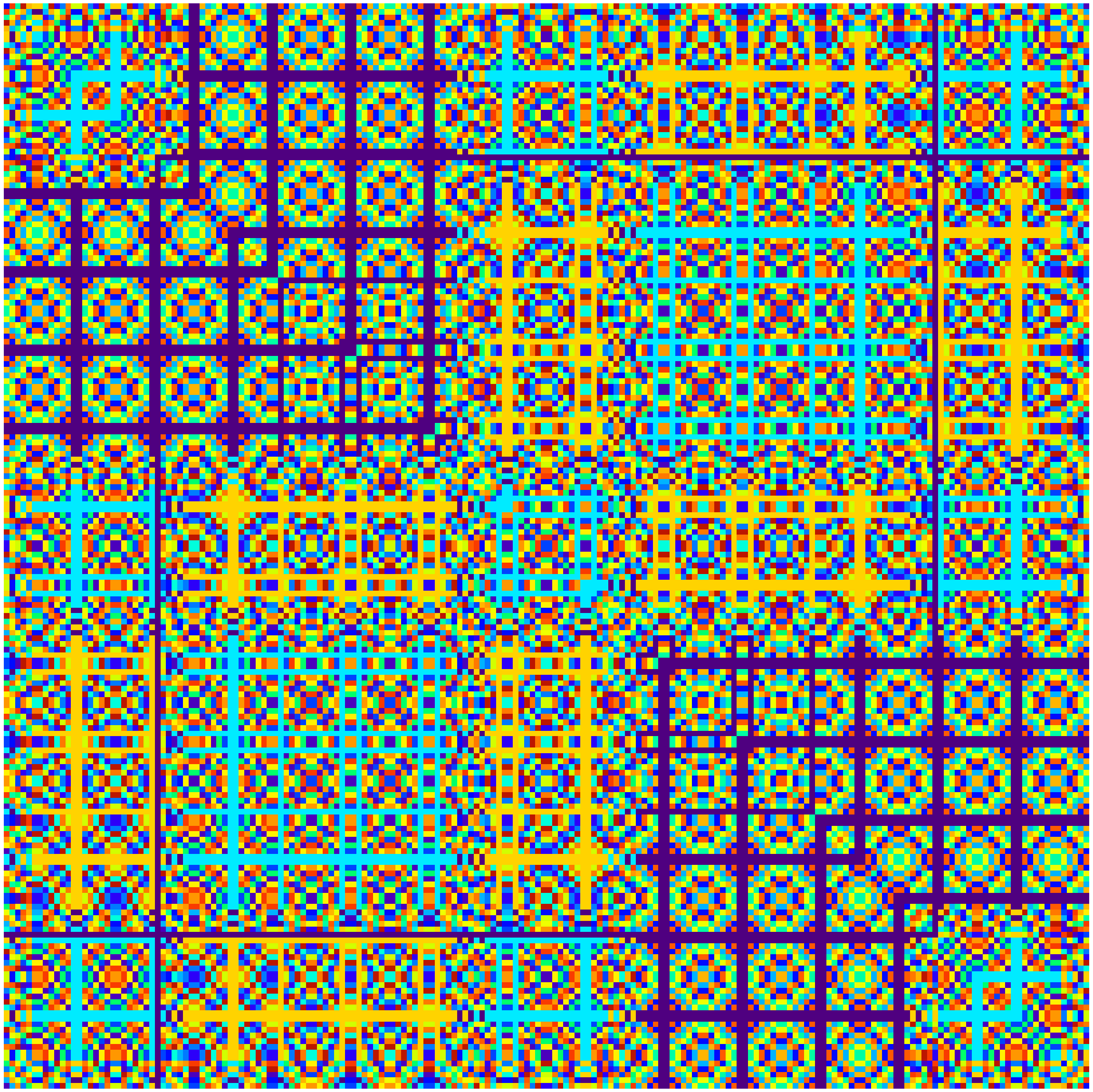
WSCONW[18,3]



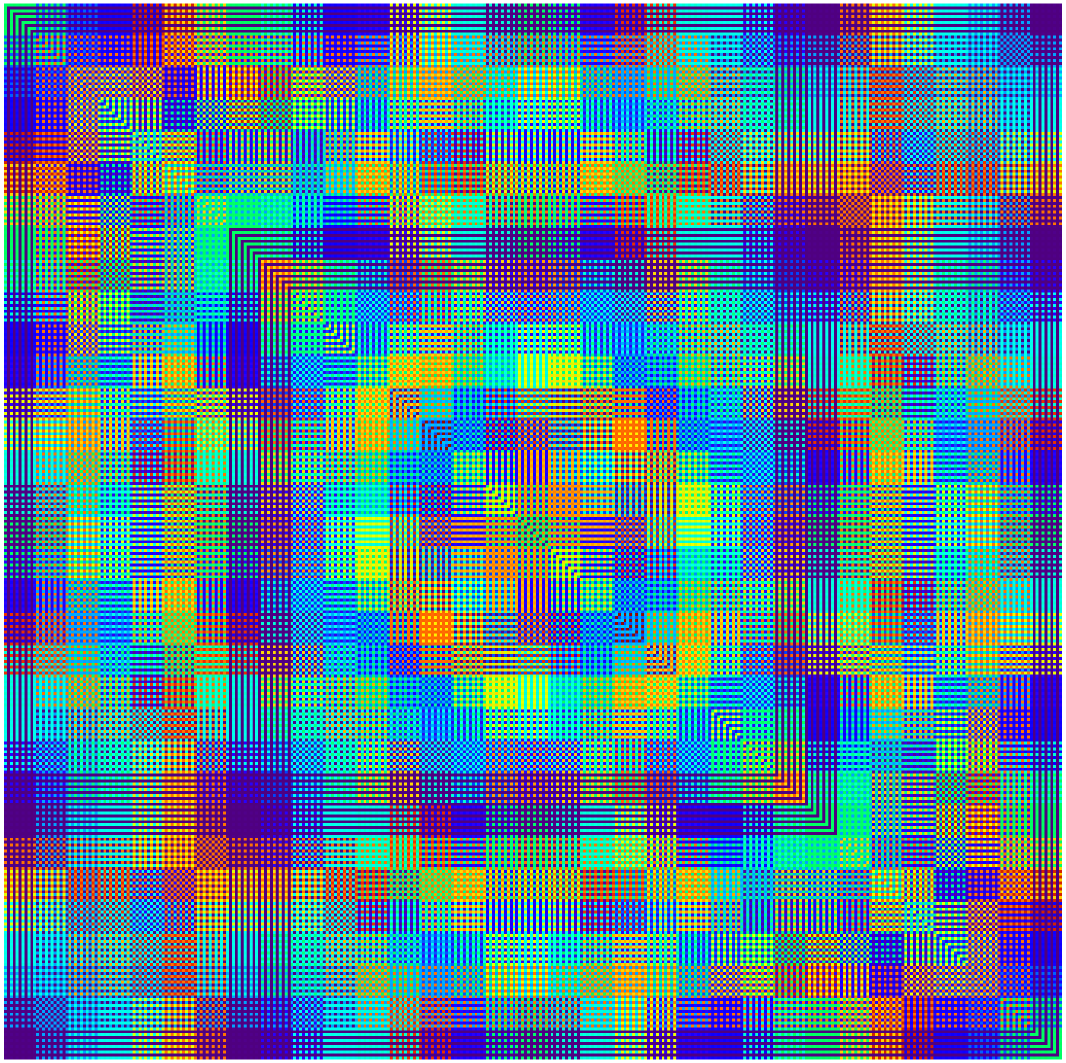
WSCONW[18,7]



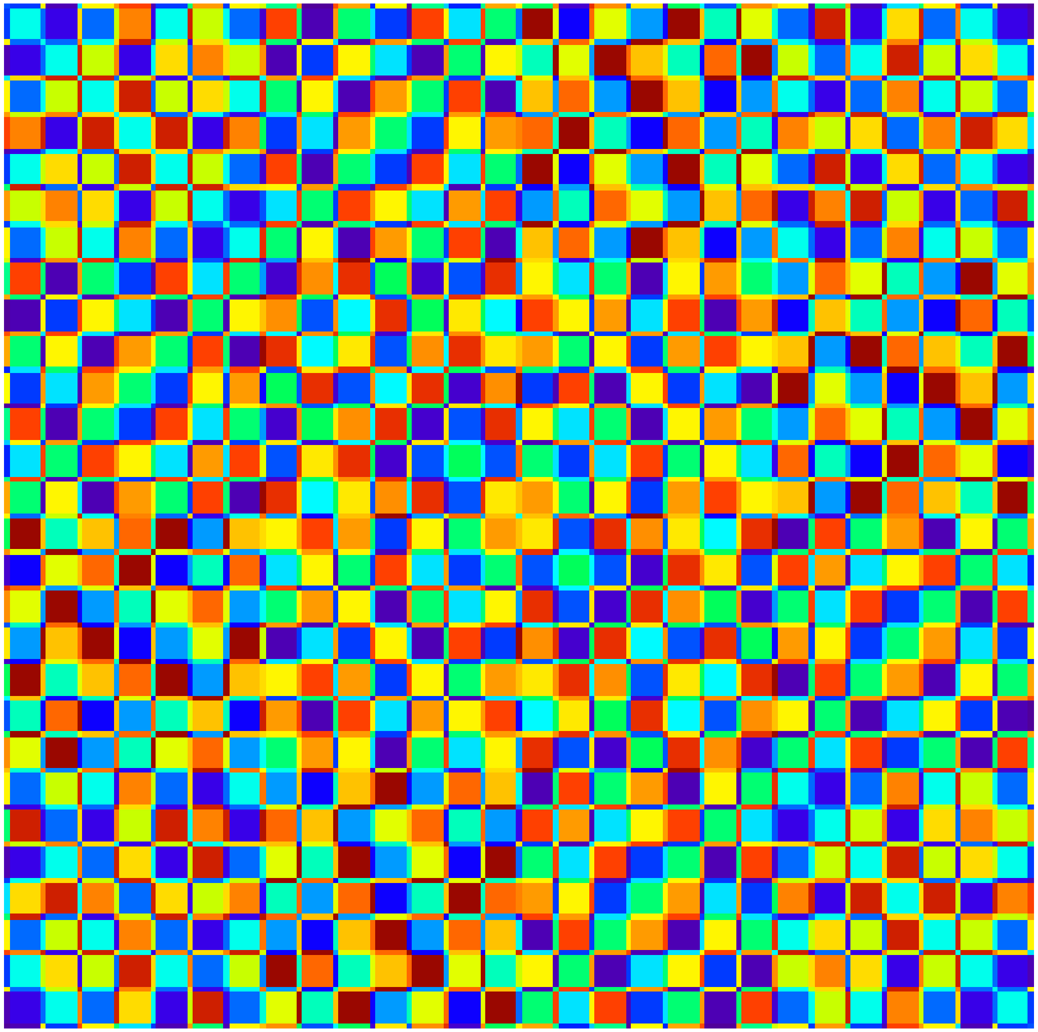
WSCONW[227,23]



WSCONW[193,21]



WSCONW[361,121]



WSCONW[196,49]

Some References

1. Gardner M. Time Travel and Other Mathematical Bewilderments. NY, 1988
2. Guibas L.J. , Odlyzko. String Overlaps , Pattern Matching, and Nontransitive Games. Journal of Combinatorial Theory, March 1981, 30, 183-208.
3. MacWilliams F.J., Sloane N.J.A.. The Theory of Error-Correcting Codes. North-Holland P.C., NY, Amsterdam, 1977
4. Graham, R. L., Knuth, D.E., Patashnik, O. Concrete Mathematics, Addison Wesley, 1989
5. Ollerton, R. L. Partial row-sums of Pascal's triangle , International Journal of Mathematical Education in Science and Technology Volume 38, Number 1/15, January 2007, pp. 124-127