On the dimension of solution spaces of full rank linear differential systems

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Given a full rank system of linear ordinary differential equations of arbitrary order, we examine the change in the dimension of its solution space due to differentiation of one of its equations.

Scalar case $L(y) = 0$:

$$(L(y))' = 0$$

The new equation has some extra solutions, if, e.g., the differential field $\mathbb{K}$ of coefficients of equations and the "functional" space Λ where we consider the solutions of equations are such that any equation of order $n \geq 0$ has the solution space of dimension n.

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$\mathrm{Mat}_n(R)$, I_n, M^T

Let

$$(\mathbb{K}, \partial)$$

($\partial ='$) be a differential field of characteristic 0 with an algebraically closed constant field

$$Const(\mathbb{K}) = \{c \in \mathbb{K} \mid \partial c = 0\}.$$

We denote by Λ a fixed *universal differential extension field* of $\mathbb{K}$ (Singer, van der Put).
This is a differential extension Λ of $\mathbb{K}$ with

$$Const(\Lambda) = Const(\mathbb{K})$$

such that any differential system

$$y' = Ay, \tag{1}$$

with $A \in \mathrm{Mat}_n(\mathbb{K})$ has a *solution space* of dimension n over constants.

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If, e.g., $\mathbb{K}$ is a subfield of the field $\mathbb{C}((x))$ of formal Laurent series with complex coefficients with $\partial = \frac{d}{dx}$ then we can consider Λ as the quotient field of the ring generated by expressions of the form

$$e^{P(x)} x^{\gamma} (\psi_0 + \psi_1 \log x + \cdots + \psi_s (\log x)^s), \tag{2}$$

where in any such an expression

- $P(x)$ is a polynomial in $x^{-1/p}$, where p is a positive integer,
- $\gamma \in \mathbb{C}$,
- s is a non-negative integer and $\psi_i \in \mathbb{C}[[x^{1/p}]]$, $i = 0, 1, \ldots, s$.

Besides first-order systems of the form (1) we will consider differential systems of order $r \geq 1$ which have the form

$$A_r y^{(r)} + A_{r-1} y^{(r-1)} + \cdots + A_0 y = 0. \tag{3}$$

The coefficient matrices

$$A_0, A_1, \ldots, A_r \tag{4}$$

belong to $\mathrm{Mat}_m(\mathbb{K})$, and A_r (the *leading matrix* of the system) is non-zero.

If A_r is invertible in $\mathrm{Mat}_m(\mathbb{K})$ then the system (3) is equivalent to the first order system having mr equations:

$$Y' = AY, \tag{5}$$

with

$$A = \begin{pmatrix} 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_m \\ \hat{A}_0 & \hat{A}_1 & \dots & \hat{A}_{r-1} \end{pmatrix}, \tag{6}$$

where $\hat{A}_k = -A_r^{-1}A_k$, $k = 0, 1, \dots, r-1$, and

$$Y = \left(y_1 \dots, y_m, \, y_1' \dots, y_m', \dots, \, y_1^{(r-1)}, \dots, y_m^{(r-1)}\right)^T. \tag{7}$$

Therefore if the leading matrix of the system (3) is invertible then the dimension of the solution space of this system is equal to mr.

System (3) can be also written as a system of m scalar linear equations

$$L_1(y_1, \dots, y_m) = 0, \quad \dots, \quad L_m(y_1, \dots, y_m) = 0. \tag{8}$$

When a system is represented in the form (8) we can rewrite it in the form (3) and vice versa. The matrix A_r is the leading matrix of the system regardless of the representation form.

We suppose that the system is of *full rank*, i.e., that equations (8) are independent over $\mathbb{K}[\partial]$.

Theorem 1

Let a system of the form (8) be of full rank. Let the system

$$L_1(y_1, \dots, y_m) = 0, \;\; \dots, \;\; L_{m-1}(y_1, \dots, y_m) = 0, \;\; \tilde{L}_m(y_1, \dots, y_m) = 0, \tag{9}$$

be such that its first $m-1$ *equations are as in the system (8) while the m-th equation is the result of differentiation of the m-th equation of (8) (thus the equation* $\tilde{L}_m(y_1, \dots, y_m) = 0$ *is equivalent to the equation* $(L_m(y_1, \dots, y_m))' = 0$*). Then the dimension of the solution space of (9) exceeds by* 1 *the dimension of the solution space of (8).*

Invertible leading matrix case

$$L_1(y_1, \dots, y_m) = 0, \quad \dots, \quad L_{m-1}(y_1, \dots, y_m) = 0,$$
$$L_m(y_1, \dots, y_m) = z, \quad z' = 0. \quad (10)$$

$Y' = AY$

System (10) is equivalent to the system $\tilde{Y}' = \tilde{A}\tilde{Y}$ where

$$\tilde{A} = \begin{pmatrix} & & & & 0 \\ & A & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the solution space of $\tilde{Y}' = \tilde{A}\tilde{Y}$ is equal to $mr + 1$, while the dimension of the solution space of $Y' = AY$ is equal to mr.

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General case

System (3) can be written as $L(y) = 0$ where

$$L = A_r \partial^r + A_{r-1} \partial^{r-1} + \cdots + A_0 \in \mathcal{D}_m. \quad (11)$$

Denote the ring $\mathrm{Mat}_m(\mathbb{K})\,[\partial]$ by $\mathcal{D}_m$.

It can be proved that if L is a full rank operator of the form (11) then there exists $N \in \mathcal{D}_m$ such that the leading matrix of LN is invertible. (In addition, N can be taken such that LN is of order r.)

Set

$$D = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \partial + \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \quad (12)$$

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The dimension of the solution space of the system

$$DLN(y) = 0 \tag{13}$$

is bigger than the dimension of the solution space of the system

$$LN(y) = 0. \tag{14}$$

This implies that there exists $\varphi \in \Lambda^m$ such that

$$N(\varphi) \tag{15}$$

is a solution of the system

$$DL(y) = 0 \tag{16}$$

but is not a solution of $L(y) = 0$.

An application: Dimension of the solution space of a given full rank system

We use the notation

$$[M]_{i,*}\,,\quad 1 \le i \le m, \tag{17}$$

for the $(1 \times m)$-matrix which is the i-th row of an $(m \times m)$-matrix M.

Let a full rank operator $L \in \mathcal{D}_m$ be of form

$$L = A_r\partial^r + A_{r-1}\partial^{r-1} + \cdots + A_0 \in \mathcal{D}_m. \tag{18}$$

If $1 \le i \le m$ then define $\alpha_i(L)$ as the maximal integer k, $1 \le k \le r$, such that $[A_k(x)]_{i,*}$ is a nonzero row.

The matrix $M(x) \in \mathrm{Mat}_m(\mathbb{K})$ such that

$$[M(x)]_{i,*} = [A_{\alpha_i(L)}]_{i,*},\quad i = 1, 2, \ldots, m, \tag{19}$$

is the *row frontal matrix* of L.

Example

$$L = \begin{pmatrix} \underline{1} & \underline{0} & \underline{0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial^3 + \begin{pmatrix} 0 & 2 & 0 \\ \underline{1} & \underline{x} & \underline{0} \\ 0 & 0 & 0 \end{pmatrix} \partial^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \underline{1} & \underline{x} & \underline{0} \end{pmatrix} \partial + \begin{pmatrix} x & 0 & x^2 + x \\ 0 & 0 & 2x^2 + 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (20)$$

The row frontal matrix is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & x & 0 \end{pmatrix} \quad (21)$$

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Theorem 2

Let the row frontal matrix of a full rank system $L(y) = 0$, $L \in \mathcal{D}_m$, be invertible. Then the dimension of the solution space of this system is $\alpha_1(L) + \alpha_2(L) + \cdots + \alpha_m(L)$.

It follows directly from Theorem 1: when we differentiate $r - \alpha_1(L)$ times the i-th equation of the given system, $i = 1, 2, \ldots, m$, we increase the dimension of the solution space by

$$mr - (\alpha_1(L) + \alpha_2(L) + \cdots + \alpha_m(L)), \tag{22}$$

and the received full rank system has the leading matrix which coincides with the row frontal matrix of the original system, therefore the obtained system has an invertible leading matrix and the dimension of its solution space is equal to mr.

An algorithm for transforming a given full rank system to an equivalent system having an invertible row frontal matrix was proposed by M.A. Barkatou, C. El Bacha, G. Labahn and E. Pflügel. Using this algorithm we can find an equivalent operator for operator (20):

$$\bar{L} = \begin{pmatrix} \underline{1} & \underline{0} & \underline{0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial^3 + \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial^2 + \begin{pmatrix} 0 & 0 & 0 \\ \underline{0} & \underline{-1} & \underline{-1} \\ \underline{1} & \underline{x} & \underline{0} \end{pmatrix} \partial + \begin{pmatrix} x & 0 & x^2+x \\ 0 & 0 & 2x^2+1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (23)$$

The row frontal matrix is invertible, and

$$\alpha_1(\bar{L}) = 3, \ \ \alpha_2(\bar{L}) = 1, \ \ \alpha_3(\bar{L}) = 1. \quad (24)$$

By Theorem 2 the dimension of the solution space of $\bar{L}(y) = 0$ is $3 + 1 + 1 = 5$. The same holds for $L(y) = 0$.

Remark. The algorithm is correct when the field $\mathbb{K}$ is *constructive*, in particular that the zero testing problem in $\mathbb{K}$ is decidable.

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A universal differential extension of $\mathbb{K}$ is not unique, and we can consider the solution space of a system only after fixing such an extension. But due to Theorem 2 it is no matter which universal extension is fixed when we consider the dimension of the solution space.

One more application: Proving algorithms termination

The idea is the following. Regardless of the fact that solutions belonging to a universal differential extensions are not possibly of our interest, each differentiation of one of equations increases the dimension of the solution space. Suppose that each step of an algorithm makes some equivalent transformation of the system and differentiates an equation after this, and that the order of the system and the number of equations do not grow up, then the number of such steps is bounded by rm.

Using this approach we propose an improved version of EG_δ algorithm by S.Abramov and D.Khmelnov, which transforms step-by-step a given full rank system into an appropriate form.

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