On Possibility of Additional Solutions of the Degenerate System Near Double Degeneration at the Special Value of the Parameter

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We consider an autonomous system of ordinary differential equations, which is solved with respect to derivatives.

To study local integrability of the system near a degenerate stationary point, we use an approach based on Power geometry and on the computation of the resonant normal form.

For the concrete planar 5-parametric system, we found the set of necessary conditions on parameters of the system for which the system is locally integrable near a degenerate stationary point.

This set consists of 4 two-parametric sets in this 5-parametric space.

Because these methods are constructive we get for these 4 sets first integrals of the system. At these set of parameters the system is globally integrable.

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But the technique supposes excluding from the parametric space the hyperplane $b^2=2/3$. Here we try to discuss possibilities studying of this domain.

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Introduction

We consider an autonomous system of ordinary differential equations

$$dx_i/dt \stackrel{\text{def}}{=} \dot{x}_i = \varphi_i(X), \quad i = 1, \ldots, n, \tag{1}$$

where $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\varphi_i(X)$ are polynomials.

In a neighborhood of the stationary point $X = X^0$, the system (1) is *locally integrable* if it has there sufficient number m of independent first integrals of the form

$$a_j(X)/b_j(X), \quad j = 1, \ldots, m,$$

where functions $a_j(X)$ and $b_j(X)$ are analytic in a neighborhood of the point $X = X^0$. Otherwise we call the system (1) *locally nonintegrable* in this neighborhood.

For a planar system m=1.

In [BrunoEdneral:2009], it was proposed a method of analysis of integrability of a system based on power transformations and computation of normal forms near stationary solutions of transformed systems [Bruno:1998].

In this report we demonstrate how this approach can be applied to the study of local integrability of the planar case (i.e. $n = 2$) of the system (1) near the stationary point $X^0 = 0$ of high degeneracy.

In the neighborhood of the stationary point $X = 0$ the system (1) can be written in the form

$$\dot{X} = A\,X + \tilde{\Phi}(X), \tag{2}$$

where $\tilde{\Phi}(X)$ has no linear in X terms.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the matrix A. If at least one of them $\lambda_i \neq 0$, then the stationary point $X = 0$ is called an *elementary* stationary point. In this case the system (2) has a normal form which is equivalent to a system of lower order [Bruno:1979]. If all eigenvalues vanish, then the stationary point $X = 0$ is called a *nonelementary* stationary point. In this case there is no normal form for the system (2). But by using power transformations, a nonelementary stationary point $X = 0$ can be blown up to a set of elementary stationary points. After that, it is possible to compute the normal form and verify that the condition A (see later) is satisfied [Bruno:1971] in each elementary stationary point.

In this paper we demonstrate how this approach can be applied to study the local and global integrability in the planar case of the system (1) near the stationary point $X^0 = 0$ of high degeneracy

$$\begin{aligned} \dot{x} &= \alpha\, y^3 + \beta\, x^3\, y + (a_0\, x^5 + a_1\, x^2 y^2) + (a_2\, x^4\, y + a_3\, x\, y^3), \\ \dot{y} &= \gamma\, x^2\, y^2 + \delta\, x^5 + (b_0\, x^4 y + b_1\, x\, y^3) + (b_2\, x^6 + b_3\, x^3\, y^2 + b_4\, y^4). \end{aligned} \qquad \text{(M)}$$

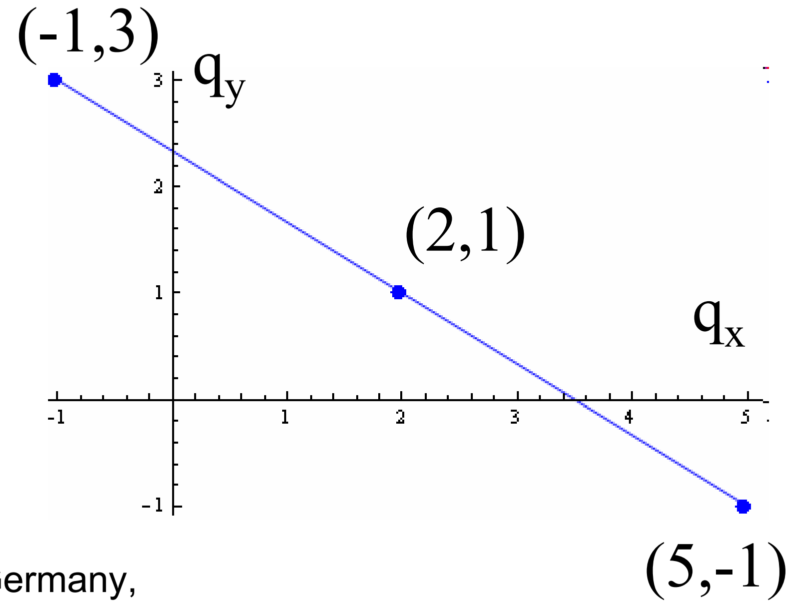
For the first case, with the additional assumption that the polynomial *H(X)* is expandable into the product of only square free factors, the problem is solved in [Algaba et.al.:2009]. Therefore here we discuss only the second case. More precisely, we study the system with *R*=(2,3) and *s*=7.

At $R = (2,3)$ and $s = 7$ the quasi-homogeneous system (4) has the form

$$\dot{x} = a\,y^3 + b\,x^3\,y, \quad \dot{y} = c\,x^2\,y^2 + d\,x^5, \tag{5}$$

where $a \neq 0$ and $d \neq 0$.

$$dx/dt = x(ax^{-1}y^3 + bx^2y)$$
$$dy/dt = y(cx^2y + dx^5y^{-1})$$



Lemma 1. *If the system (5) with $b \neq 0$ and $c \neq 0$ has the first integral*

$$I = \alpha\, y^4 + \beta\, x^3\, y^2 + \gamma\, x^6, \quad \beta \neq 0, \tag{6}$$

then

$$(a\,d - b\,c)(3\,b + 2\,c) = 0. \tag{7}$$

Proof. A derivative of the integral (6) with respect to the system (5) has the form

$$\partial I/\partial x\,(a\,y^3 + b\,x^3 y) + \partial I/\partial y\,(c\,x^2 y^2 + d\,x^5) =$$
$$= (3\,\beta\,a + 4\,\alpha\,c)x^2 y^5 + (6\,\gamma\,a + 3\,\beta\,b + 2\,\beta\,c + 4\,\alpha\,d)x^5 y^3 +$$
$$+(6\,\gamma\,b + 2\,\beta\,d)x^8 y \equiv 0,$$

thus coefficients at three monomials $x^p y^q$ are equal to zero, i.e.

$$\begin{aligned} &3\,\beta\,a + 4\,\alpha\,c = 0, \quad 6\,\gamma\,b + 2\,\beta\,d = 0, \\ &6\,\gamma\,a + 3\,\beta\,b + 2\,\beta\,c + 4\,\alpha\,d = 0. \end{aligned} \tag{8}$$

From the first two equations (8), we obtain

$$\alpha = -\frac{3\,\beta\,a}{4\,c}, \quad \gamma = -\frac{\beta\,d}{3\,b}. \tag{9}$$

Substituting these values in the third equations (8), cancelling the factor β, multiplying $(b\,c)$, and simplifying we obtain the equality (7). □

More detail Consideration

We will study the case $R = (2, 3)$ and $s = 7$, when the quasi-homogeneous system (6) has the form

$$\dot{x} = \alpha\, y^3 + \beta\, x^3\, y, \quad \dot{y} = \gamma\, x^2\, y^2 + \delta\, x^5, \tag{7}$$

where $\alpha \neq 0$ and $\delta \neq 0$. We can fix two nonzero parameters in (7)

be at least locally integrable. But each autonomous planar quasi-homogeneous system looks like (7) has an integral, but it can have not the form (2) with analytic a_1 and b_1. So we need to have the local integrability of (7) in the sense (2).

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be at least locally integrable. But each autonomous planar quasi-homogeneous system looks like (7) has an integral, but it can have not the form (2) with analytic a_1 and b_1. So we need to have the local integrability of (7) in the sense (2).

Theorem D *In the case* $D \stackrel{\text{def}}{=} (3\,b + 2\,c)^2 - 24 \neq 0$, *the system (8) is locally integrable if and only if the number* $(3\,b - 2\,c)/\sqrt{D}$ *is rational.*

Proof. After the power transformation

$$x = u\,v^2, \quad y = u\,v^3 \tag{9}$$

and time rescaling

$$dt = u^2 v^7 d\tau,$$

we obtain the system (8) in the form

$$\dot{u} = -u\,(3 + (3\,b + 2\,c)u + 2\,u^2), \quad \dot{v} = v\,(1 + (b + c)\,u + u^2).$$

So

$$\frac{d \log v}{d\,u} = -\frac{1 + (b + c)\,u + u^2}{4\,[3 + (3\,b + 2\,c)\,u + 2\,u^2]}. \tag{10}$$

The number D is the discriminant of the polynomial $3+(3\,b+2\,c)\,u+2\,u^2$. In our case $D \neq 0$, so the polynomial has two different roots $u_1 \neq u_2$, and the right hand part of (10) has the form

$$\frac{\xi}{u}+\frac{\eta}{u-u_1}+\frac{\zeta}{u-u_2},$$

where ξ, η and ζ are constants. Direct computation shoes that

$$\xi=-\frac{1}{3}, \quad \eta+\zeta=-\frac{1}{6}, \quad \zeta=-\frac{1+(3\,b-2\,c)/\sqrt{D}}{12}. \tag{11}$$

The first integral of (10) is

$$u^{\xi}\,(u-u_1)^{\eta}\,(u-u_2)\zeta\,v^{-1}$$

According to (11), its integral power can have the form (2) if and only if the number

$$\frac{3\,b - 2\,c}{\sqrt{D}} \tag{12}$$

is rational. The same is true for the integral in variables x, y, because

$$u = \frac{x^3}{y^2}, \quad v = \frac{y}{x}.$$

Proof is finished.

In accordance with Lemma 1.1, the system (5) has the first integral (6) in the two cases:

1. $3\,b + 2\,c = 0$, then in accordance with equalities (9) the integral (6) has the form

$$I = (-\frac{3}{2} a\,y^4 + 2\,c\,x^3 y^2 + d\,x^6)\frac{\beta}{2\,c} \qquad (10)$$

and Hamiltonian function $H = -I\,c/(3\,\beta)$;

2. $a\,d - b\,c = 0$, if $3\,b + 2\,c \neq 0$, then the integral $c_0\,I$ is not a Hamiltonian function for any constant c_0; if $3\,b + 2\,c = 0$, then the integral (10) and Hamiltonian are proportional to the square $(c_1 y^2 + c_2 x^3)^2$, where $c_1, c_2 =$ const.

Multiplying x and y in the system (5) by the constants, we can reduce 2 from 4 parameters a, b, c, d. For example it is possible to take $a = d = 1$.

In [Algaba et al.:2009], systems (3), (5) were studied in the case 1 above. We study them in the case 2.

The Simplest Nontrivial Example

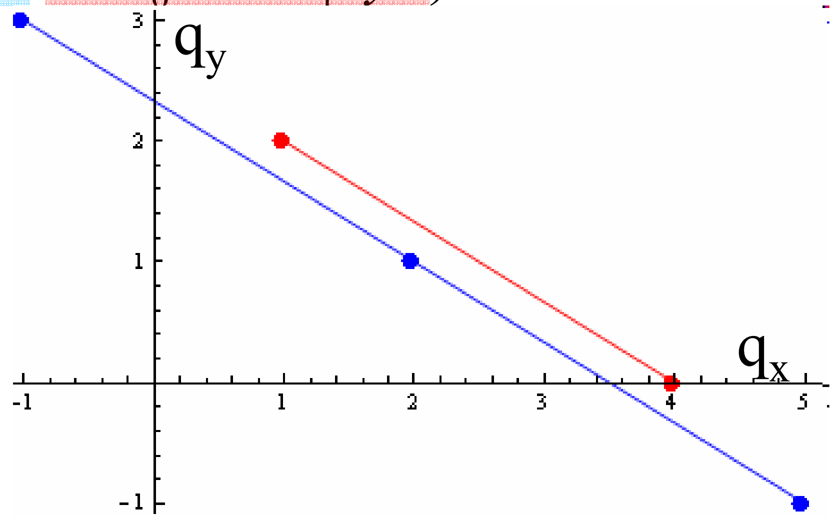
We consider the system

$$
\begin{aligned}
dx/dt &= -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2, \\
dy/dt &= (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3,
\end{aligned} \tag{18}
$$

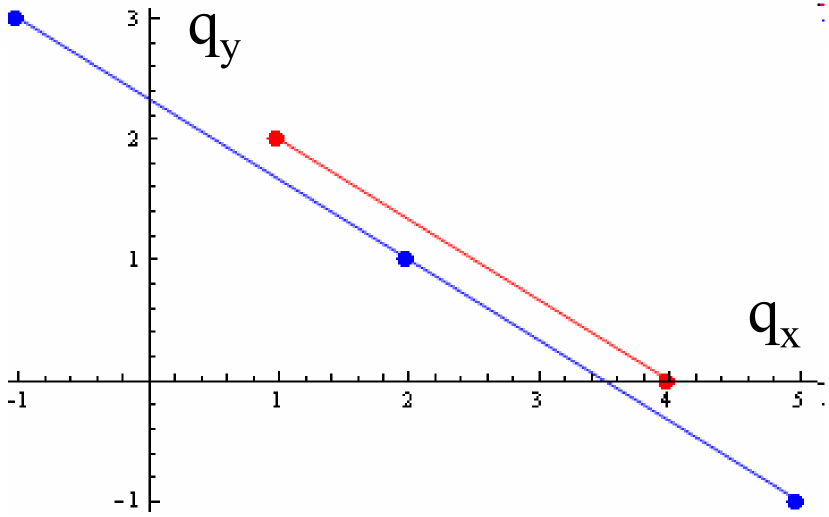
with arbitrary complex parameters a_i, b_i and $b \neq 0$.

$$dx/dt = x(-x^{-1}y^3 - bx^2y + a_0x^4 + a_1xy^2)$$

$$dy/dt = y((1/b)x^2y + x^5y^{-1} + b_0x^4 + b_1xy^2)$$



Systems with a nilpotent matrix of the linear part are thoroughly studied by Lyapunov et. al. In our example, there is no linear part, and the first approximation is not homogeneous but quasi homogeneous. This is the simplest case of a planar system without linear part with Newton's open polygon consisting of a single edge. In our case the system corresponds to the quasi homogeneous first approximation with $R = (2, 3)$, $s = 7$. In general case such problems have not been studied, and the authors do not know of any applications of the system (18).



System (3) has a quasi-homogeneous initial approximation if there exists an integer vector $R = (r_1, r_2) > 0$ and a number s such that the scalar product

$$\langle Q, R \rangle \stackrel{\text{def}}{=} q_1 r_1 + q_2 r_2 \geq s = \text{const}$$

for nonzero $|\phi_Q| + |\psi_Q| \neq 0$, and between vectors Q with $\langle Q, R \rangle = s$ there are vectors of the form $(q_1, -1)$ and $(-1, q_2)$. In this case, the system (3) takes the form

$$\begin{aligned} \dot{x}_1 &= x_1[\phi_s(X) + \phi_{s+1}(X) + \phi_{s+2}(X) + \ldots], \\ \dot{x}_2 &= x_2[\psi_s(X) + \psi_{s+1}(X) + \psi_{s+2}(X) + \ldots], \end{aligned}$$

where $\phi_k(X)$ is the sum of terms $\phi_Q X^Q$ for which $\langle Q, R \rangle = k$. And the same holds for the $\psi_k(X)$. Then the initial approximation of (3) is the quasi-homogeneous system

$$\begin{aligned} \dot{x}_1 &= x_1\, \phi_s(X), \\ \dot{x}_2 &= x_2\, \psi_s(X). \end{aligned} \tag{4}$$

We study the problem: what are the conditions on parameters under which the full system (18) is locally integrable. The local integrability of the first approximation (4) is necessary for this. For an autonomous planar system *m* = 1; so there are two cases:

1. System (4) is Hamiltonian, i.e. it has the form

$$\dot{x}_1 = \partial H(X)/\partial x_2, \quad \dot{x}_2 = -\partial H(X)/\partial x_1,$$

   where $H(X)$ is a quasi-homogeneous polynomial.
2. System (4) is non Hamiltonian, but it has the first integral $F(X)$:

$$\frac{\partial F(X)}{\partial x_1} x_1 \phi_s + \frac{\partial F(X)}{\partial x_2} x_2 \psi_s = 0,$$

   where $F(X)$ is a quasi-homogeneous polynomial.

About Normal Form and the Condition **A**

Let the linear transformation

$$X = BY \tag{11}$$

bring the matrix A to the Jordan form $J = B^{-1}AB$ and (2) to

$$\dot{Y} = JY + \tilde{\tilde{\Phi}}(Y). \tag{12}$$

Let the formal change of coordinates

$$Y = Z + \Xi(Z), \tag{13}$$

where $\Xi = (\xi_1, \ldots, \xi_n)$ and $\xi_j(Z)$ are formal power series, transform (12) in the system

$$\dot{Z} = JZ + \Psi(Z). \tag{14}$$

We write it in the form

$$\dot{z}_j = z_j g_i(Z) = z_j \sum g_{jQ} Z^Q \quad \text{over} \quad Q \in \mathbb{N}_j,\ j = 1, \ldots, n, \tag{15}$$

where $Q = (q_1, \ldots, q_n)$, $Z^Q = z_1^{q_1} \ldots z_n^{q_n}$,

$$\mathbb{N}_j = \{Q : Q \in \mathbb{Z}^n,\ \ Q + E_j \geq 0\},\ j = 1, \ldots, n,$$

E_j means the unit vector. Denote

$$\mathbb{N} = \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_n. \tag{16}$$

The diagonal $\Lambda = (\lambda_1, \ldots, \lambda_n)$ of J consists of eigenvalues of the matrix A.

System (14), (15) is called the *resonant normal form* if:

a) J is the Jordan matrix,

b) in writing (15), there are only the *resonant terms*, for which the scalar product

$$\langle Q, \Lambda \rangle \stackrel{\text{def}}{=} q_1\lambda_1 + \ldots + q_n\lambda_n = 0. \tag{17}$$

Theorem 1 (Bruno [4]) *There exists a formal change (13) reducing (12) to its normal form (14), (15).*

In [Bruno:1971] was proved that there are conditions on the normal form (15), which guarantee the convergence of the normalizing transformation (13).

Condition A. *In the normal form (15)*

$$g_j = \lambda_j \alpha(Z) + \bar{\lambda}_j \beta(Z), \quad j = 1, \dots, n,$$

where $\alpha(Z)$ and $\beta(Z)$ are some power series.

Let

$$\omega_k = \min |\langle Q, \Lambda \rangle| \ \text{ over } \ Q \in \mathbb{N}, \ \ \langle Q, \Lambda \rangle \neq 0, \ \ \sum_{j=1}^{n} q_j < 2^k, \ \ k = 1, 2, \dots$$

Condition ω (on small divisors). *The series*

$$\sum_{k=1}^{\infty} 2^{-k} \log \omega_k > -\infty,$$

i.e. it converges.

It is fulfilled for almost all vectors Λ.

Theorem 2. Bruno 1971.*If vector Λ satisfies Condition ω and the normal form* (2.6) *satisfies Condition A then the normalizing transformation (13) con-verges*

The algorithm of a calculation of the normal form, the normalizing transformation and the corresponding computer program are described in [Edneral:2007].

After the power transformation

$$x = u\,v^2, \quad y = u\,v^3 \tag{19}$$

and time rescaling

$$dt = u^2 v^7 d\tau,$$

we obtain the system (18) in the form

$$\begin{aligned} du/d\tau &= -3\,u - [3\,b + (2/b)]u^2 - 2\,u^3 + (3\,a_1 - 2\,b_1)u^2 v + (3\,a_0 - 2\,b_0)u^3 v, \\ dv/d\tau &= v + [b + (1/b)]u\,v + u^2 v + (b_1 - a_1)u\,v^2 + (b_0 - a_0)u^2 v^2. \end{aligned} \tag{20}$$

Under the power transformation (19), the point $x = y = 0$ blows up into two straight lines $u = 0$ and $v = 0$. Along the line $u = 0$ the system (20) has a single stationary point $u = v = 0$. Along the second line $v = 0$ this system has three elementary stationary points

$$u = 0, \quad u = -\frac{1}{b}, \quad u = -\frac{3b}{2}. \tag{21}$$

Lemma 2. *Near the point $u = v = 0$, the system (20) is locally integrable.*

Proof. In accordance with Chapter 2 of the book **[Bruno:1979], the support of the system (20) consists of the five points $Q = (q_1, q_2)$**

$$(0,0), \quad (1,0), \quad (2,0), \quad (1,1), \quad (2,1). \tag{22}$$

At the point $u = v = 0$ eigenvalues of the system (20) are $\Lambda = (\lambda_1, \lambda_2) = (-3, 1)$. Only for the first point from (22) $Q = 0$, the scalar product $\langle Q, \Lambda \rangle$ is zero, for the remaining four points (22) it is negative, so these four points lie on the same side of the straight line $\langle Q, \Lambda \rangle = 0$. In accordance with the remark at the end of Subsection 2.1 of Chapter 2 of the book , in such case the normal form consists only of the terms of a right side of the system (20) such that their support Q lies on the straight line $\langle Q, \Lambda \rangle = 0$. But only linear terms of the system (20) satisfy this condition. Therefore at the point $u = v = 0$ the normal form of the system is linear

$$dz_1/d\tau = -3\, z_1, \quad dz_2/d\tau = z_2.$$

$$(0,0), \quad (1,0), \quad (2,0), \quad (1,1), \quad (2,1). \qquad (22)$$

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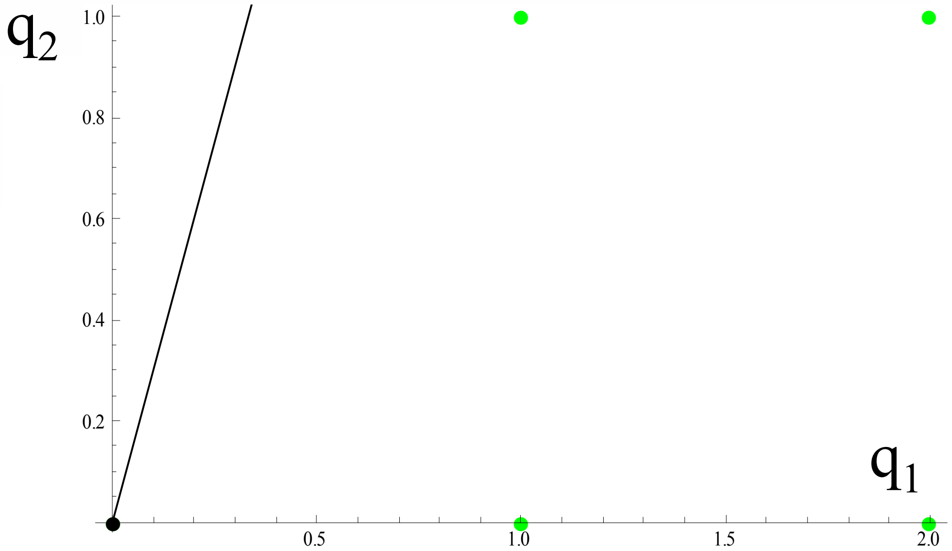

It is obvious that this normal form satisfies the condition A. So the normalizing transformation converges, and at the point $u = v = 0$ the system (20) has the analytic first integral

$$z_1 \, z_2^3 = \text{const.} \qquad \square$$

The proof of local integrability at the point $u = \infty, v = 0$ is similar.

Thus if we must find conditions of local integrability at two other stationary points (21). We will have the conditions of local integrability of the system (18) near the point $X = 0$.

Let us consider the stationary point $u = -1/b, v = 0$. Below we restrict ourselves to the case $b^2 \neq 2/3$ when a linear part of the system (20), after the shift $u = \tilde{u} - 1/b$, has non-vanish eigenvalues. At $b^2 = 2/3$ the shifted system in new variables $\tilde{u}$ and v has Jordan cell with both zero eigenvalues as the linear part. This case can be studied by using one more power transformation.

To simplify eigenvalues, we change the time at this point once more with the factor $d\tau = (2 - 3\,b^2)/b^2\, d\tau_1$. After that we obtain the vector of eigenvalues of the system (20) at this point as $(\lambda_1, \lambda_2) = (-1, 0)$. So the normal form of the system will become

$$\begin{aligned} dz_1/d\tau_1 &= -z_1 + z_1\, g_1(z_2), \\ dz_2/d\tau_1 &= \qquad\quad z_2\, g_2(z_2), \end{aligned} \tag{23}$$

where $g_{1,2}(x)$ are formal power series in x. Coefficients of these series are rational functions of the parameters of the system a_0, a_1, b_0, b_1 and b. It can be proved that denominator of each of these rational functions is proportional to some integer degree $k(n)$ of the polynomial $(2 - 3b^2)$. Their numerators are polynomials in parameters of the system

$$g_{1,2}(x) = \sum_{n=1}^{\infty} \frac{p_{1,2;n}(b, a_0, a_1, b_0, b_1)}{(2 - 3\,b^2)^{k(n)}} x^n.$$

The condition A of integrability for the equation (23) is $g_2(x) \equiv 0$. It is equivalent to the infinite polynomial system of equations

$$p_{2,n}(b, a_0, a_1, b_0, b_1) = 0, \quad n = 1, 2, \ldots . \tag{24}$$

According to the Hilbert's theorem on bases in polynomial ideals [Siegel:1956], this system has a finite basis.

We computed the first three polynomials $p_{2,1}$, $p_{2,2}$, $p_{2,3}$ by our program [Edneral:2007]. There are 2 solutions of a corresponding subset of equations (24) at $b \neq 0$

$$a_0 = 0, \quad a_1 = -b_0\, b, \quad b_1 = 0, \quad b^2 \neq 2/3 \tag{25}$$

and

$$a_0 = a_1\, b, \quad b_0 = b_1\, b, \quad b^2 \neq 2/3. \tag{26}$$

Addition of the forth equation $p_{2,4} = 0$ to the subset of equations does not change these solutions.

A calculation of polynomials $p_{2,n}(b, a_0, a_1, b_0, b_1)$ in generic case is technically a very difficult problem. But we can verify some of these equations from the set (24) on solutions (25) and (26) for several fixed values of the parameter b. We verified solutions of subset of equations

$$p_{2,n}(b,\, a_0 = a_1 b,\, a_1,\, b_0 = b_1 b,\, b_1) = 0, \quad n = 1, 2, \ldots, 28.$$

at $b = 1$ and $b = 2$. All equations are satisfied, so we can assume that (25) and (26) satisfy the condition A at the stationary point $u = -1/b, v = 0$.

Let us consider the stationary point $u = -3\,b/2, v = 0$. We rescale time at this point with the factor $d\tau = (2 - 3\,b^2)\,d\tau_2$. After that we get the vector of eigenvalues of the system (20) at this point as $(-1/4, 3/2)$. So the normal form has a resonance of the seventh order

$$\begin{aligned} dz_1/d\tau_2 &= -(1/4)\,z_1 + z_1\,r_1(z_1^6\,z_2), \\ dz_2/d\tau_2 &= \quad (3/2)\,z_2 + z_2\,r_2(z_1^6\,z_2), \end{aligned} \tag{27}$$

where $r_{1,2}(x)$ are also formal power series, and in (27) they depend on single "resonant" variable $z_1^6 z_2$. Coefficients of these series are rational functions of system parameters a_0, a_1, b_0, b_1 and b again. The denominator of each of these functions is proportional to some integer degree $l(n)$ of the polynomial $(2-3\,b^2)$. Their numerators are polynomials in parameters of the system

$$r_{1,2}(x) = \sum_{n=1}^{\infty} \frac{q_{1,2;n}(b, a_0, a_1, b_0, b_1)}{(2 - 3\,b^2)^{l(n)}} x^n.$$

The condition A for the equation (27) is $6\,r_1(x) + r_2(x) = 0$. It is equivalent to the infinite polynomial system of equations

$$6\,q_{1,n}(b, a_0, a_1, b_0, b_1) + q_{2,n}(b, a_0, a_1, b_0, b_1) = 0, \quad n = 7, 14, \ldots . \tag{28}$$

We computed polynomials $q_{1,7}, q_{2,7}$ and solved the lowest equation from the set (28) for the parameters of the solution (26). We have found 5 different two-parameter (b and a_1) solutions. With the (26) they are

$$\begin{array}{lllll} 1) & b_1 = -2\,a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3, \\ 2) & b_1 = (3/2)\,a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3, \\ 3) & b_1 = (8/3)\,a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3 \end{array} \tag{29}$$

and

$$\begin{array}{lllll} 4) & b_1 = \frac{197-7\sqrt{745}}{24} a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3, \\ 5) & b_1 = \frac{197+7\sqrt{745}}{24} a_1, & a_0 = a_1 b, & b_0 = b_1 b & b^2 \neq 2/3. \end{array} \tag{30}$$

We verified (28) up to $n = 49$ for solutions (29) for $b = 1$ and $b = 2$ and arbitrary a_1. They are correct. We verified the solution (25) in the same way. It is also correct.

Solutions (30) starting from the order $n = 14$ are correct only for the additional condition $a_1 = 0$. But for this condition, solutions (30) are a special case of solutions (29). So in accordance with the main supposition we can formulate the theorem.

Corollary 1. *If $b^2 \neq 2/3$, the equalities (25) and (29) form a complete set of necessary and sufficient conditions of local integrability of the system (20) at all stationary points on the manifold $v = 0$ and thus at the corresponding values of parameters (25) and (29), then the system (18) is locally integrable near the*

So,

Theorem 3. *Equalities (25), and (29) form a full set of necessary conditions of a local integrability of the system (20) in all its stationary points and a local integrability of system (18) at stationary point $x = y = 0$.*

About sufficient conditions of an integrability

The conditions resulted in the theorem 3.2 are necessary for local integrability of the system (18) in the zero stationary point. They can be considered from the point of view "experimental mathematics" as candidates of conditions of local integrability. However it is necessary to prove the sufficiency of these conditions by independent methods. It is necessary to do it for each of four conditions (25), (29) in each of stationary points u = -3b/2; v = 0 and u = -1/b; v = 0.

1. $a_0 = 0, \quad a_1 = -b_0\, b, \quad b_1 = 0, \quad b^2 \neq 2/3$
2. $b_1 = -2\, a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3,$
3. $b_1 = (3/2)\, a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3,$
4. $b_1 = (8/3)\, a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3$

With essential assistance by Prof. V. Romanovski [8] we found first integrals for all cases (25), (29) mainly by method of the Darboux factor of the system (20), see [9]. We opened 4 families of solutions which exhausted all integrable cases except possible the case $b^2 = 2/3$.

$$\text{At} \quad a_0 = 0, \quad a_1 = -b_0\, b, \quad b_1 = 0:$$
$$I_{1uv} = u^2(3\,b + 2\,u)v^6 \ ,$$
$$I_{1xy} = 2\,x^3 + 3\,b\,y^2 \ .$$

$$\text{At} \quad b_1 = -2a_1,\ a_0 = a_1 b,\ b_0 = b_1 b:$$
$$I_{2uv} = u^2\, v^6\,(3\,b + u\,(2 - 6\,a_1\,b\,v)) \ ,$$
$$I_{2xy} = 2\,x^3 - 6\,a_1\,b\,x^2\,y + 3\,b\,y^2 \ .$$

$$\text{At} \quad b_1 = 3a_1/2,\ a_0 = a_1 b,\ b_0 = b_1 b:$$

$$I_{3uv} = \frac{4 - 4a_1\,u\,v + 3^{5/6} a_1\,{}_2F_1(2/3,1/6;5/3;-2u/(3b))\,u\,v\,(3+2u/b)^{1/6}}{u^{1/3} v\,(3b+2u)^{1/6}} \ ,$$

$$I_{3xy} = \frac{a_1 x^2 (-4 + 3^{5/6}\,{}_2F_1\left(2/3,1/6;5/3;-2\,x^3/(3\,b\,y^2)\right)(3+2x^3/(b\,y^2))^{1/6}) + 4y}{y^{4/3}(3\,b + 2\,x^3/y^2)^{1/6}} \ ,$$

At $b_1 = 8a_1/3,\ a_0 = a_1 b,\ b_0 = b_1 b:$

$$I_{4u,v} = \frac{u\,(3+2\,a_1^2 bu)+6\,a_1\,b\,v}{3\,u\,[u^3(6+a_1^2 b\,u)+6\,a_1^2 b\,u^2 v+9\,b\,v^2]^{1/6}} - \frac{8\,a_1\sqrt{-b}}{3^{5/3}} B_{6+a_1\sqrt{-6\,b\,u}+3\,v\sqrt{-6\,b/u^3}}(5/6, 5/6)\ ,$$

where the B is the incomplete beta function.

In the paper [8] there are printed corresponding solutions for the integrals above. The integrals and solutions have not any singularities near the points $b^2 = 2/3$, but the approach in which these solutions was received has the corresponding limitation, so theoretically there is possible of existence additional solutions at this point. So we need to study the domains of $b^2 = 2/3$ separately.

Case $b^2 = 2/3$.

At this value of b, the both stationary points (u =-3b/2, have instead(\ref{SystemT}) after the shift $u \to w$-1/b = w-√6/2 the degenerate system again

$$
\begin{aligned}
dw/d\tau &= v\left(-\tfrac{9}{2}\sqrt{\tfrac{3}{2}}\,a_0 + \tfrac{9}{2}\,a_1 + 3\sqrt{\tfrac{3}{2}}\,b_0 - 3\,b_1\right) \\
&+wv\left(\tfrac{27}{2}\,a_0 - 3\sqrt{6}\,a_1 - 9\,b_0 + 2\sqrt{6}\,b_1\right) \\
&+\sqrt{6}\,w^2 + w^2v\left(-9\sqrt{\tfrac{3}{2}}\,a_0 + 3\,a_1 + 3\sqrt{6}\,b_0 - 2\,b_1\right) \\
&-2\,w^3 + w^3v(3\,a_0 - 2\,b_0)\ , \\
dv/d\tau &= -\tfrac{\sqrt{6}}{6}wv + v^2\left(-\tfrac{3}{2}\,a_0 + \sqrt{\tfrac{3}{2}}\,a_1 + \tfrac{3}{2}\,b_0 - \sqrt{\tfrac{3}{2}}\,b_1\right) \\
&+w^2v + wv^2\left((\sqrt{6}\,a_0 - a_1 - \sqrt{6}\,b_0 + b_1\right) \\
&+w^2v^2(-a_0 + b_0)\ .
\end{aligned}
$$

On the other hand, we should apply a power transformation once more. We choose here the transformation

$$v \to u^2 w, \quad v' \to 2u'uw + u^2 w' \;,$$

with the scaling $d\tau = u/\sqrt{6}d\bar{\tau}$ **. So after this scaling we have equation (20) in the new variables *u* and *w* in the form**

$$\begin{aligned}
du/d\tilde{\tau} = {} & \tfrac{1}{2}\sqrt{\tfrac{3}{2}}u(4\sqrt{6} + 3(-3\sqrt{6}a_0 + 6a_1 + 2\sqrt{6}b_0 - 4b_1)w + \\
& 2u(-4 + (27a_0 - 6\sqrt{6}a_1 - 18b_0 + 4\sqrt{6}b_1 + \\
& u(-9\sqrt{6}a_0 + 6a_1 + 6\sqrt{6}b_0 - 4b_1 + (6a_0 - 4b_0)u))w)) \ , \\
dw/d\tilde{\tau} = {} & \tfrac{-1}{2}w(26 - 6(9a_0 - 3\sqrt{6}a_1 - 6b_0 + 2\sqrt{6}b_1)w + \\
& u(-10\sqrt{6} + (57\sqrt{6}a_0 - 78a_1 - 39\sqrt{6}b_0 + 54b_1 + \\
& 2u(-60a_0 + 7\sqrt{6}a_1 + 42b_0 - 5\sqrt{6}b_1 + \sqrt{6}(7a_0 + 5b_0)u))w)) \ .
\end{aligned}$$

On the invariant line *u* = 0 this equation has 2 singular points: *w* = 0 and *w* = 13/(3(9a_0 – 3√6 a_1 – 6b_0 +2√6 b_1)). Near the first point we have a resonance of 19 level (13 : 6) and near the second point a resonance of 27 level (1 : 26).

The calculation of the normal form of the equation above up to 19 order produced a single equation for the necessary condition of integrability A as polynomial in variables a_0; a_1; b_0 and b_1. This calculation was produced by RLISP system compartible with the REDUCE [11] system. It takes about 6.5 hours and 600 Mytes of RAM on the 3 GGc Pentium IV processor. The normalizing transformation concludes 226145 terms and the normal form has 1174 terms.

We have obtained the condition A as the long polynomial *p*(a_0, a_1, b_0, b_1). This polynomial was the sixth-order homogenous polynomial

p:=23005478902714560000000*sqrt(6)*a0**6 -
1331442050024872128000*sqrt(6)*a0**5*b0 +
40226785177971499200000*sqrt(6)*a0**4*a1**2 -
1218593816594559972000*sqrt(6)*a0**4*a1*b1 +
97261108766693455360000*sqrt(6)*a0**4*b0**2 -
20887053949788648000000*sqrt(6)*a0**4*b1**2 -
88437430500022337280000*sqrt(6)*a0**3*a1**2*b0 +
15529781352143773920000*sqrt(6)*a0**3*a1*b0*b1 +
1649261434287556096000*sqrt(6)*a0**3*b0**3 +

6217786690276645176000*sqrt(6)*a0**3*b0*b1**2 +
1941686573597117412480*sqrt(6)*a0**2*a1**4 -
3132005383893107121120*sqrt(6)*a0**2*a1**3*b1 +
1833603743893730891520*sqrt(6)*a0**2*a1**2*b0**2 -
3770969989774833254520*sqrt(6)*a0**2*a1**2*b1**2 +
5169734129372720175360*sqrt(6)*a0**2*a1*b0**2*b1 +
2485280180118279147480*sqrt(6)*a0**2*a1*b1**3 -
996896012323306403840*sqrt(6)*a0**2*b0**4 -
925270976215868906880*sqrt(6)*a0**2*b0**2*b1**2 +
74462461939798015680*sqrt(6)*a0**2*b1**4 -
524123564091628253184*sqrt(6)*a0*a1**4*b0 -
1194754357857233361024*sqrt(6)*a0*a1**3*b0*b1 +

243647586737387226624*sqrt(6)*a0*a1**2*b0**3 +
1629616656945795239376*sqrt(6)*a0*a1**2*b0*b1**2 -
754425993869902562688*sqrt(6)*a0*a1*b0**3*b1 +
867915164207913847056*sqrt(6)*a0*a1*b0*b1**3 +
3618421863956788224*sqrt(6)*a0*b0**5 -
772619961956143446336*sqrt(6)*a0*b0**3*b1**2 -
438818680673003917224*sqrt(6)*a0*b0*b1**4 +
62646023758716980352*sqrt(6)*a1**6 +
9159099848800905312*sqrt(6)*a1**5*b1 -
24464696304634479360*sqrt(6)*a1**4*b0**2 -
254534097203995557060*sqrt(6)*a1**4*b1**2 +
130801547413334926368*sqrt(6)*a1**3*b0**2*b1 -

1361191183556503662O*sqrt(6)*a1**3*b1**3 -
3747045239840612352O*sqrt(6)*a1**2*b0**4 +
193420319940986808456*sqrt(6)*a1**2*b0**2*b1**2 +
215791821335366287695*sqrt(6)*a1**2*b1**4 -
132762007686584411328*sqrt(6)*a1*b0**4*b1 -
292670009299507757976*qrt(6)*a1*b0**2*b1**3 -
74484700774007658462*sqrt(6)*a1*b1**5 +
21949306985425741824*sqrt(6)*b0**6 +
80314491397893109248*sqrt(6)*b0**4*b1**2 +
21593660315481388512*sqrt(6)*b0**2*b1**4 +
1354565584806144408*sqrt(6)*b1**6 +
828197240497724160000O*a0**5*a1 -
2011346500174812960000*a0**5*b1 -

36573738838205248128000*a0**4*a1*b0 +
6003154009788539808000*a0**4*b0*b1 +
8613056866550417222400*a0**3*a1**3 -
20644245769284583027200*a0**3*a1**2*b1 +
17983242016812128755200*a0**3*a1*b0**2 -
8750813781982920292800*a0**3*a1*b1**2 +
16959446950264477420800*a0**3*b0**2*b1 +
4391629028622064497600*a0**3*b1**3 -
7711079140985454570240*a0**2*a1**3*b0 -
3850186094266425770880*a0**2*a1**2*b0*b1
+ 1003252142962580728320*a0**2*a1*b0**3 +
14283635922380792436480*a0**2*a1*b0*
b1**2 - 5631985753778872446720*a0**2*b0**3*b1 +
509028593643436364640*a0**2*b0*
b1**3 + 1394659279178180889984*a0*a1**5 -

1147713270022925239680*a0*a1**4*b1 +
1713014061322239469568*a0*a1**3*b0**2 -
4204379270818300054320*a0*a1**3*b1**2 +
3703884433039501479936*a0*a1**2*b0**2*b1 +
1901508492617094508200*a0*a1**2*b1**3
- 934730485667321783808*a0*a1*b0**4 +
2755838910434761434 24*a0*a1*b0**2*b1**2 +
113392232113389385536 0*a0*a1*b1**4 -
847059605374837085952*a0*b0**4*b1 -
1803910228089693344928*a0*b0**2*b1**3 -
366298334949014384544*a0*b1**5 -
36281763677008879488*a1**5*b0 -
492125652475057868544*a1**4*b0*b1 +
782100558696812313 60*a1**3*b0**3 +
1426873682174732364 96*a1**3*b0*b1**2 -

5963962097766947456*a1**2*b0**3*b1 +
8854188271076916988856*a1**2*b0*b1**3 -
2204203340674944153 6*a1*b0**5 -
7036096645305724250 88*a1*b0**3*b1**2 -
4629416365306073175 84*a1*b0*b1**4 +
1755017294620822348 80*b0**5*b1 +
1285678361786635059 84*b0**3*b1**3 +
1083652467844915526 4*b0*b1**5
1361191183556503662 0*sqrt(6)*a1**3*b1**3 -
3747045239840612352 0*sqrt(6)*a1**2*b0**4 +
1934203199409868084 56*sqrt(6)*a1**2*b0**2*b1**2 +
2157918213353662876 95*sqrt(6)*a1**2*b1**4 -
1327620076865844113 28*sqrt(6)*a1*b0**4*b1 -

29267000929950775797 6*sqrt(6)*a1*b0**2*b1**3 -

744847007740076584 62*sqrt(6)*a1*b1**5 +

219493069854257418 24*sqrt(6)*b0**6 +

803144913978931092 48*sqrt(6)*b0**4*b1**2 +

215936603154813885 12*sqrt(6)*b0**2*b1**4 +

135456558480614440 8*sqrt(6)*b1**6 +

828197240497724160 0000*a0**5*a1 -

201134650017481296 00000*a0**5*b1 -

365737388382052481 28000*a0**4*a1*b0 +

600315400978853980 8000*a0**4*b0*b1 +

861305686655041722 2400*a0**3*a1**3 -

206442457692845830 27200*a0**3*a1**2*b1 +

179832420168121287 55200*a0**3*a1*b0**2 -

875081378198292029 2800*a0**3*a1*b1**2 +

169594469502644774208OO*a0**3*b0**2*b1 +
439162902862206449760O*a0**3*b1**3 -
771107914098545457024O*a0**2*a1**3*b0 -
385018609426642577088O*a0**2*a1**2*b0*b1 +
100325214296258072832O*a0**2*a1*b0**3 +
142836359223807924364 8O*a0**2*a1*b0*b1**2 -
563198575377887244672O*a0**2*b0**3*b1 +
509028593643436364640*a0**2*b0*b1**3 +
1394659279178180889984*a0*a1**5 -
1147713270022925239680*a0*a1**4*b1 +
1713014061322394695 68*a0*a1**3*b0**2 -
420437927081830005432O*a0*a1**3*b1**2 +
370388443303950147993 6*a0*a1**2*b0**2*b1 +
190150849261709450820O*a0*a1**2*b1**3 -
93473048566732178380 8*a0*a1*b0**4 +

27558389104347614342 4*a0*a1*b0**2*b1**2 +
11339223211338938553 60*a0*a1*b1**4 -
84705960537483708595 2*a0*b0**4*b1 -
18039102280896933449 28*a0*b0**2*b1**3 -
36629833494901438454 4*a0*b1**5 -
36281763677008879488*a1**5*b0 -
49212565247505786854 4*a1**4*b0*b1 +
78210055869681231360*a1**3*b0**3 +
14268736821747323649 6*a1**3*b0*b1**2 -
59639620977766947456*a1**2*b0**3*b1 +
88541882710769169885 6*a1**2*b0*b1**3 -
22042033406749441536*a1*b0**5 -
70360966453057242508 8*a1*b0**3*b1**2 -
46294163653060731758 4*a1*b0*b1**4 +
17550172946208223488 0*b0**5*b1 +
12856783617866350598 4*b0**3*b1**3 +
10836524678449155264*b0*b1**5

The obtained condition A is consistent with condition (25) and each from (\29), i.e., with the all four solutions. So, if you put a_0=a_1 b and b_0=b_1 b as in (29) and solve the equation p=0 you will receive exactly solution (29), (30) but without the limitation $b^2 \neq 2/3$.

```
b:=sqrt(6)/3$
aa:=sub(a0=a1*b,b0=b1*b,p)$
solve(aa=0,b1);

{b1= - 2*a1,
b1=(7*sqrt(745)*abs(a1) + 197*a1)/24,
b1=( - 7*sqrt(745)*abs(a1) + 197*a1)/24,
b1=(8*a1)/3,
b1=(3*a1)/2}
```

Solution (30) is not realized thereafter, of course.

So our result is compatible with previous results, but the result which we have really obtained is that solutions (29) do exist at b^2 =2/3, and there are no more solutions of such a type in any case.

Also the $p(a_0, a_1, b_0, b_1)$ is equal to zero under conditions \ref{Sol1T}).

So, we have proved

Theorem 3. ***All solutions (\ref{Sol1T}),(\ref{Sol2}) are right without limitation $b^2 \neq 2/3$.***

For searching additional first integrals we need to calculate the condition A at the point with the resonance $(1:26)$.

There are also other transformations with corresponding resonances. Its studying is a very heavy technical problem, and we work on it at present.

Conclusions

For the particular 5-parametrical case the planar system (\ref{System1}), non-Hamiltonian at $c=1/b$, we have found the necessary sets of conditions on parameters at which system(18) is locally integrable near degenerate point $X=0$. They form 4 sets of conditions on parameters of the system. So these sets of conditions are also sufficient for local and global integrability of system (M). The case of point $b^2 = 2/3$ at $c=1/b$ and different from $c=1/b$ but with a real rational value D of in the Theorem D cases should be studied separately. But we have proved here that the limitation $b^2 \neq 2/3$ can be excluded from the solutions obtained previously in the paper [EdneralRomanovski:2010]. At the moment, we have not found the additional first integrals.

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