On Consistency of Finite Difference Approximations to the Navier-Stokes Equations

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Approximate solution

In the finite difference method (FDM) partial differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme.

Requirements for FDA

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$$\Downarrow$$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$$\Downarrow$$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

S-consistency

FDA is s-consistent with PDE(s) if any differential consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

Navier-Stokes PDE system

Involutive PDE system of the Navier-Stokes equations for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity can be written in the following form (G.,Blinkov CASC-2009) obtained by the method suggested in (G.,Blinkov, Mozzhilkin'06)

$$F := \begin{cases} f_1 := u_x + v_y = 0\,, \\ f_2 := u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}) = 0\,, \\ f_3 := v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) = 0\,, \\ f_4 := u_x^2 + 2v_x u_y + v_y^2 + p_{xx} + p_{yy} = 0\,. \end{cases}$$

Here

- f_1 - the continuity equation,
- f_2, f_3 - the proper Navier-Stokes equations,
- f_4 - the pressure Poisson equation which is the integrability condition for $\{f_1, f_2, f_3\}$,
- (u, v) - the velocity field,
- p - the pressure,
- Re - the Reynolds number.

Divergence form

The involutive Navier-Stokes system admits two-dimensional conservation law form

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial x} + \frac{\partial \mathbf{R}}{\partial y} = 0\,.$$

In terms of $\{f_1, f_2, f_3, f_4\}$ this form reads

Conservation law form

$$\begin{cases} f_1 : \; \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0\,, \\ f_2 : \; \frac{\partial}{\partial t} u + \frac{\partial}{\partial x}\left(u^2 + p - \frac{1}{\mathrm{Re}} u_x\right) + \frac{\partial}{\partial y}\left(vu - \frac{1}{\mathrm{Re}} u_y\right) = 0\,, \\ f_3 : \; \frac{\partial}{\partial t} v + \frac{\partial}{\partial x}\left(uv - \frac{1}{\mathrm{Re}} v_x\right) + \frac{\partial}{\partial y}\left(v^2 + p - \frac{1}{\mathrm{Re}} v_y\right) = 0\,, \\ f_4 : \; \frac{\partial}{\partial x}\left(uu_x + vu_y + p_x\right) + \frac{\partial}{\partial y}\left(vv_y + uv_x + p_y\right) = 0\,. \end{cases}$$

Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring R

$$f_i = 0 \ (1 \le i \le 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p]\,,$$

where $\mathbb{K} := \mathbb{Q}(\mathrm{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0,\ h > 0,\ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\} \mid_{x=jh, y=kh, t=\tau n}\,.$$

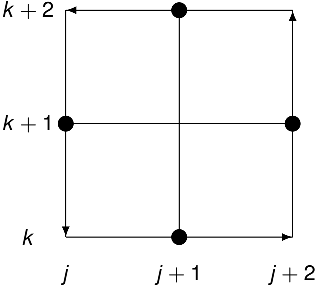
We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t),\ \sigma_y \circ \phi = \phi(x, y + h, t),\ \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$.

Integration contour

To discretize NSS on the grid choose the integration contour Γ in the (x, y) plane



The Navie-Stokes system in integral form

Integral conservation law form

$$\begin{cases} \oint\limits_{\Gamma} -v dx + u dy = 0\,, \\ \int\limits_{x_j}^{x_{j+2}} \int\limits_{y_k}^{y_{k+2}} u dx dy \Big|_{t_n}^{t_{n+1}} - \int\limits_{t_n}^{t_{n+1}} \left(\oint\limits_{\Gamma} \left(vu - \frac{1}{\mathrm{Re}} u_y \right) dx - \left(u^2 + p - \frac{1}{\mathrm{Re}} u_x \right) dy \right) dt = 0\,, \\ \int\limits_{x_j}^{x_{j+2}} \int\limits_{y_k}^{y_{k+2}} v dx dy \Big|_{t_n}^{t_{n+1}} - \int\limits_{t_n}^{t_{n+1}} \left(\oint\limits_{\Gamma} \left(v^2 + p - \frac{1}{\mathrm{Re}} v_y \right) dx - \left(uv - \frac{1}{\mathrm{Re}} v_x \right) dy \right) dt = 0\,, \\ \oint\limits_{\Gamma} - \left((v^2)_y + (uv)_x + p_y \right) dx + \left((u^2)_x + (vu)_y + p_x \right) dy = 0\,. \end{cases}$$

Additional relations

Now we add integral relations between dependent variables and derivatives

Exact integral relations

$$
\begin{cases}
\int_{x_j}^{x_{j+1}} (u^2)_x dx = u(x_{j+1}, y)^2 - u(x_j, y)^2 \,, \quad \int_{y_k}^{y_{k+1}} (v^2)_y dy = v(x, y_{k+1})^2 - v(x, y_k)^2 \,, \\
\int_{x_j}^{x_{j+1}} (uv)_x dx = u(x_{j+1}, y) v(x_{j+1}, y) - u(x_j, y) v(x_j, y) \,, \\
\int_{y_k}^{y_{k+1}} (uv)_y dy = u(x, y_{k+1}) v(x, y_{k+1}) - u(x, y_k) v(x, y_k) \,, \\
\int_{x_j}^{x_{j+1}} u_x dx = u(x_{j+1}, y) - u(x_j, y) \,, \quad \int_{y_k}^{y_{k+1}} u_y dy = u(x, y_{k+1}) - u(x, y_k) \,, \\
\int_{x_j}^{x_{j+1}} v_x dx = v(x_{j+1}, y) - u(x_j, y) \,, \quad \int_{y_k}^{y_{k+1}} v_y dy = v(x, y_{k+1}) - u(x, y_k) \,, \\
\int_{x_j}^{x_{j+1}} p_x dx = p(x_{j+1}, y) - u(x_j, y) \,, \quad \int_{y_k}^{y_{k+1}} p_y dy = p(x, y_{k+1}) - u(x, y_k) \,.
\end{cases}
$$

Finite difference approximation 1

By using the midpoint integration approximation for the integrals over x and y and the top-left corner approximation for integration over t. Then elimination of partial derivatives from the obtained difference system gives the following FDA with a 5×5 stencil (G.,Blinkov CASC-2009)

$$
\mathrm{FDA\ 1} = \left\{
\begin{aligned}
{e_1}^n_{j,k} &:= \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1} - v^n_{j,k-1}}{2h} = 0, \\
{e_2}^n_{j,k} &:= \frac{u^{n+1}_{jk} - u^n_{jk}}{\tau} + \frac{{u^n_{j+1,k}}^2 - {u^n_{j-1,k}}^2}{2h} + \frac{v^n_{j,k+1} u^n_{j,k+1} - v^n_{j,k-1} u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k} - p^n_{j-1,k}}{2h} \\
&\quad - \frac{1}{\mathrm{Re}} \left(\frac{u^n_{j+2,k} - 2u^n_{jk} + u^n_{j-2,k}}{4h^2} + \frac{u^n_{j,k+2} - 2u^n_{jk} + u^n_{j,k-2}}{4h^2} \right) = 0, \\
{e_3}^n_{j,k} &:= \frac{v^{n+1}_{jk} - v^n_{jk}}{\tau} + \frac{u^n_{j+1,k} v^n_{j+1,k} - u^n_{j-1,k} v^n_{j-1,k}}{2h} \frac{{v^n_{j,k+1}}^2 - {v^n_{j,k-1}}^2}{2h} + \frac{p^n_{j,k+1} - p^n_{j,k-1}}{2h} \\
&\quad - \frac{1}{\mathrm{Re}} \left(\frac{v^n_{j+2,k} - 2v^n_{jk} + v^n_{j-2,k}}{4h^2} + \frac{v^n_{j,k+2} - 2v^n_{jk} + v^n_{j,k-2}}{4h^2} \right) = 0, \\
{e_4}^n_{j,k} &:= \frac{{u^n_{j+2,k}}^2 - 2{u^n_{j,k}}^2 + {u^n_{j-2,k}}^2}{4h^2} + \frac{{v^n_{j,k+2}}^2 - 2{v^n_{j,k}}^2 + {v^n_{j,k-2}}^2}{4h^2} \\
&\quad + 2\,\frac{u^n_{j+1,k+1} v^n_{j+1,k+1} - u^n_{j+1,k-1} v^n_{j+1,k-1} - u^n_{j-1,k+1} v^n_{j-1,k+1} + u^n_{j-1,k-1} v^n_{j-1,k-1}}{4h^2} \\
&\quad + \frac{p^n_{j+2,k} - 2p^n_{jk} + p^n_{j-2,k}}{4h^2} + \frac{p^n_{j,k+2} - 2p^n_{jk} + p^n_{j,k-2}}{4h^2} = 0\,.
\end{aligned}
\right.
$$

Finite difference approximation 2

If one applies the trapezoidal approximation to the integral relations for $u_x, u_y, v_x, v_y, u^2)_x, (v^2)_y$ and p instead of the midpoint approximation, then it produces FDA with a 3×3 stencil (G.,Blinkov CASC-2009)

$$
\text{FDA 2} = \left\{
\begin{aligned}
& {e_1}_{j,k}^n := \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
& {e_2}_{j,k}^n := \frac{u_{jk}^{n+1} - u_{jk}^n}{\tau} + u_{jk}^n \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + v_{jk}^n \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
& \qquad - \frac{1}{\mathrm{Re}} \left(\frac{u_{j+1,k}^n - 2u_{jk}^n + u_{j-1,k}^n}{h^2} + \frac{u_{j,k+1}^n - 2u_{jk}^n + u_{j,k-1}^n}{h^2} \right) = 0, \\
& {e_3}_{j,k}^n := \frac{v_{jk}^{n+1} - v_{jk}^n}{\tau} + u_{jk}^n \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} + v_{jk}^n \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j,k+1}^n - p_{j,k-1}^n}{2h} \\
& \qquad - \frac{1}{\mathrm{Re}} \left(\frac{v_{j+1,k}^n - 2v_{jk}^n + v_{j-1,k}^n}{h^2} + \frac{v_{j,k+1}^n - 2v_{jk}^n + v_{j,k-1}^n}{h^2} \right) = 0, \\
& {e_4}_{j,k}^n := \left(\frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} \right)^2 + 2 \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \left(\frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} \right)^2 \\
& \qquad + \frac{p_{j+1,k}^n - 2p_{jk}^n + p_{j-1,k}^n}{h^2} + \frac{p_{j,k+1}^n - 2p_{jk}^n + p_{j,k-1}^n}{h^2} = 0
\end{aligned}
\right.
$$

Finite difference approximation 3

The third approximation with 3×3 stencil is obtained from NSS by the conventional discretization what consists of replacing the temporal derivatives with the forward differences and the spatial derivatives with the central differences.

$$
\text{FDA 3} = \begin{cases}
{e_1}_{j,k}^{n} := \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
{e_2}_{j,k}^{n} := \frac{u_{jk}^{n+1} - u_{jk}^n}{\tau} + u_{jk}^n \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + v_{jk}^n \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left(\frac{u_{j+1,k}^n - 2u_{jk}^n + u_{j-1,k}^n}{h^2} + \frac{u_{j,k+1}^n - 2u_{jk}^n + u_{j,k-1}^n}{h^2} \right) = 0, \\
{e_3}_{j,k}^{n} := \frac{v_{jk}^{n+1} - v_{jk}^n}{\tau} + u_{jk}^n \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} + v_{jk}^n \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j,k+1}^n - p_{j,k-1}^n}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left(\frac{v_{j+1,k}^n - 2v_{jk}^n + v_{j-1,k}^n}{h^2} + \frac{v_{j,k+1}^n - 2v_{jk}^n + v_{j,k-1}^n}{h^2} \right) = 0, \\
{e_4}_{j,k}^{n} := \left(\frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} \right)^2 + 2 \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \left(\frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} \right)^2 \\
\qquad + \frac{p_{j+1,k}^n - 2p_{jk}^n + p_{j-1,k}^n}{h^2} + \frac{p_{j,k+1}^n - 2p_{jk}^n + p_{j,k-1}^n}{h^2} = 0
\end{cases}
$$

Differential and difference consequences

A perfect difference ideal $[\![\tilde{F}]\!]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1}(\sigma_y \circ \tilde{f})^{k_2}(\sigma_t \circ \tilde{f})^{k_3} \in [\![\tilde{F}]\!] \Longrightarrow \tilde{f} \in [\![\tilde{F}]\!] .$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset R$ (NSS) generates radical differential ideal $[\![F]\!]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0 , \quad \tilde{F} := \{\tilde{f}_1, \ldots \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to F. Generally, p needs not to be equal 4.

Differential and difference consequences

A differential (resp. difference) polynomial $f \in R$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of F (resp. $\tilde{F}$) if $f \in [\![F]\!]$ (resp. $\tilde{f} \in [\![\tilde{F}]\!]$).

Conventional (weak) consistency of FDA

We shall say that a difference equation $\tilde{f} = 0$ implies (in the continuous limit) the differential equation $f = 0$ and write $\tilde{f} \rhd f$ if f does not contain the grid spacings h, τ and the Taylor expansion about a grid point $(u^n_{j,k}, v^n_{j,k}, p^n_{j,k})$ transforms equation $\tilde{f} = 0$ into $f + O(h, \tau) = 0$ where $O(h, \tau)$ denotes expression which vanishes when h and τ go to zero.

Definition

The difference approximation $\tilde{F}$ is (weakly or w-)consistent with F if $p = 4$ and

$$(\forall \tilde{f} \in \tilde{F}) \, (\exists f \in F) \, [\tilde{f} \rhd f].$$

The requirement of w-consistency which has been universally accepted in the literature, is not satisfactory by the following two reasons:

1. The cardinality of FDA to a system of differential equations may be different from that in the system.
2. A w-consistent FDA may not be good in view of inheritance of properties of the underlying differential equation(s) at the discrete level.

Strong consistency

Definition

An FDA to PDE(s) is strongly consistent or s-consistent if

$$(\,\forall \tilde{f} \in [\![\tilde{F}]\!]\,)\,(\,\exists f \in [F]\,)\,[\,\tilde{f} \rhd f\,]\,.$$

The algorithmic approach (G'12) to verification of s-consistency is based on the following statement.

Theorem

A difference approximation $\tilde{F} \subset \mathcal{R}$ to $F \subset R$ is s-consistent iff a (reduced) standard basis G of the difference ideal $[\tilde{F}]$ satisfies

$$(\,\forall g \in G\,)\,(\,\exists f \in [F]\,)\,[\,g \rhd f\,]\,.$$

Given a differential polynomial $f \in R$, one can algorithmically check its membership in $[\![F]\!]$ by performing the involutive Janet reduction.

S-consistency analysis of FDA 1,2 and 3

All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

$$\tilde{F} := \{ {e_1}_{j,k}^{n}, {e_2}_{j,k}^{n}, {e_3}_{j,k}^{n}, {e_4}_{j,k}^{n} \}$$

about the grid point $\{hj, hk, n\tau\}$ when the grid spacings h and τ go to zero.

Proposition

Among weakly consistent FDAs 1,2, and 3 only FDA 1 is strongly consistent.

Corollary

A standard basis G of the difference ideal generated by the set of polynomials in FDA 1 satisfies the condition

$$(\forall g \in G)\,(\exists f \in [F])\,[\,g \triangleright f\,]\,.$$

Numerical problem

Suppose that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$\begin{aligned} u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y)\,, \\ v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y)\,, \\ p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4\,. \end{aligned}$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the (x, y)-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \dots m+1$, and $h = \pi/(m+1)$.

Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute u_{jk}^{n+1} and v_{jk}^{n+1} for $j, k = 1, \dots, m$.

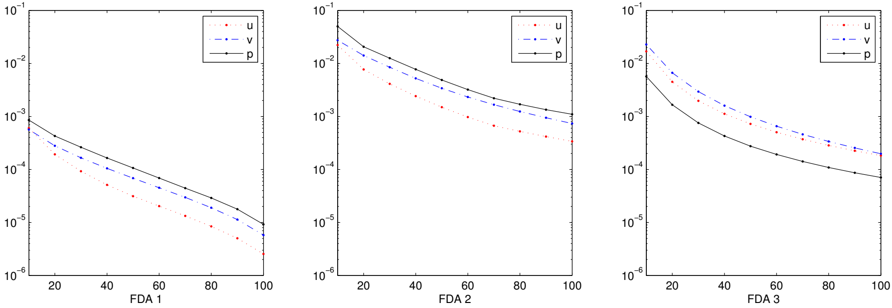
The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns p_{jk}^{n+1} for $j, k = 1, \dots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \dots, N$ being $t_f = N\tau$ the end point of the time interval.

Relative error for $\mathrm{Re} = 10^5$

We computed error by means of the formula

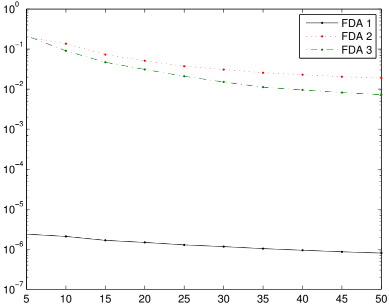
$$e_g = \max_{j,k} \frac{|g_{j,k}^N - g(x_j, y_k, t_f)|}{1 + |g(x_j, y_k, t_f)|}\,.$$

where $g \in \{u, v, p\}$ and $g(x, y, t)$ belongs to the exact solution.



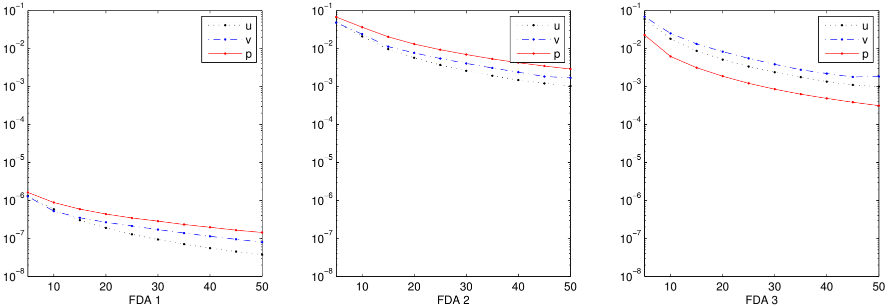
Relative error for $N = 10$, $t_f = N\tau = 1$, $\mathrm{Re} = 10^5$ and varying m from 5 to 50

Computed value of $u_x + v_y$



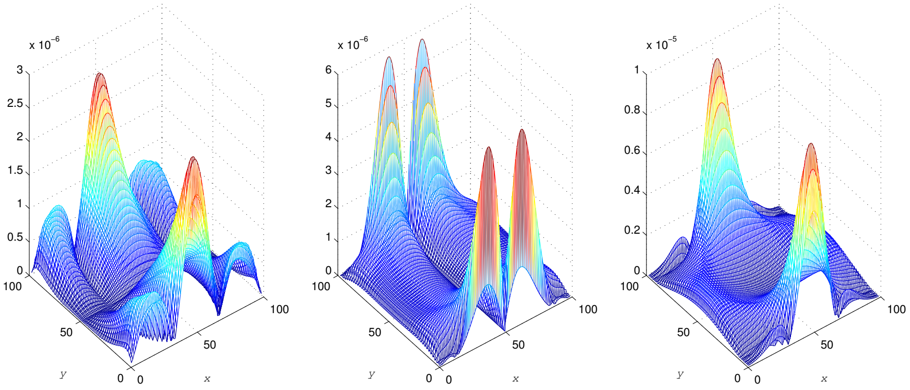
Computed value of f_1 in NSS for FDA 1, FDA 2 and FDA 3 with $N = 10$, $t_f = 1$, $Re = 10^5$ and varying m from 5 to 50

Relative error for Re = 10^2



Computed errors in u, v and p for FDA 1 (left), FDA 2 (middle) and FDA 3 (right): $N = 40$, $t_f = 1$, $Re = 10^2$ and varying m from 10 to 100

Relative error in u, v and p with FDA 1 for $\mathrm{Re} = 10^2$



Computed error with FDA 1 (u, v and p, respectively): $N = 40$, $t_f = 1$, $Re = 10^2$ and $m = 100$

Conclusions

Main results obtained

- We investigated s-consistency of three finite difference approximations to the Navier-Stokes equations for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- By using algorithmic methods of differential and difference algebra we shown that one of the approximations which is characterized by a 5×5 stencil is s-consistent whereas the other two with a 3×3 stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

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