Complexity in Tropical Algebra

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Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.

If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.

If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;

• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;

• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;

$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j \{Q_j\}$ is attained for at least two different values of j.

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Historical sources of the tropical algebra

Logarithmic scaling of the reals (mathematical physics)

Define two operations on positive reals, replacing addition and multiplication:

$a, b \to t \cdot \log(\exp(a/t) + \exp(b/t)), \quad \lim_{t\to 0} = \max\{a, b\}$

$a, b \to t \cdot \log(\exp(a/t) \cdot \exp(b/t)) = a + b$

Thus, the "dequantization" of the logarithmic scaling is a tropical semi-ring

Solving systems of polynomial equations in Puiseux series (algebraic geometry)

The field of Puiseux series $F((t^{1/\infty})) \ni a_0 \cdot t^{i/q} + a_1 \cdot t^{(i+1)/q} + \cdots, \ 0 < q \in \mathbb{Z}$ over an algebraically closed field F is algebraically closed. In the (Newton) algorithm for solving a system of polynomial equations $f_i(X_1, \dots, X_n) = 0, \ 1 \le i \le k$ with $f_i \in F((t^{1/\infty}))[X_1, \dots, X_n]$ in Puiseux series the leading exponents i_j/q_j in $X_j = a_{0j} \cdot t^{i_j/q_j} + \cdots$ satisfy a tropical polynomial system (due to cancelation of the leading terms).

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Minimal weights of paths in a graph (computer science)

For a graph with weights w_{ij} on edges (i, j) for any k to compute for each pair of vertices i, j the minimal weight of paths between i and j. This is equivalent to computing the tropical k-th power of matrix (w_{ij}).

Scheduling

Let several jobs i should be executed by means of several machines j with times of execution t_{ij}. The restrictions like that job i_0 should be executed after job i are imposed. Denoting by unknown x_{ij} a starting moment of execution of i by j, the latter restriction is expressed as $x_{i_0,j_0} \geq \min_j\{x_{ij} + t_{ij}\}$. Another sort of restrictions is that a machine can't execute two jobs simultaneously, i. e. $x_{i_1,j} \geq x_{ij} + t_{ij}$. It leads to a system of min-plus linear inequalities, the problem being equivalent to tropical linear systems.

This approach is employed in scheduling of Dutch and Korean railways.

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Tropical linear systems

If T is an ordered semi-group then tropical linear function over T can be written as $\min_{1\leq i\leq n}\{a_i + x_i\}$.

Tropical linear system

$$\min_{1\leq j\leq n}\{a_{i,j} + x_j\},\ 1 \leq i \leq m$$

(or $(m \times n)$-matrix $A = (a_{i,j})$) has a *tropical solution* $x = (x_1 \ldots, x_n)$ if for every row $1 \leq i \leq m$ there are two columns $1 \leq k < l \leq n$ such that

$$a_{i,k} + x_k = a_{i,l} + x_l = \min_{1\leq j\leq n}\{a_{i,j} + x_j\}$$

Coefficients $a_{i,j} \in \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$. Not all $x_j = \infty$. For $a_{i,j} \in \mathbb{Z}$ we assume $0 \leq a_{i,j} \leq M$.

$n \times n$ matrix $(a_{i,j})$ is **tropically non-singular** if $\min_{\pi\in S_n}\{a_{1,\pi(1)} + \cdots + a_{n,\pi(n)}\}$ is attained for a *unique* permutation π

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If T is an ordered semi-group then tropical linear function over T can be written as $\min_{1 \le i \le n}\{a_i + x_i\}$.

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Theorem

One can solve an $m \times n$ tropical linear system A within complexity polynomial in n, m, M. (Akian-Gaubert-Guterman; G.)
Moreover, the algorithm either finds a solution over $\mathbb{Z}_\infty$ or produces an $n \times n$ tropically nonsingular submatrix of A.

Corollary

The problem of solvability of tropical linear systems is in the complexity class $NP \cap coNP$.

Remark

• *My algorithm has also a complexity bound polynomial in $2^{nm}, \log M$ (as well as an obvious algorithm which invokes linear programming);*
• *Davydov: an example of A with $M \asymp 2^n \asymp 2^m$ for which my algorithm runs with exponential complexity $\Omega(M)$;*
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The theorem on complexity of solving tropical linear systems implies

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- *The corollary holds for matrices over $\mathbb{R}_\infty$.*
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- *For matrices A with* **finite** *coefficients from $\mathbb{R}$ it was proved by Develin-Santos-Sturmfels.*
- *Equivalence of 1) and 2) was established by Izhakian-Rowen.*

Solvability of a tropical linear system and rank(s)

The theorem on complexity of solving tropical linear systems implies

Corollary

The following statements are equivalent

1) a tropical linear system with $m \times n$ matrix A has a solution;

2) $trk(A) < n$;

3) $Krk_R(A) < n$.

Remark

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Computing dimension of a tropical system

Proposition

One can test uniqueness (in the tropical projective space) of a solution of a tropical linear system (i. e. whether the dimension of a tropical linear space equals 0) within complexity polynomial in n, m, M.

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Computing the dimension of a tropical linear space (being a union of polyhedra) is NP-complete (G.-Podol'ski)

Proposition

One can test solvability of a tropical nonhomogeneous linear system $\min_{1\leq j\leq n}\{a_{i,j}+x_j, a_i\}, 1\leq i\leq m$ *within complexity* $(n\cdot m\cdot M)^{O(1)}$.

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Testing equivalence of tropical linear systems

Two tropical linear systems are equivalent if their spaces of solutions coincide.

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One can reduce within polynomial, so $(n \cdot m \cdot \log M)^{O(1)}$ complexity testing equivalence of a pair of tropical linear systems to solving tropical linear systems. (G.-Podol'ski using Allamigeon-Gaubert-Katz)

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Min-plus linear system has a form

$$\min_{1\le j\le n}\{a_{i,j}+x_j\}=\min_{1\le j\le n}\{b_{i,j}+x_j\},\ 1\le i\le m$$

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One can test solvability of a min-plus linear system within complexity polynomial in M, n, m. If the system is solvable the algorithm yields its solution (Butkovic-Zimmermann).

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Complexities of the following 4 problems coincide up to a polynomial: solvability, equivalence of min-plus and of tropical linear systems (G.-Podol'ski using Allamigeon-Gaubert-Katz).

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Min-atom problem and mean payoff games

Min-atom problem is a system of inequalities of the form $\min\{x, y\} + c \leq z$, $c \in \mathbb{Z}$ (min-plus linear programming).

Mean payoff games

A bipartite graph (V, W, E) with integer weights a_{ij} on edges $e_{ij} \in E$ is given. Two players in turn move a token between nodes $V \cup W$ of the graph. The first player moves from a (current) node $i \in V$ to a node $j \in W$ (respectively, the second player moves from W to V). Weight a_{ij} is assigned to this move. Mean sum of assigned weights after k moves is computed: $(\sum a_{ij})/k$.

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Solvability of tropical polynomial systems is NP-complete (Theobald)

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Solvability of min-plus polynomial systems $f_i = 0$, $1 \leq i \leq m$ where f_i are min-plus polynomials, is NP-complete (G.-Shpilrain).

How to reduce tropical polynomial systems to tropical linear ones?

In the classical algebra for this aim serves Nullstellensatz.

In the tropical world the direct version of Nullstellensatz is false even for linear univariate polynomials: $X \oplus 0$, $X \oplus 1$ do not have a tropical solution, while their (tropical) ideal does not contain 0 or any other monomial (tropical monomials are the only polynomials without tropical zeroes).

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"Dual" (classical) Nullstellensatz

For polynomials $g_1, \ldots, g_s \in \mathbb{C}[X_1, \ldots, X_k]$ consider an infinite Cayley matrix C with the columns indexed by monomials X^I and the rows by shifts $X^J \cdot g_i$

Nullstellensatz: system $g_1 = \cdots = g_s = 0$ has no solution iff a linear combination of the rows of a suitable *finite* submatrix C_0 of C (generated by a set of rows of C) equals vector $(1, 0, \ldots, 0)$.

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Remark

Nullstellensatz deals with ideal $\langle g_1, \ldots, g_s \rangle$, while dual Nullstellensatz forgets the ideal, therefore, gives a hope to hold in the tropical setting

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Tropical dual effective Nullstellensatz

Assume w.l.o.g. that for tropical polynomials $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ in k variables which we consider, function $J \to a_J$ is concave on $\mathbb{R}^k$. This assumption does not change tropical varieties.

For tropical polynomials $h_1, \dots, h_s$ consider (infinite in all 4 directions) Cayley matrix H with the rows indexed by $X^{\otimes I} \otimes h_i$ for $I \in \mathbb{Z}^k$, $1 \le i \le s$.

Theorem

*Tropical polynomials $h_1, \dots, h_s$ have a solution iff tropical linear system $H_0 \otimes (z_0, \dots, z_N)$ has a solution with $z_0 \neq \infty$ where H_0 is (finite) submatrix of H generated by its rows $X^{\otimes I} \otimes h_i$ for $0 \le |I| \le (k+2) \cdot (\mathrm{trdeg}(h_1) + \cdots + \mathrm{trdeg}(h_s))$, $1 \le i \le s$. (**G.-Podolskii**)*

Conjecture is that the latter bound is $O(trdeg(h_1) + \cdots + trdeg(h_s))$.

For two tropical polynomials ($s = 2$) the bound $trdeg(h_1) + trdeg(h_2)$ (**Tabera**) holds using the classical resultant and **Kapranov's** theorem: for a polynomial $f \in R((t^{1/\infty}))[x_1, \dots, x_k]$ we have $\mathrm{Variety}(\mathrm{Trop}(f)) = \mathrm{Trop}(\mathrm{Variety}(f))$

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(Convex)-geometrical rephrasing of the tropical dual Nullstellensatz

For a tropical polynomial $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ consider its extended Newton polyhedron G being the convex hull of the graph $\{(J, a) : a \leq -a_J\} \subset \mathbb{R}^{k+1}$. As vertices of G consider all the points of the form (I, c), $I \in \mathbb{Z}^k$ on the boundary of G. Let G_i correspond to h_i, $1 \leq i \leq s$. Denote by $G^{(I)} := G + (I, 0)$ a horizontal shift of G. Solution $Y := \{(J, y_J)\} \subset \mathbb{R}^{k+1}$ of a tropical linear system $H \otimes Y$ treat also as a graph on $\mathbb{R}^k$.

The tropical dual (infinite) Nullstellensatz is equivalent to the following.

For any I, i take the maximal $b := b_{I,i}$ such that a vertical shift $G_i^{(I)} + (0, b) \leq Y$ (pointwise as graphs).

Assume that $G_i^{(I)} + (0, b)$ has at least two common points with Y. Then there is a hyperplane in $\mathbb{R}^{k+1}$ (not containing the vertical line) which supports (after a parallel shift) each G_i, $1 \leq i \leq s$ at least at two points.

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Dual effective Nullstellensatz for min-plus equations

Let a system of non-linear min-plus equations be given

$$f_i(X_1, \dots, X_k) = g_i(X_1, \dots, X_k),\ 1 \leq i \leq s \qquad (1)$$

where f_i, g_i are min-plus polynomials.

Consider an (infinite) Cayley matrix B with 2 sets of columns both corresponding to $X^{\otimes I}$ with entries in the row J, i being coefficients in the expansion of $X^{\otimes J} \otimes f_i$ and of $X^{\otimes J} \otimes g_i$ in two sets, correspondingly. B_N denotes a finite submatrix of B spanned by $|J| \leq N$.

Theorem

*System (1) has a solution iff min-plus linear system with matrix B_N, $N \leq (k+2) \cdot (\mathrm{trdeg}(f_1) + \cdots + \mathrm{trdeg}(f_s) + \mathrm{trdeg}(g_1) + \cdots + \mathrm{trdeg}(g_s))$ has a solution (**G.-Podolskii**).*

Question. Is it possible to get rid of factor $k+2$?

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Algorithm for solving tropical linear systems: finite coefficients

First assume that the coefficients of a tropical linear system $A = (a_{i,j})$ are finite: $0 \leq a_{i,j} \leq M$, $1 \leq i \leq n$, $1 \leq j \leq m$.
Induction on m. Suppose that (tropical) vector $x := (x_1, \ldots, x_n)$ fulfils $m - 1$ equations (except, perhaps, the first one).

The algorithm modifies x and either produces a solution of A or finds $n \times n$ tropically nonsingular submatrix of A (in the latter case A has no solution).
After each step of modification a vector is produced (we keep for it the same notation x) such that it still fulfils $m - 1$ equations, and $m \times n$ matrix $B := (a_{i,j} + x_j)$ (after suitable permutations of rows and columns) has a form below.

If $a_{i,j} + x_j = \min_{1 \leq l \leq n}\{a_{i,l} + x_l\}$ mark entry i, j with $*$. The first row contains a single $*$ (otherwise, x is a solution of A and every other row contains at least two $*$.

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First assume that the coefficients of a tropical linear system $A = (a_{i,j})$ are finite: $0 \leq a_{i,j} \leq M$, $1 \leq i \leq n$, $1 \leq j \leq m$.
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Continuation: producing a candidate for solution

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \\ B_5 & B_6 \end{pmatrix}$$

• a square matrix B_1 contains $*$ on the diagonal and no $*$ above the diagonal. Hence B_1 is tropically nonsingular.
• B_2, B_4 contain no $*$.
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Modify vector $x_1, \dots, x_n$ adding (classically) to it a vector $(b, \dots, b, 0, \dots, 0)$ for integer $b = \max_i \{a_{i,j} + x_j - a_{i,l} - x_l\}$ where j runs right columns, l runs left columns, i runs rows from matrices $(B_1\ B_2)$ and $(B_3\ B_4)$.
The modified vector (keeping for it the notation x) still fulfils $m - 1$ equations and $b \geq 1$.

If the first row of the modified matrix B contains at least two $*$, x is a solution of A.

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Termination of the algorithm

Construct recursively a set L of the left columns by augmenting. As a base of recursion the first column belongs to L.

For current L if there exists a row with single $*$ in a column off L, join this column to L. These rows and columns form matrix B_1.

If L coincides with the set of all the columns then B_1 is $n \times n$ tropically nonsingular submatrix of B and therefore, A has no solution. This completes the description of the algorithm.

Tropical norm and complexity bound

To estimate the number of steps of the algorithm define a *tropical norm* of a vector (in the tropical projective space) $(y_1, \dots, y_n)$ as

$$\sum_{1 \le i \le n} (y_i - \min_{1 \le j \le n} \{y_j\}).$$

After every modification step the tropical norm of vector $(a_{1,1} + x_1, \dots, a_{1,n} + x_n)$ (corresponding to the first row) drops.

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Solving tropical linear systems over $\mathbb{Z}_\infty$

For the inductive (again on m) hypothesis assume that $(m-1)\times n$ matrix A' (obtained from A by removing its first row) has a block form (after permuting its rows and columns)

$$\begin{pmatrix} A_{1,1} & \infty & \cdots & \infty & \infty \\ A_{2,1} & A_{2,2} & \cdots & \infty & \infty \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{t-1,1} & A_{t-1,2} & \cdots & A_{t-1,t-1} & \infty \\ \hline A_{t,1} & A_{t,2} & \cdots & A_{t,t-1} & A_{t,t} \end{pmatrix}$$

where each entry of upper-triangular blocks equals ∞.

A finite vector $y=(y_1,\dots,y_n)=:(y^{(1)},\dots,y^{(t)})\in\mathbb{Z}^n$ is produced (where $y^{(1)},\dots,y^{(t)}$ is its partition corresponding to the block structure) such that each diagonal block $A_{p,p}$, $1\le p\le t-1$ has $*$ (with respect to vector $y^{(p)}$) everywhere on its diagonal and no $*$ above the diagonal. Matrix $A_{p,p}$ is of size $u_p\times v_p$ with $u_P\ge v_p$.
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Continuation: modifying candidate for a solution

To be closer to the finite case $\mathbb{Z}$ extend the lowest block $\overline{A_{t,1}}\ \overline{A_{t,2}} \cdots \overline{A_{t,t-1}}\ \overline{A_{t,t}}$ of A' by joining to it the first row of A as its first row. The resulting extension of matrix $\overline{A_{t,t}}$ denote by C.

Again as in the finite case assume (after a permutation of the columns) that a single $*$ (with respect to vector $y^{(t)}$) in the first row of C is located in the first column.

The algorithm modifies vector $y^{(t)}$ keeping it to be a solution of $\overline{A_{t,t}}$ and keeping the same notation for the modified vectors.
If $y^{(t)}$ is a solution of C then vector $(\infty, \dots, \infty, y^{(t)})$ is a solution of A and the algorithm terminates the inductive step.

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In addition, the algorithm considers an oriented graph with the nodes being the coordinates of vector $y^{(t)} =: (y_1^{(t)}, \ldots, y_s^{(t)})$ and with an edge from node $y_j^{(t)}$ to $y_l^{(t)}$ when $y_j^{(t)} - y_l^{(t)} \leq M$ (remind that all finite coefficients of matrix A satisfy $0 \leq a_{i,j} \leq M$).

Denote by S the set of nodes of the graph reachable from the first node $y_1^{(t)}$.

Lemma

$L \subset S$ and in the course of the algorithm while modifying S, the next S is a subset of the previous one.

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In addition, the algorithm considers an oriented graph with the nodes being the coordinates of vector $y^{(t)} =: (y_1^{(t)}, \dots, y_s^{(t)})$ and with an edge from node $y_j^{(t)}$ to $y_l^{(t)}$ when $y_j^{(t)} - y_l^{(t)} \leq M$ (remind that all finite coefficients of matrix A satisfy $0 \leq a_{i,j} \leq M$).
Denote by S the set of nodes of the graph reachable from the first node $y_1^{(t)}$.

Lemma

$L \subset S$ and in the course of the algorithm while modifying S, the next S is a subset of the previous one.

The algorithm modifies $y^{(t)}$ while $L \neq S$.

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Termination of the algorithm

$$C = \begin{pmatrix} C_1 & \infty \\ C_2 & \infty \\ C_3 & C_4 \end{pmatrix}$$

• L are columns of a square matrix C_1;
• (tropically nonsingular) C_1 contains $*$ everywhere on the diagonal and no $*$ above it;
• each row of C_2 and of C_4 contains at least two $*$

This completes the inductive step of the algorithm and constructing a new block structure of matrix A.
Vector $y^{(t)} =: (y^{(t)}, y^{(t+1)})$ (abusing the notations) and vector $(\infty, \dots, \infty, y^{(t+1)})$ is a solution of A.

The algorithm terminates if either all the columns or all the rows are exhausted. If all the columns are exhausted then A has no solution.
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