Symbolic-Numerical Investigation of Gyrostat Satellite Dynamics

Gutnik S.A.

Moscow Institute of Physics and Technology, MGIMO

Sarychev V. A.

Keldysh Institute of Applied Mathematics, RAS

The 15th International Workshop on Computer Algebra in Scientific Computing September 9 - 13, 2013 Berlin, Germany

1. Equations of motion

The equations of the satellite's attitude motion under the influence of gravitational and gyrostatic torques in a circular orbit take the form:

$$A\dot{p} + (C-B)qr - 3\omega_0^2(C-B)a_{32}a_{33} + \bar{H}_3 q - \bar{H}_2 r = 0,$$
$$B\dot{q} + (A-C)rp - 3\omega_0^2(A-C)a_{33}a_{31} + \bar{H}_1 r - \bar{H}_3 p = 0, \qquad (1)$$
$$C\dot{r} + (B-A)pq - 3\omega_0^2(B-A)a_{31}a_{32} + \bar{H}_2 p - \bar{H}_1 q = 0;$$

$$p = \dot{\psi} a_{31} + \dot{\vartheta}\cos\varphi + \omega_0 a_{21} = \bar{p} + \omega_0 a_{21},$$
$$q = \dot{\psi} a_{32} - \dot{\vartheta}\sin\varphi + \omega_0 a_{22} = \bar{q} + \omega_0 a_{22}, \qquad (2)$$
$$r = \dot{\psi} a_{33} + \dot{\varphi} + \omega_0 a_{23} = \bar{r} + \omega_0 a_{23}.$$

Here ***A, B, C****; are the principal central moments of inertia of the satellite;* p, q, r *– are the projections of the satellite's angular velocity in the axes* Ox, Oy, Oz; ψ, ϑ, *and* φ *- are the* Euler angles *;* a_{ij} - the direction cosines of the axis Ox, Oy, Oz *in the orbital reference frame,* $H_1 = \bar{H}_1/\omega_0$, $H_2 = \bar{H}_2/\omega_0$, $H_3 = \bar{H}_3/\omega_0$; are the projections of the vector gyrostatic moment at the body reference frame. ω_0 – is the angular velocity of the satellite in the circular orbit.

2. Equilibrium orientations

Putting in (1) and (2) $\psi = \psi_0, \vartheta = \vartheta_0, \varphi = \varphi_0$, (are constants) we get the stationeries equations:

$$\begin{aligned}
&(C-B)(a_{22}a_{23}-3a_{32}a_{33})-H_2a_{13}+H_3a_{12}=0,\\
&(A-C)(a_{23}a_{21}-3a_{33}a_{31})-H_3a_{11}+H_1a_{13}=0,\\
&(B-A)(a_{21}a_{22}-3a_{31}a_{32})-H_1a_{12}+H_2a_{11}=0,
\end{aligned} \qquad (3)$$

or equivalent system

$$\begin{aligned}
&Aa_{21}a_{31}+Ba_{22}a_{32}+Ca_{23}a_{33}=0,\\
&3(Aa_{11}a_{31}+Ba_{12}a_{32}+Ca_{13}a_{33})+(H_1a_{31}+H_2a_{32}+H_3a_{33})=0,\\
&(Aa_{11}a_{21}+Ba_{12}a_{22}+Ca_{13}a_{23})-(H_1a_{21}+H_2a_{22}+H_3a_{23})=0,
\end{aligned} \qquad (4)$$

with orthogonal conditions

$$\begin{aligned}
&a_{11}^2+a_{12}^2+a_{13}^2=1, \quad a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}=0,\\
&a_{21}^2+a_{22}^2+a_{23}^2=1, \quad a_{11}a_{31}+a_{12}a_{32}+a_{13}a_{33}=0,\\
&a_{31}^2+a_{32}^2+a_{33}^2=1, \quad a_{21}a_{31}+a_{22}a_{32}+a_{23}a_{33}=0
\end{aligned} \qquad (5)$$

the system of equations (4), (5) determine all equilibrium orientations of a satellite when A, B. C, H_1, H_2, H_3 are given.

In the general case $A \neq B \neq C,\ h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0.$ After the introduction of the dimensionless parameters $h_1 = \frac{H_1}{B-C},\ h_2 = \frac{H_2}{B-C},\ h_3 = \frac{H_3}{B-C},\ \nu = \frac{B-A}{B-C}.$ the system (4), takes the form:

$$-(\nu a_{21}a_{31} + a_{23}a_{33}) + (h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0,$$
$$\nu a_{11}a_{31} + a_{13}a_{33} = 0, \qquad (6)$$
$$\nu a_{11}a_{21} + a_{13}a_{23} - (h_1 a_{11} + h_2 a_{12} + h_3 a_{13}) = 0.$$

The system (5), (6) can be solved for the variables $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ in the following form:

$$a_{11} = \frac{-4a_{32}a_{33}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \qquad a_{21} = \frac{4[\nu a_{32}^2 - (1-\nu)a_{33}^2]a_{31}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}},$$
$$a_{12} = \frac{4(1-\nu)a_{33}a_{31}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \qquad a_{22} = \frac{-4(\nu a_{31}^2 + a_{33}^2)a_{32}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \qquad (7)$$
$$a_{13} = \frac{4\nu a_{31}a_{32}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \qquad a_{23} = \frac{4[(1-\nu)a_{31}^2 + a_{32}^2]a_{33}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}.$$

Substituting solutions (7) in (6), we obtain the system of algebraic equations from 3 unknowns a_{31}, a_{32}, a_{33}, or from Groebner basis (5), (6)

$$16[a_{32}^2 a_{33}^2 + (1-\nu)^2 a_{33}^2 a_{31}^2 + \nu^2 a_{31}^2 a_{32}^2] = (h_1 a_{31} + h_2 a_{32} + h_3 a_{33})^2 (a_{31}^2 + a_{32}^2 + a_{33}^2),$$
$$4\nu(1-\nu) a_{31} a_{32} a_{33} + [h_1 a_{32} a_{33} - h_2 (1-\nu) a_{33} a_{31} - h_3 \nu a_{31} a_{32}](h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0,$$
$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1. \qquad (8)$$

Let us introduce the values $x = \dfrac{a_{31}}{a_{33}}, \; y = \dfrac{a_{32}}{a_{33}}$. Then we will have 2 equations:

$$a_0 y^2 + a_1 y + a_2 = 0,$$
$$b_0 y^4 + b_1 y^3 + b_2 y^2 + b_3 y + b_4 = 0. \qquad (9)$$

Here
$$a_0 = h_2 (h_1 - \nu h_3 x),$$
$$a_1 = h_1 h_3 + [4\nu(1-\nu) + h_1^2 - (1-\nu) h_2^2 - \nu h_3^2] x - \nu h_1 h_3 x^2,$$
$$a_2 = -(1-\nu) h_2 (h_1 x + h_3) x,$$
$$b_0 = h_2^2,$$
$$b_1 = 2 h_2 (h_1 x + h_3),$$
$$b_2 = (h_2^2 + h_3^2 - 16) + 2 h_1 h_3 x + (h_1^2 + h_2^2 - 16\nu^2) x^2,$$
$$b_3 = 2 h_2 (h_1 x + h_3)(1 + x^2),$$
$$b_4 = (h_1 x + h_3)^2 (1 + x^2) - 16(1-\nu)^2 x^2.$$

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Resultant of equations (9) has the form of the 12-th order algebraic equation:

$$p_0 x^{12} + p_1 x^{11} + p_2 x^{10} + p_3 x^9 + p_4 x^8 + p_5 x^7 + p_6 x^6 + \qquad (10)$$

$$+ p_7 x^5 + p_8 x^4 + p_9 x^3 + p_{10} x^2 + p_{11} x + p_{12} = 0,$$

Where the coefficients $p_i = p_i(h_1, h_2, h_3, \nu)$ have the form:

$$p_0 = -h_1^4 h_3^4 \nu^6,$$

$$p_1 = 2h_1^3 h_3^3 \nu^5 \left[2h_1^2 - h_2^2(\nu - 1) - 2\nu\left(h_3^2 + 2\nu - 2\right)\right],$$

$$p_2 = -h_1^2 h_3^2 \nu^4 \left\{6h_1^4 + h_2^4(\nu-1)^2 - h_2^2(\nu-1)\left[16\left(\nu^3 - \nu^2\right) + (\nu - 1) + h_3^2(1 - 7\nu)\right] + \right.$$

$$+ h_1^2\left[\left(-25\nu^2 + 26\nu - 1\right) + h_3^2\left(\nu^2 - 16\nu + 1\right) + h_2^2\left(\nu^2 - 8\nu + 7\right)\right] +$$

$$\left. + 2\nu^2\left[3h_3^4 + 8(\nu-1)^2 - 4h_3^2\left(2\nu^2 - 7\nu + 5\right)\right]\right\},$$

$$p_3 = 2h_1 h_3 \nu^3 \left\{2h_1^6 + h_1^4\left[\left(-13\nu^2 + 14\nu - 1\right) + 2h_3^2\left(\nu^2 - 6\nu + 1\right) + h_2^2\left(\nu^2 - 5\nu + 4\right)\right] + \right.$$

$$+ h_3^2\left[-h_2^4(\nu-1)^2(2\nu - 1) + h_2^2(\nu-1)\nu\left[h_3^2(1 - 4\nu) + \left(16\nu^3 - 16\nu^2 + \nu - 1\right)\right] + \right.$$

$$\left. + 2\nu^3\left[-h_3^4 + 8(\nu-1)^2(4\nu - 5) + 2h_3^2\left(7 - 11\nu + 4\nu^2\right)\right]\right] -$$

$$- h_1^2\left[h_2^4(\nu-2)(\nu-1)^2 + h_2^2(\nu-1)\left[\left(16\nu^3 - 16\nu^2 + \nu - 1\right) + h_3^2\left(3\nu^2 - 13\nu + 3\right)\right] + \right.$$

$$\left.\left. + 2\nu\left[-2(\nu-1)^2(5\nu - 1) + h_3^2\left(\nu^2 - 6\nu + 1\right) + h_3^2\left(18\nu^3 - 53\nu^2 + 38\nu - 3\right)\right]\right]\right\} \quad \ldots$$

3. Analysis of equilibrium orientations

The numerical calculations it is possible to provide for the case $B > A > C \ (0 < \nu < 1)$.

For the limiting parameters values $\nu = 0$ and $\nu = 1$, it is possible to define analytically a boundary of regions with the equal number of equilibria.

In the case $\nu = 0 \ (A = B)$ the boundary has the form:

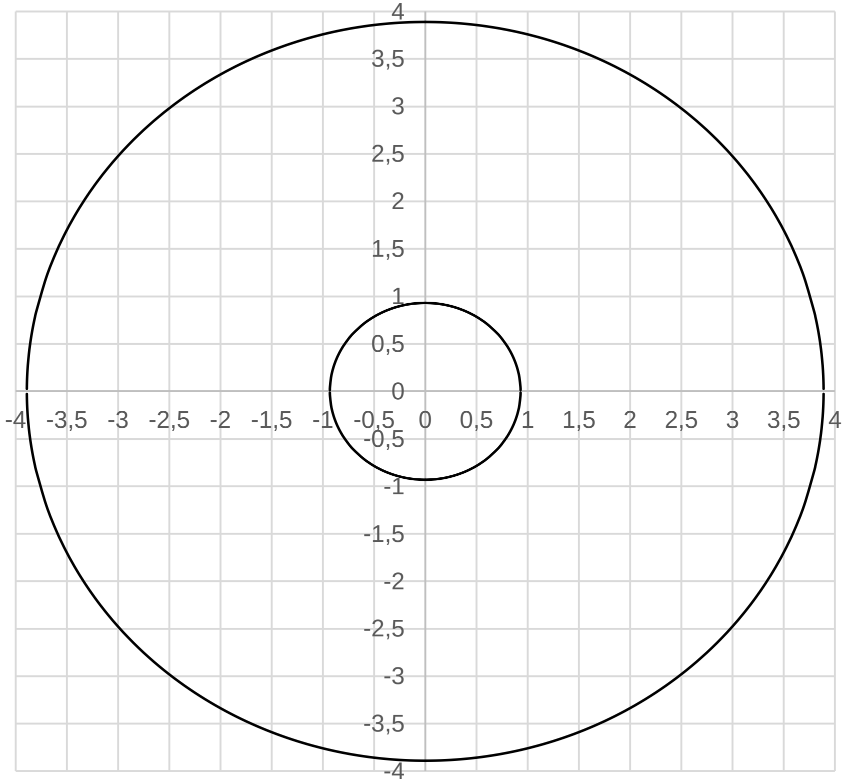
$$h_1^2 + h_2^2 = (4^{2/3} - h_3^{2/3})^3,$$
$$h_1^2 + h_2^2 = (1 - h_3^{2/3})^3. \qquad (11)$$

In the case $\nu = 1 \, (A = C)$ the boundary has the form:

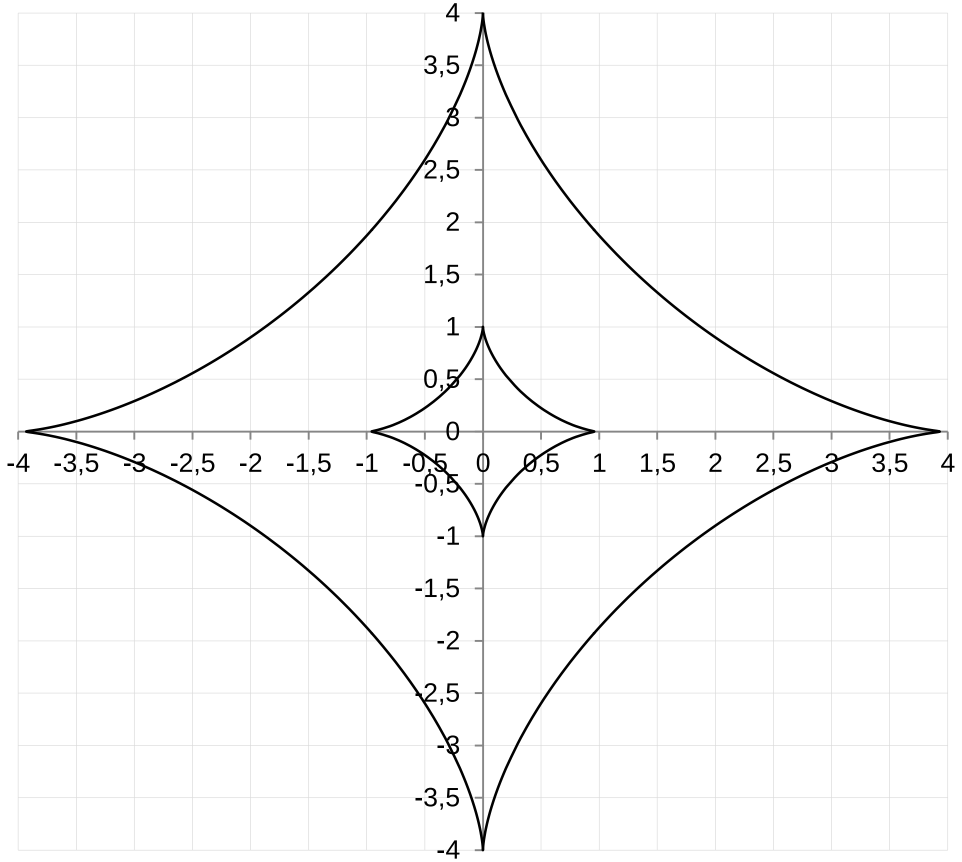
$$h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} = 4^{2/3},$$
$$h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} = 1. \qquad (12)$$

3. Analytical results- axisymmetrical cases

v=0 (A=B) (11)

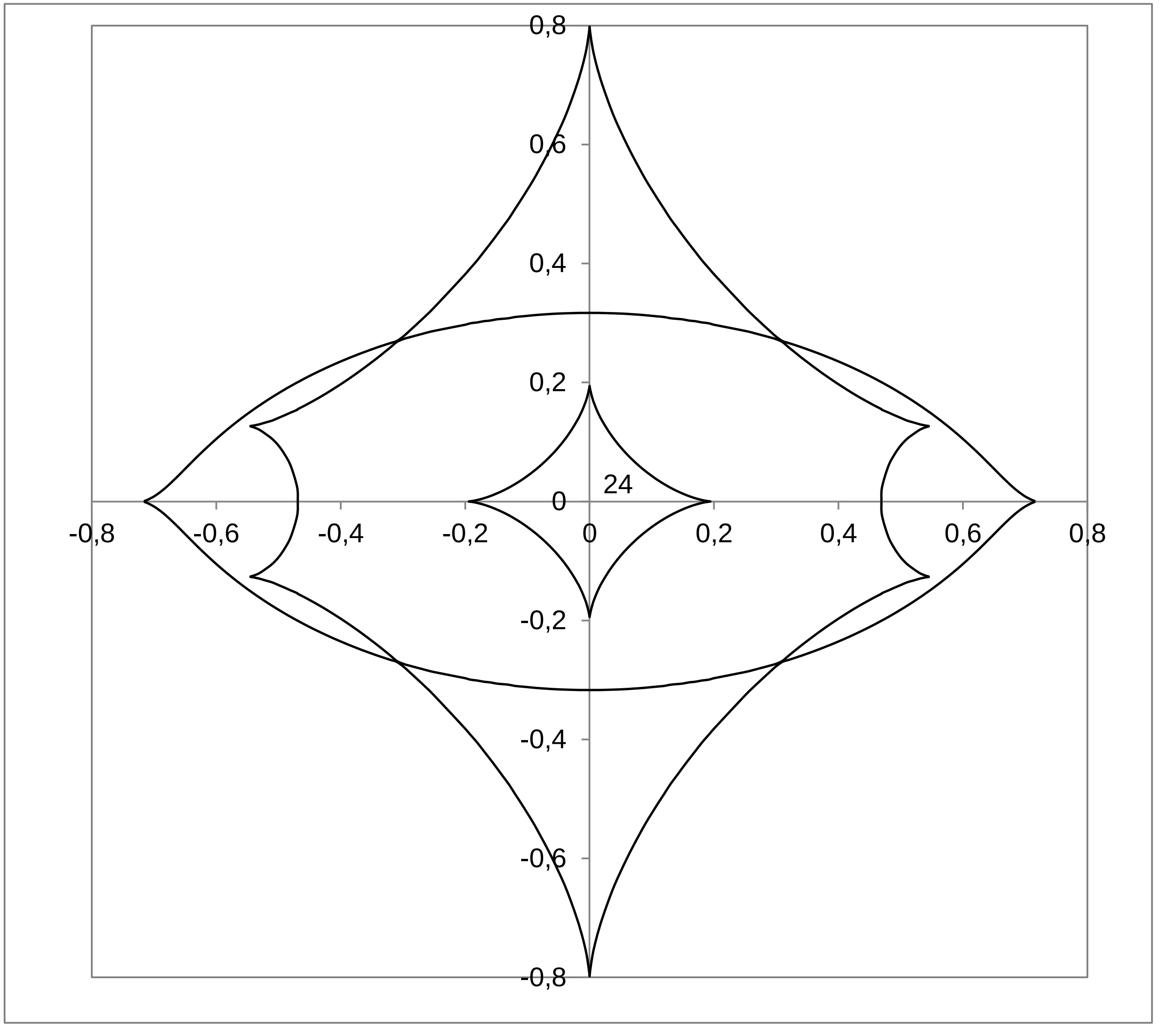


v=1 (A=C) (12)



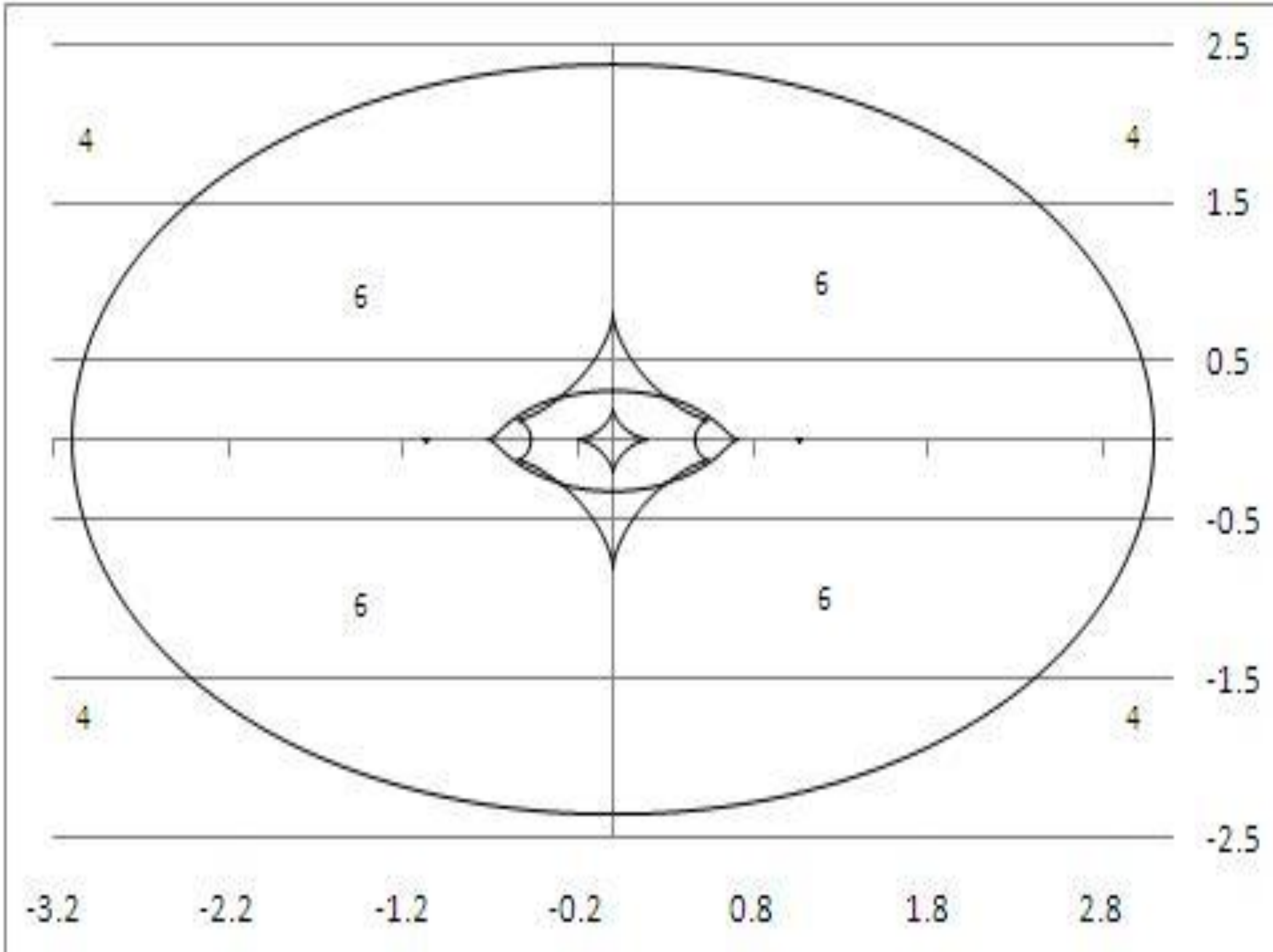
$h_3 = 0.01$

4. Numerical results (Central part)



v=0.2; h_3 = 0.25

4. Numerical results



***v*=0.2; h_3 = 0.25**

4. Numerical results

Regions of equilibria $\boldsymbol{v}$	**24/20**	**20/16**	**16/12**	**12/8**
0,01	h_3=0,990	h_3=0,999	h_3=3,959	h_3=4,0
0,1	h_3=0,900	h_3=1,021	h_3=3,610	h_3=4,0
0,2	h_3=0,800	h_3=1,048	h_3=3,264	h_3=4,0
0,3	h_3=0,700	h_3=1,082	h_3=2,950	h_3=4,0
0,4	h_3=0,600	h_3=1,124	h_3=2,669	h_3=4,0
0,5	h_3=0,500	h_3=1,182	h_3=2,412	h_3=4,0
0,6	h_3=0,400	h_3=1,186	h_3=2,167	h_3=4,0

4. Numerical results

Regions of equilibria v	**24/20**	**20/16**	**16/12**	**12/8**
0,7	h_3=0,300	h_3=1,105	h_3=1,915	h_3=4,0
0,8	h_3=0,200	h_3=0,909	h_3=1,629	h_3=4,0
0,9	h_3=0,100	h_3=0,676	h_3=1,245	h_3=4,0
0,99	h_3=0,010	h_3=0,168	h_3=0,997	h_3=4,0

5. Stability analysis of equilibria

Let us use the Hamiltonian of (1), (2) as Lyapunov's function in order to obtain the sufficient conditions of stability

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B-C)\omega_0^2\{3[(1-\nu)a_{31}^2 + a_{32}^2] + \quad (13)$$
$$+(\nu a_{21}^2 + a_{23}^2) - 2(h_1 a_{21} + h_2 a_{22} + h_3 a_{23})\} = const.$$

Let us present angles $\psi = \psi_0 + \overline{\psi},\ \vartheta = \vartheta_0 + \overline{\vartheta},\ \varphi = \varphi_0 + \overline{\varphi},$ then, Hamiltonian (1) can be presented in the form:

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B-C)\omega_0^2(A_{\psi\psi}\overline{\psi}^2 + A_{\vartheta\vartheta}\overline{\vartheta}^2 + A_{\varphi\varphi}\overline{\varphi}^2 + \quad (14)$$
$$+2A_{\psi\vartheta}\overline{\psi}\overline{\vartheta} + 2A_{\psi\varphi}\overline{\psi}\overline{\varphi} + 2A_{\vartheta\varphi}\overline{\vartheta}\overline{\varphi}) + \Sigma = const.$$

5. Stability analysis of equilibria

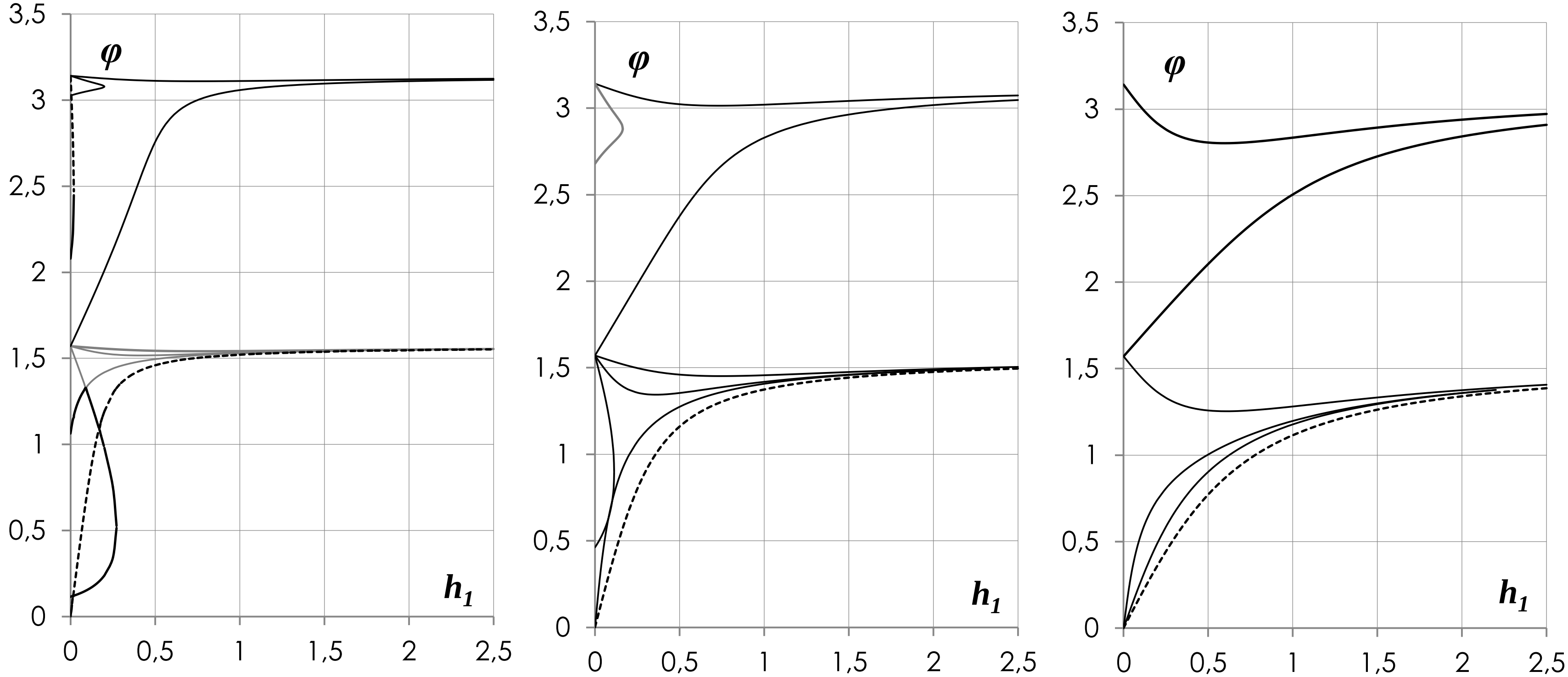
Where:

$$\begin{aligned}
A_{\psi\psi} &= \nu(a_{11}^2 - a_{21}^2) + (a_{13}^2 - a_{23}^2) + h_1 a_{21} + h_2 a_{22} + h_3 a_{23},\\
A_{\vartheta\vartheta} &= (3 + \cos^2\psi_0)(1 - \nu\sin^2\varphi_0)\cos 2\vartheta_0 -\\
&- \frac{1}{4}\nu\sin 2\psi_0\cos\vartheta_0\sin 2\varphi_0 + (h_1\sin\varphi_0 + h_2\cos\varphi_0)\cos\psi_0\cos\vartheta_0 + h_3 a_{23},\\
A_{\varphi\varphi} &= \nu[(a_{22}^2 - a_{21}^2) - 3(a_{32}^2 - a_{31}^2)] + h_1 a_{21} + h_2 a_{22},\\
A_{\psi\vartheta} &= -\frac{1}{2}\sin 2\psi_0\sin 2\vartheta_0 + \nu(a_{11}a_{23} + a_{13}a_{21}) - \sin\psi_0(h_1 a_{31} + h_2 a_{32} + h_3 a_{33}),\\
A_{\psi\varphi} &= \nu(a_{11}a_{22} + a_{12}a_{21}) - h_1 a_{12} + h_2 a_{11},\\
A_{\vartheta\varphi} &= -\frac{3}{2}\nu\sin 2\vartheta_0\sin 2\varphi_0 + \nu(a_{21}\cos\varphi_0 + a_{22})a_{23} - (h_1\cos\varphi_0 - h_2\sin\varphi_0)a_{23}.
\end{aligned} \qquad (15)$$

It follows form the Lyapunov's theorem that the equilibrium solution is stable if the quadratic form (14), (15) for this solution is positive definite i.e. the following inequalities take place:

$$\begin{aligned}
&A_{\psi\psi} > 0,\\
&A_{\psi\psi}A_{\vartheta\vartheta} - (A_{\psi\vartheta})^2 > 0,\\
&A_{\psi\psi}A_{\vartheta\vartheta}A_{\varphi\varphi} + 2A_{\psi\vartheta}A_{\psi\varphi}A_{\vartheta\varphi} - A_{\psi\psi}(A_{\vartheta\varphi})^2 - A_{\vartheta\vartheta}(A_{\psi\varphi})^2 - A_{\varphi\varphi}(A_{\psi\vartheta})^2 > 0.
\end{aligned} \qquad (16)$$

6. Numerical results of the stability of equilibria



$\nu = 0.1$, $h_2 = 0.05$, $h_3 = 0.45$ $\nu = 0.1$, $h_2 = 0.2$, $h_3 = 0.45$ $\nu = 0.1$, $h_2 = 0.5$, $h_3 = 0.45$

6. Numerical results of the stability of equilibria

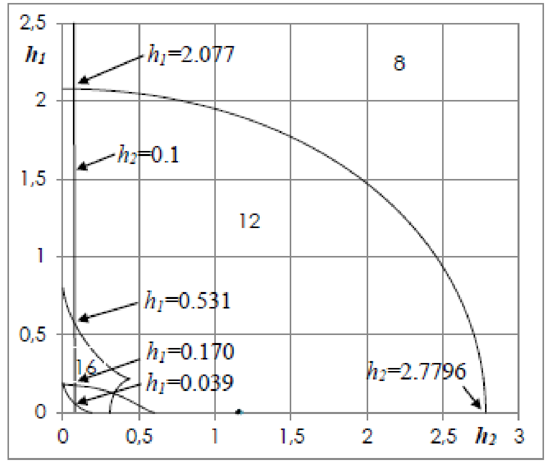


Fig. 25. ν=0.2, $h_3 = 0.4$

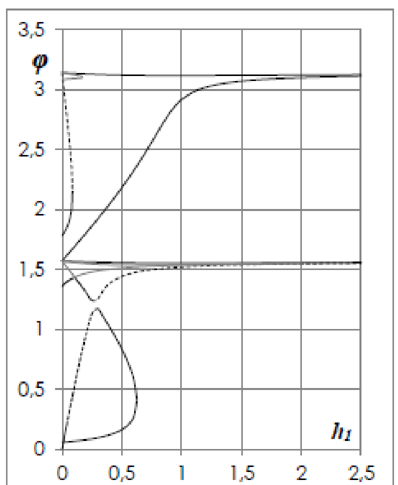


Fig. 26. ν=0.2, $h_2 = 0.05$, $h_3 = 0.4$
(24 equilibria, 4 stable)

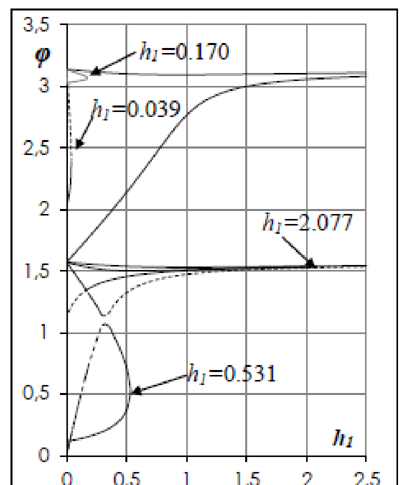


Fig. 27. ν=0.2, $h_2 = 0.1$, $h_3 = 0.4$
(24 equilibria, 4 stable)

6. Numerical results of the stability of equilibria

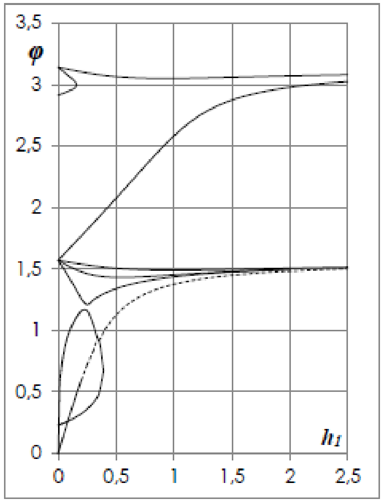


Fig. 28. ν=0.2, h_2 = 0.2, h_3 = 0.4
(20 equilibria, 2 stable)

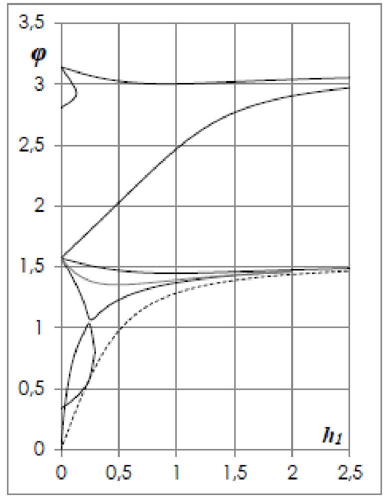


Fig. 29. ν=0.2, h_2 = 0.3, h_3 = 0.4
(20 equilibria, 2 stable)

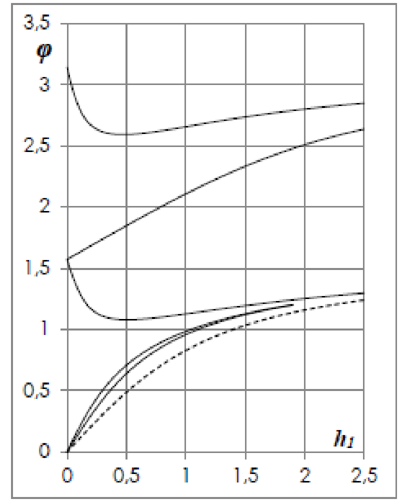


Fig. 32. ν=0.2, h_2 = 1.0, h_3 = 0.4
(12 equilibria, 2 stable)

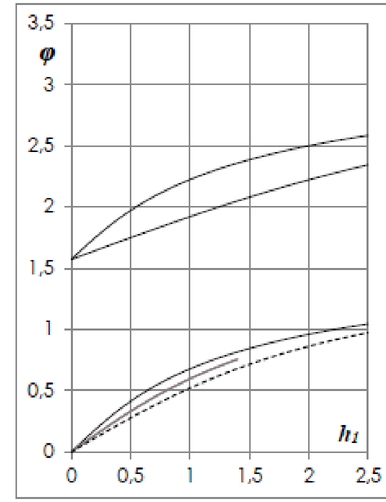


Fig. 33. ν=0.2, h_2 = 2.0, h_3 = 0.4
(12 equilibria, 2 stable)

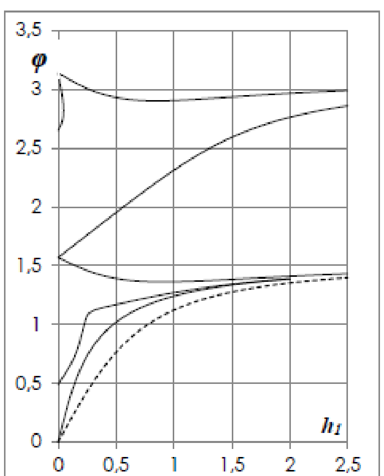


Fig. 30. ν=0.2, h_2 = 0.5, h_3 = 0.4
(16 equilibria, 2 stable)

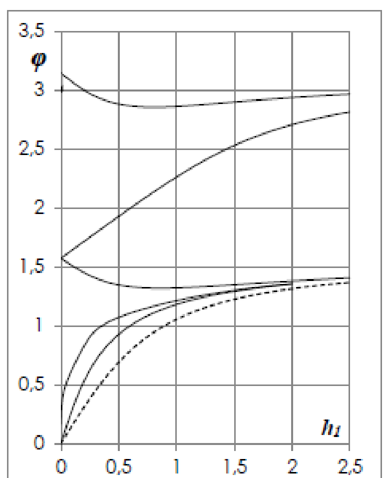


Fig. 31. ν=0.2, h_2 = 0.6, h_3 = 0.4
(12 equilibria, 2 stable)

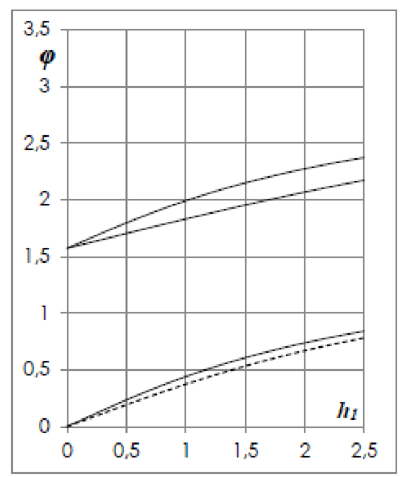


Fig. 34. ν=0.2, h_2 = 3.0, h_3 = 0.4
(8 equilibria, 2 stable)

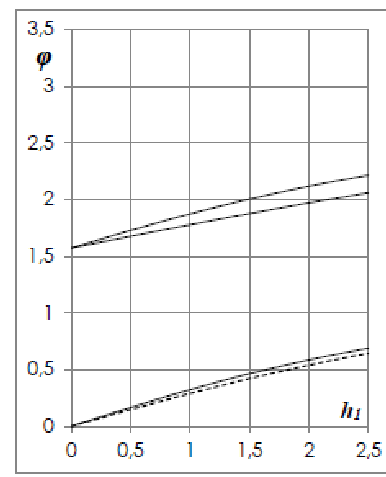


Fig. 35. ν=0.2, h_2 = 4.0, h_3 = 0.4
(8 equilibria, 2 stable)

7. Conclution

- Numerical-analytical method for the determination of the equilibrium orientations in the gyrostat satellite is proposed in general case $A \neq B \neq C, \quad h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0.$
- Evolution of domains in the space of parameters which correspond to various numbers of equilibria are carried out in detail
- Relationship with axisymmetrical cases of satellite gyrostat is considered.
- It is shown that the number of equilibria of the gyrostat satellite in general case not be less than 8 and not more than 24
- It is shown that the number of stable equilibria of the gyrostat satellite in general case changes from 4 to 2 with the increasing the absolute value of gyrostatic torque.

8. References

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Annex. Numerical results ν=0.01

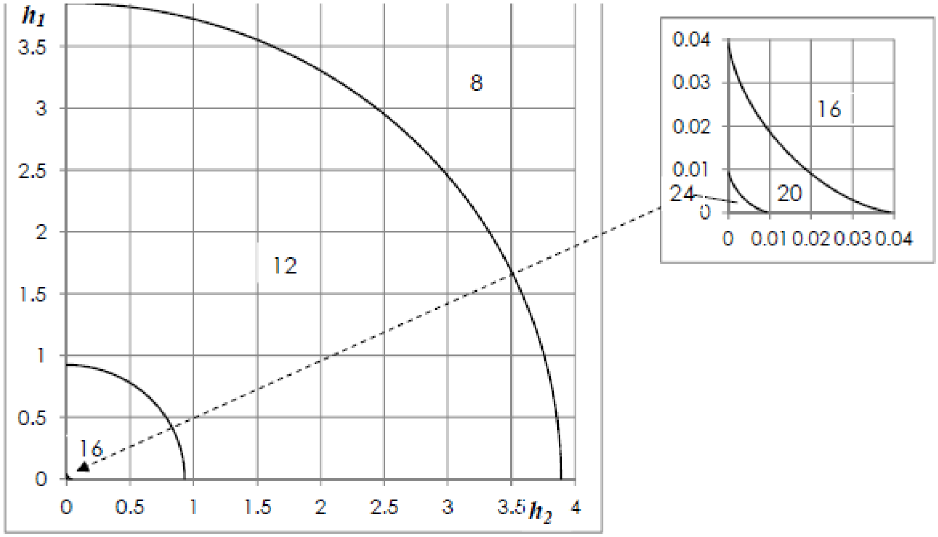

Fig. 1. ν=0.01, $h_3 = 0.01$

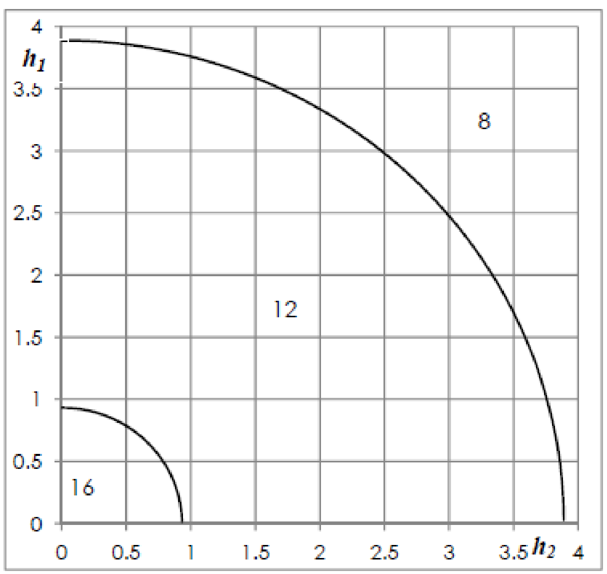

Fig. 2. ν=0, $h_3 = 0.01$

Numerical results v=0.01

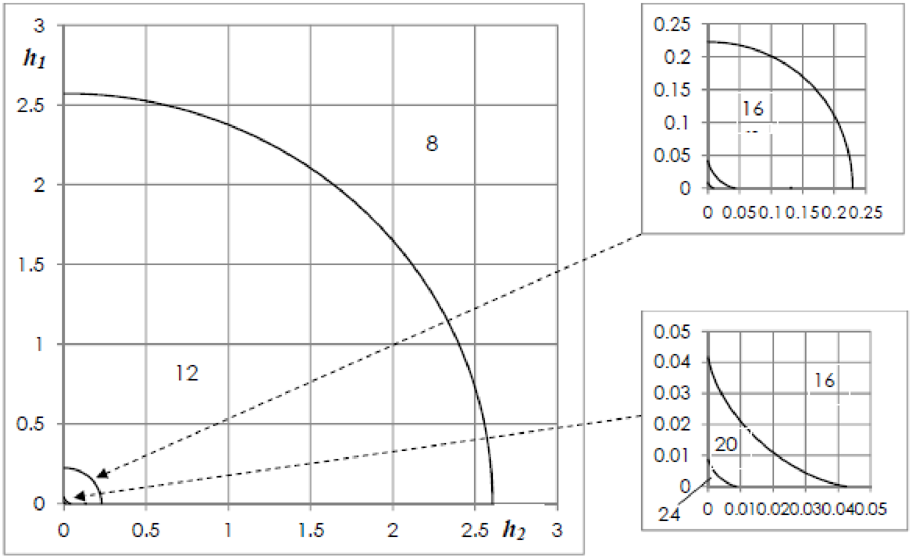

Fig. 3. v=0.01, $h_3 = 0.495$

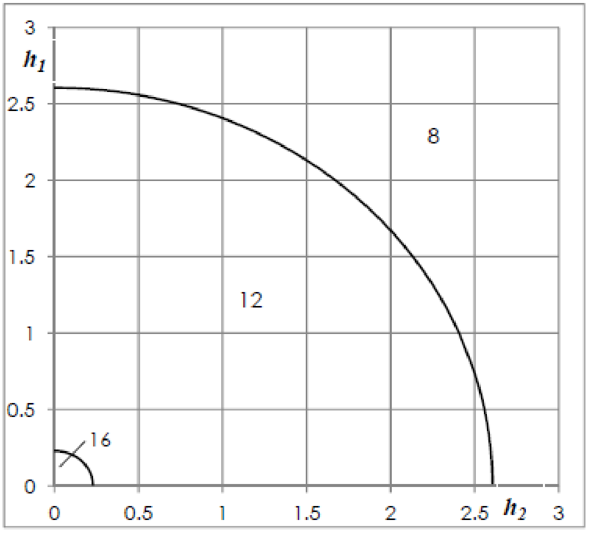

Fig. 4. v=0, $h_3 = 0.495$

Numerical results v=0.01

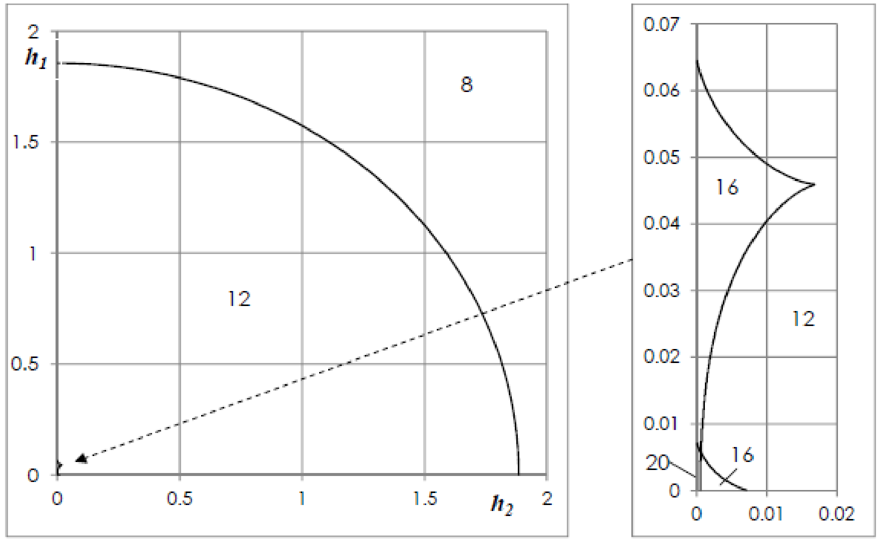

Fig. 5. v=0.01, $h_3 = 0.99$

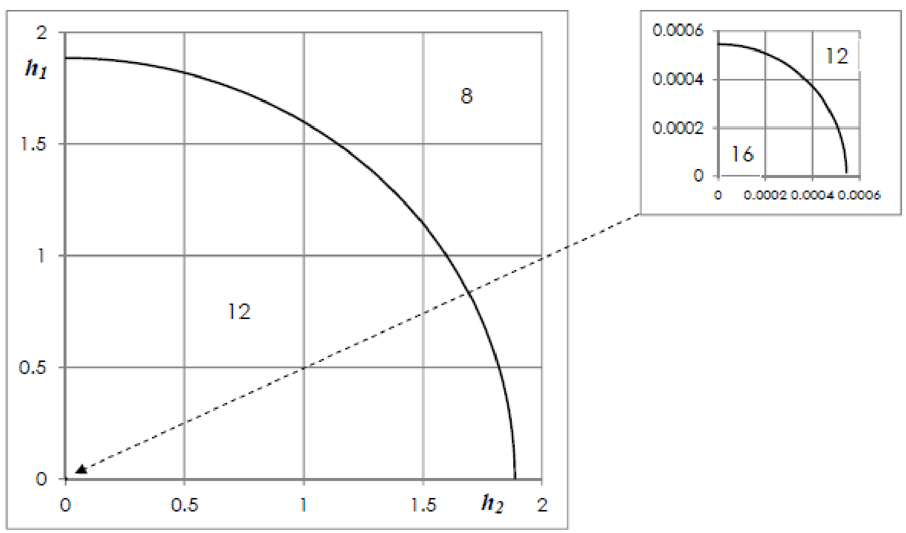

Fig. 6. v=0, $h_3 = 0.99$

Numerical results v=0.1

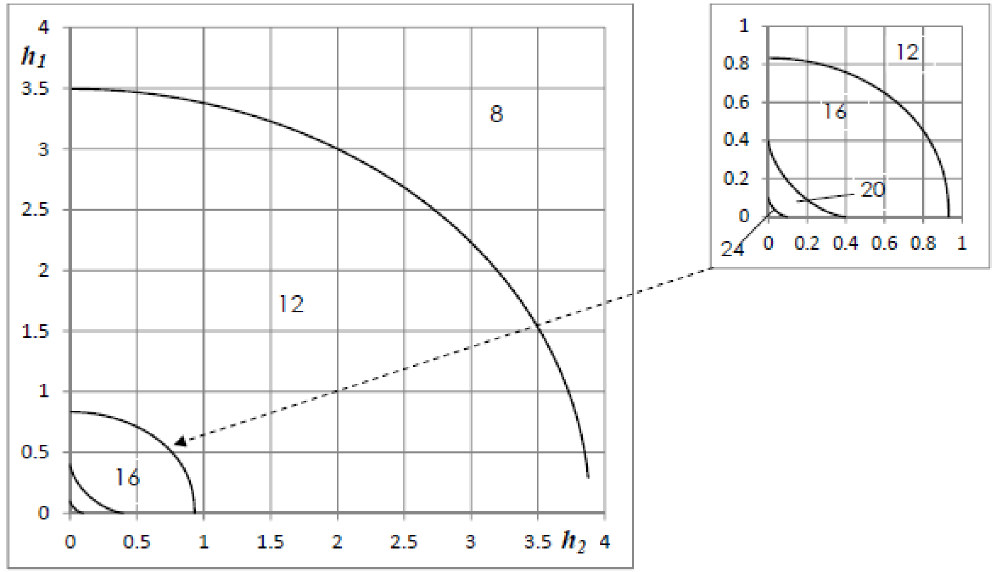

Fig. 7. v=0.1, $h_3 = 0.01$

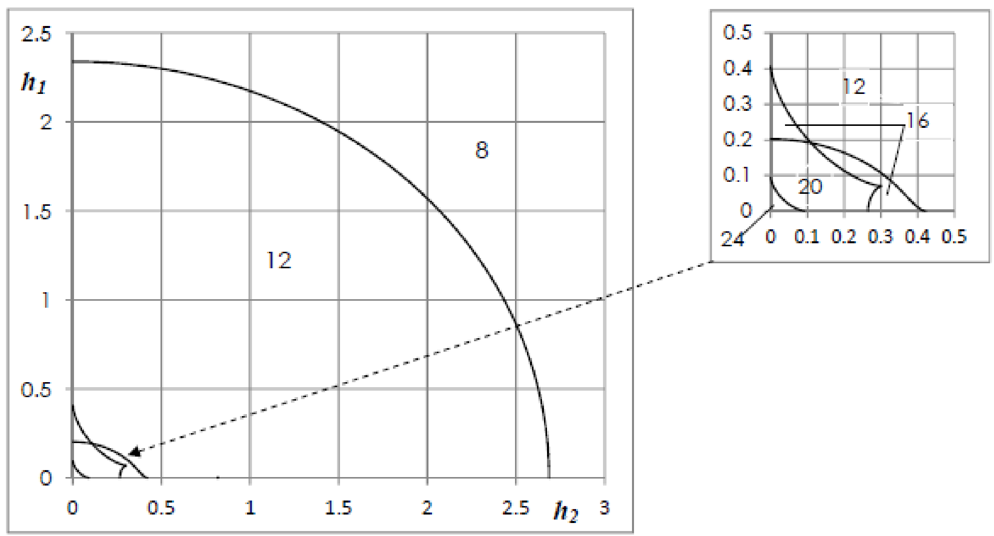

Fig. 8. v=0.1, $h_3 = 0.495$

Numerical results v=0.1

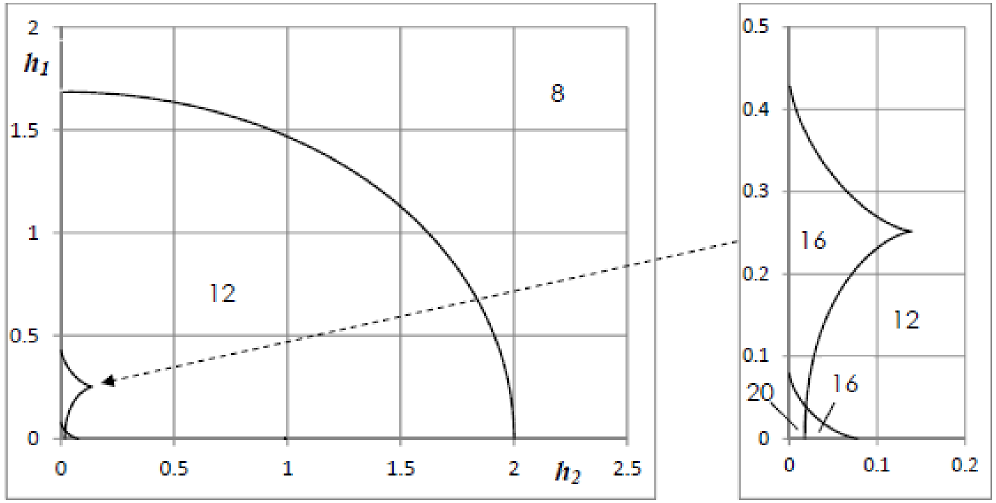

Fig. 9. v=0.1, $h_3 = 0.9$

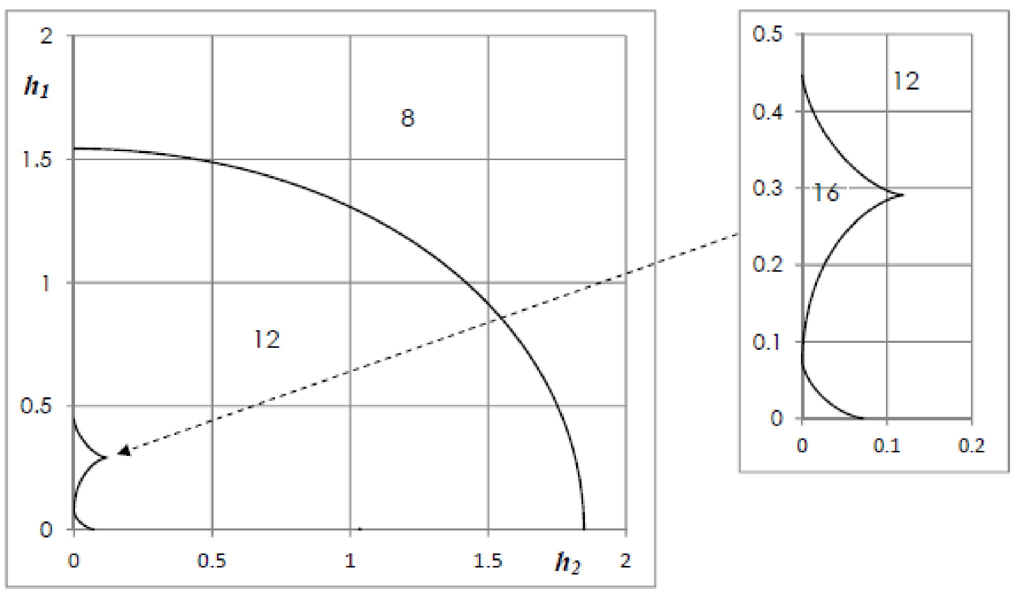

Fig. 10. v=0.1, $h_3 = 1.021$

Numerical results ν=0.1

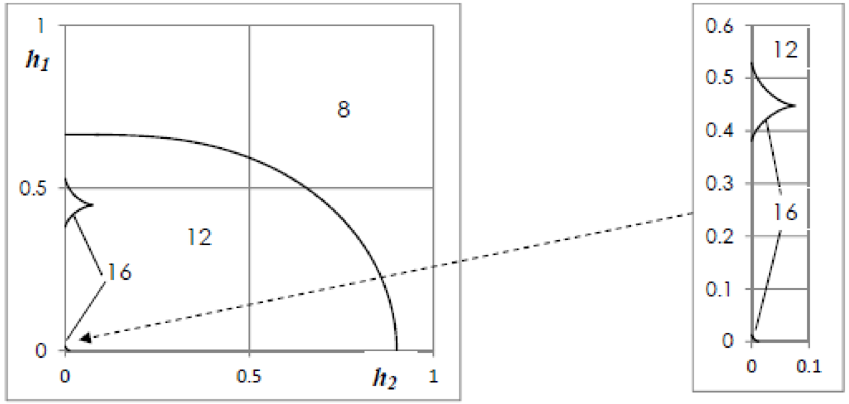

Fig. 11. ν=0.1, $h_3 = 2.0$

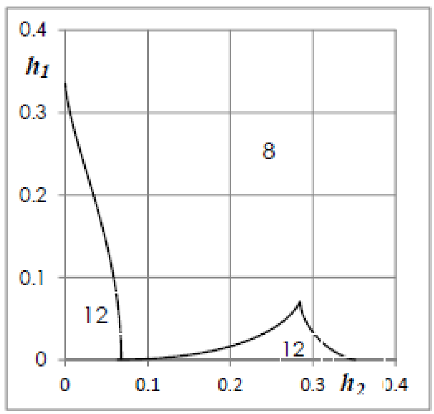

Fig. 12. ν=0.1, $h_3 = 3.61$

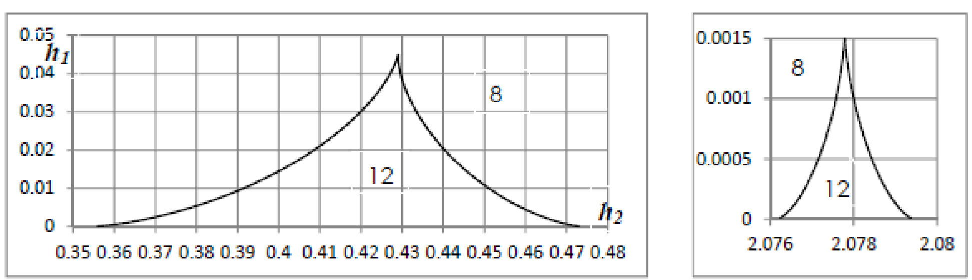

Fig. 13. ν=0.1, $h_3 = 4.0$

Numerical results v=0.2

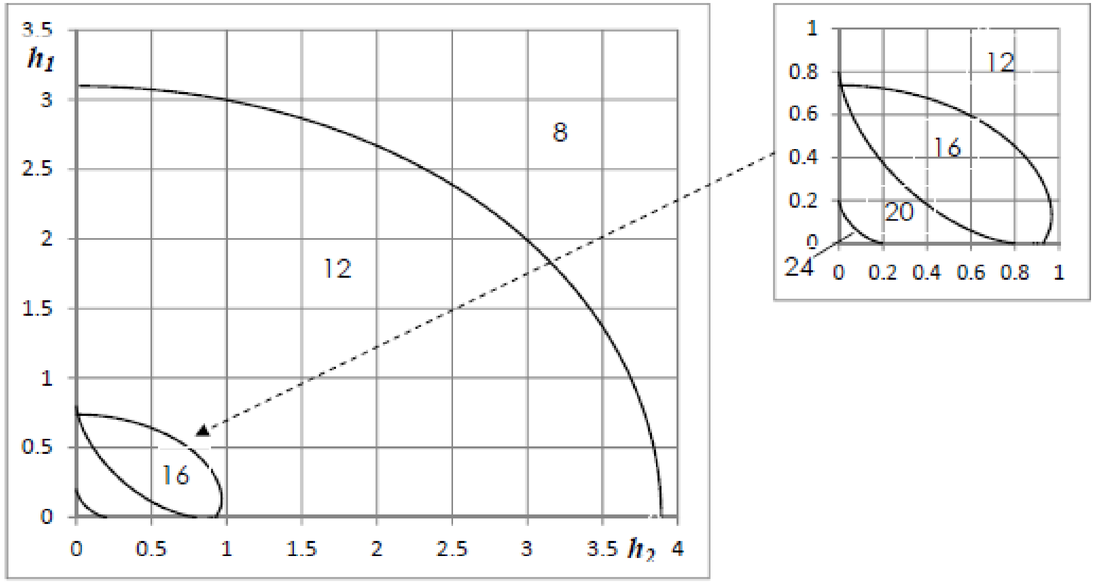

Fig. 14. v=0.2, $h_3 = 0.01$

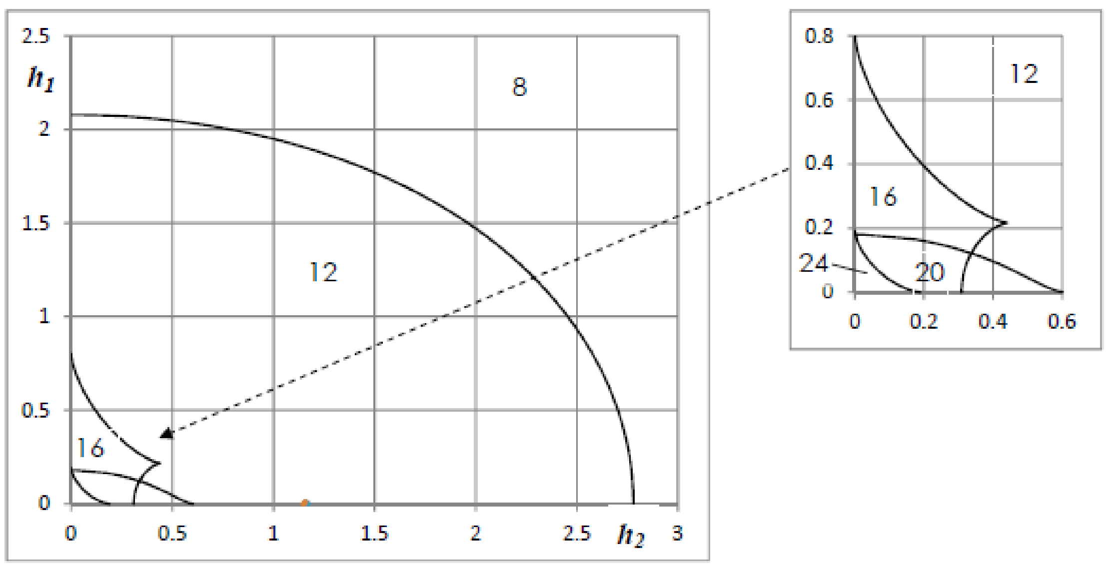

Fig. 15. v=0.2, $h_3 = 0.4$

Numerical results ν=0.2

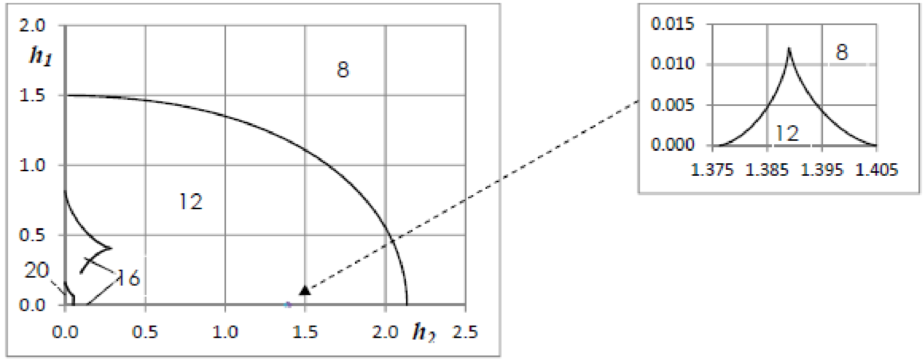

Fig. 16. ν=0.2, h_3 = 0.8

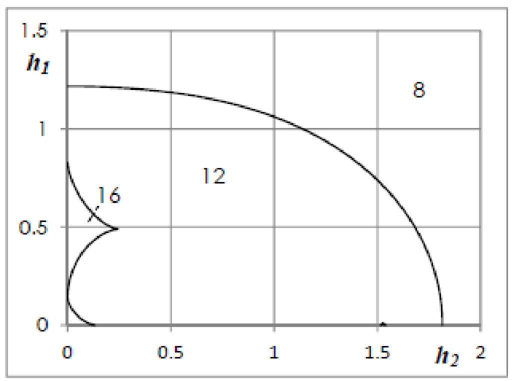

Fig. 17. ν=0.2, h_3 = 1.048

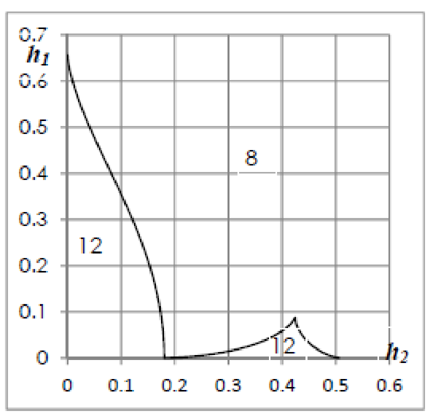

Fig. 18. ν=0.2, h_3 = 3.264

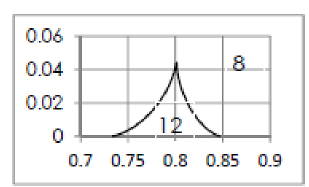

Fig. 19. ν=0.2, h_3 = 4

Numerical results *v*=0.5

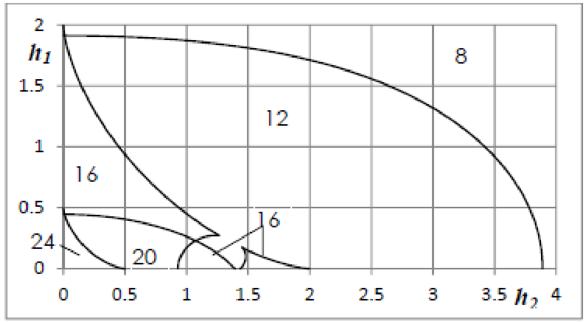

Fig. 28. v=0.5, h_3 = 0.01

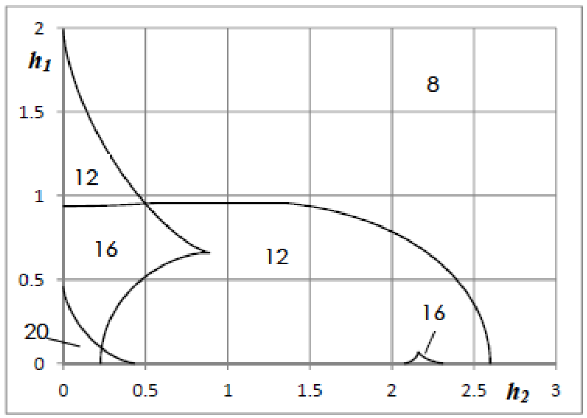

Fig. 29. v=0.5, h_3 = 0.5

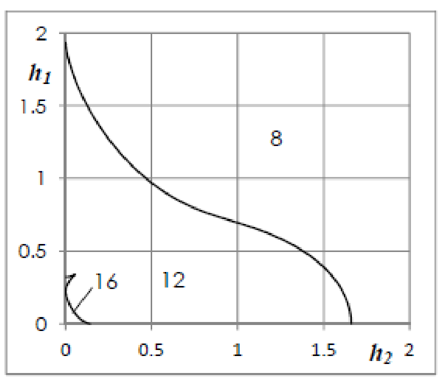

Fig. 30. v=0.5, h_3 =1.182

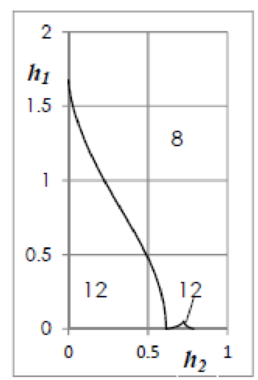

Fig.31. v=0.5, h_3=2.412

Numerical results ν=0.8

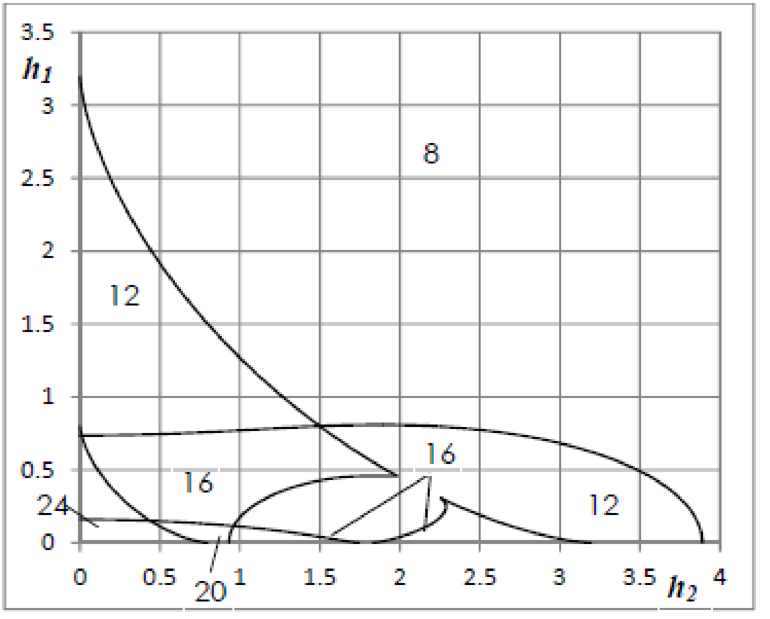

Fig. 40. ν=0.8, h_3 = 0.01

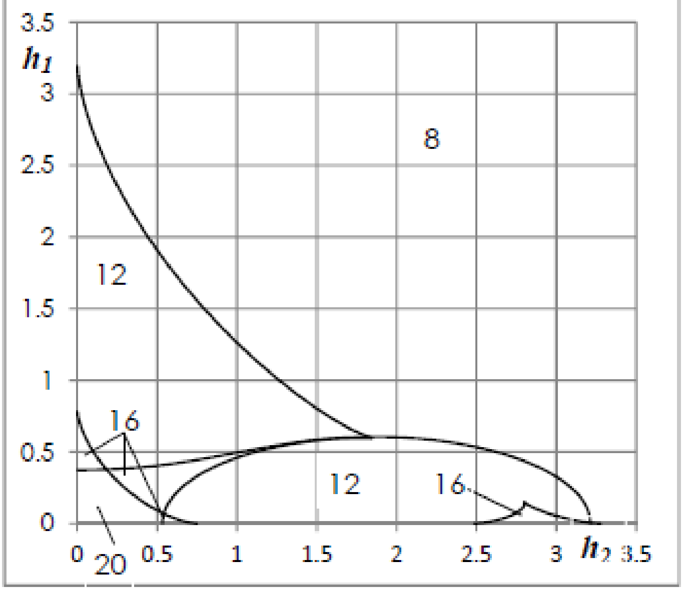

Fig. 41. ν=0.8, h_3 = 0.2

Numerical results ν=0.8

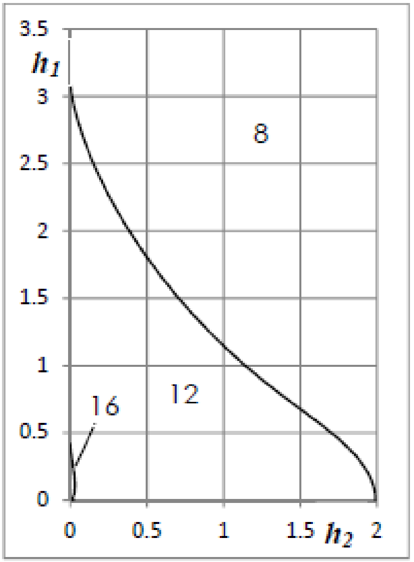

Fig. 42. ν=0.8, h_3 = 0.909

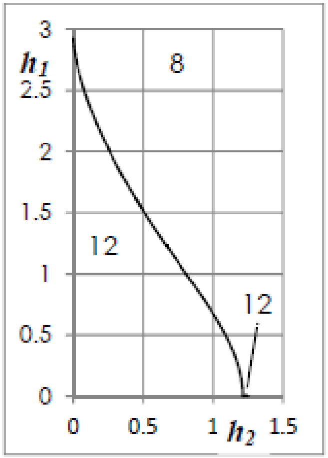

Fig. 43. ν=0.8, h_3=1.629

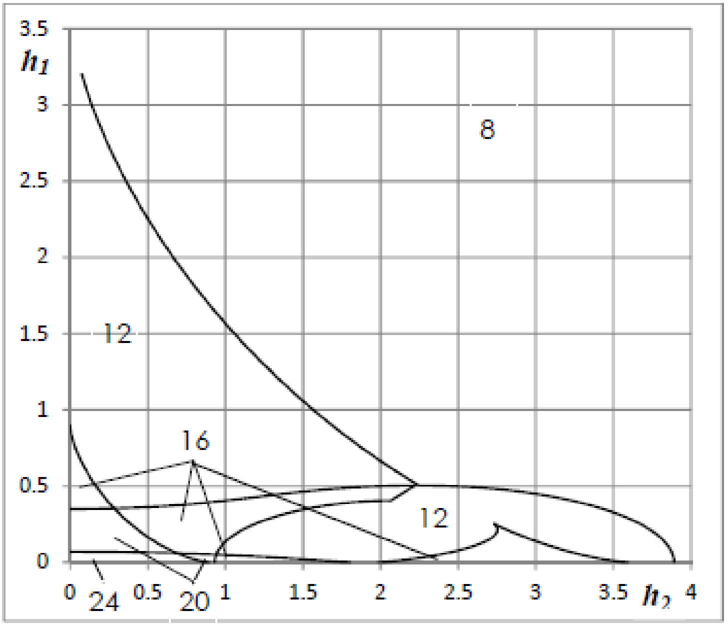

Fig. 44. ν=0.9, h_3 = 0.01

Numerical results v=0.9

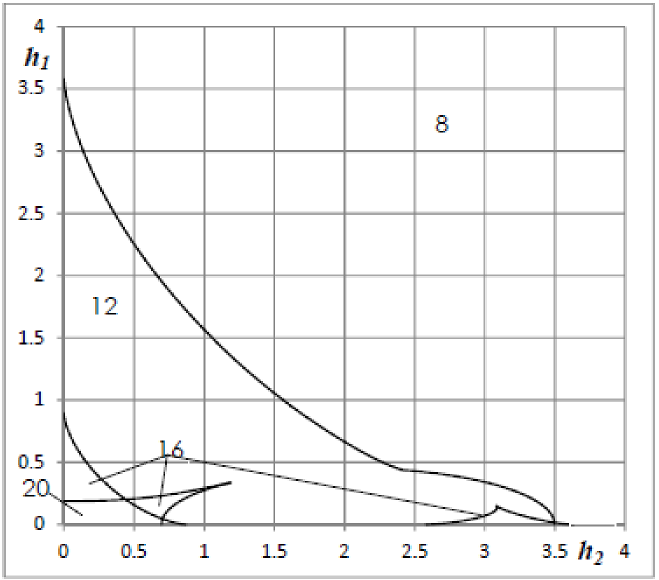

Fig. 45. v=0.9, h_3 = 0.1

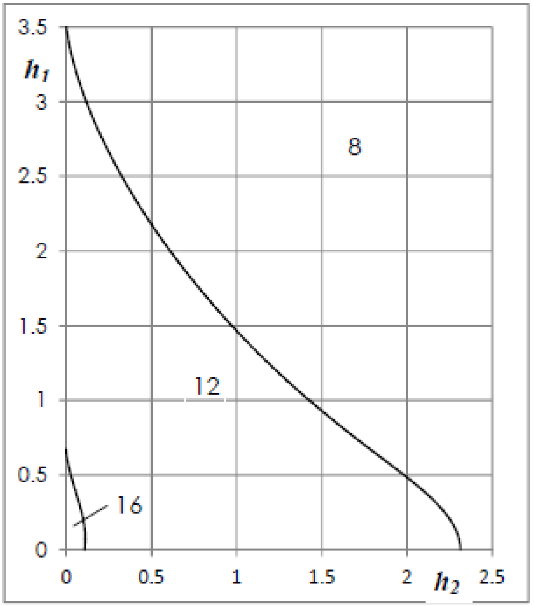

Fig. 46. v=0.9, h_3 = 0.676

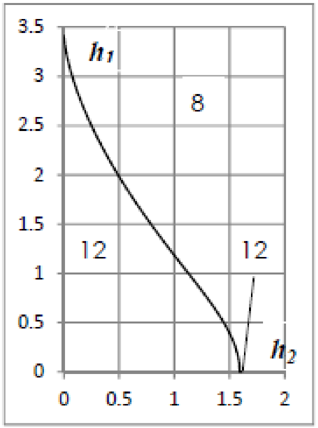

Fig. 47. v=0.9, h_3=1.245

Numerical results v=0.9

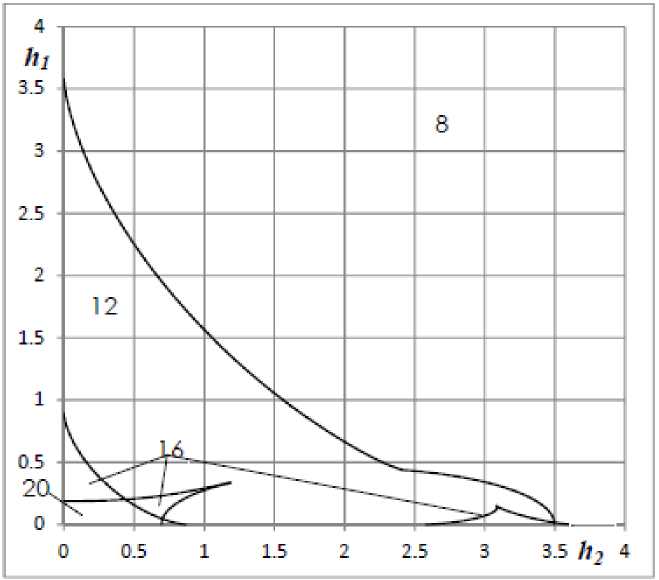

Fig. 45. v=0.9, h_3 = 0.1

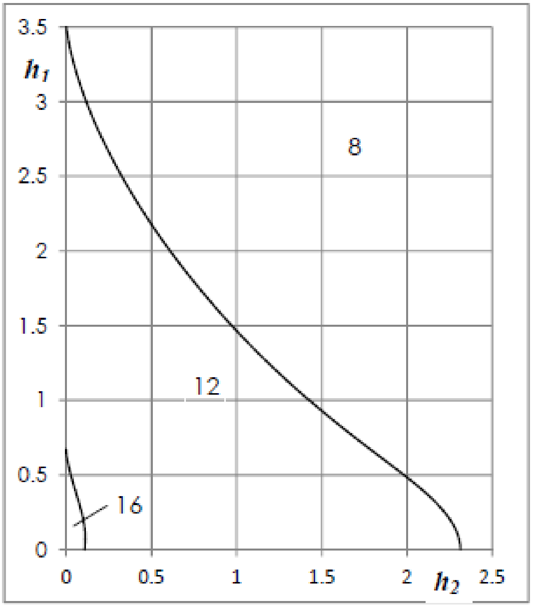

Fig. 46. v=0.9, h_3 = 0.676

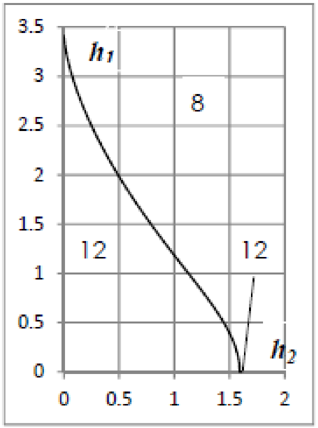

Fig. 47. v=0.9, h_3=1.245

Numerical results ν=0.99

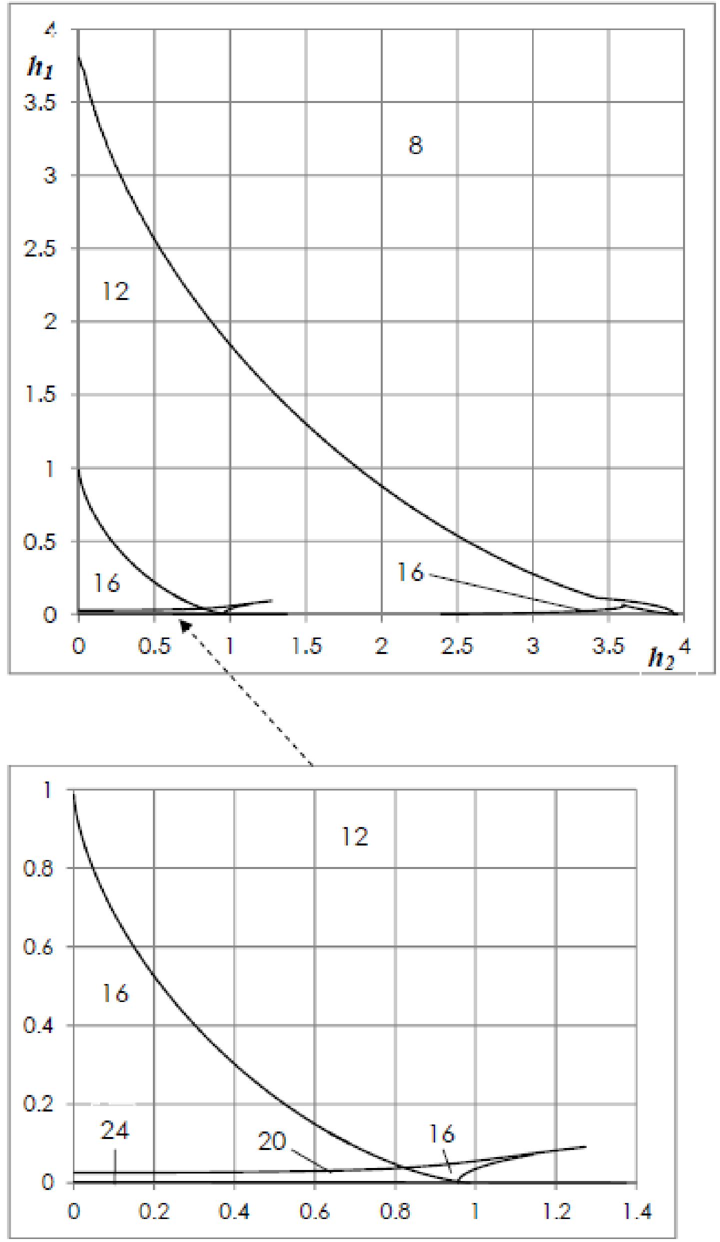

Fig.48. ν=0.99, h_3= 0.005

Numerical results v=0.99

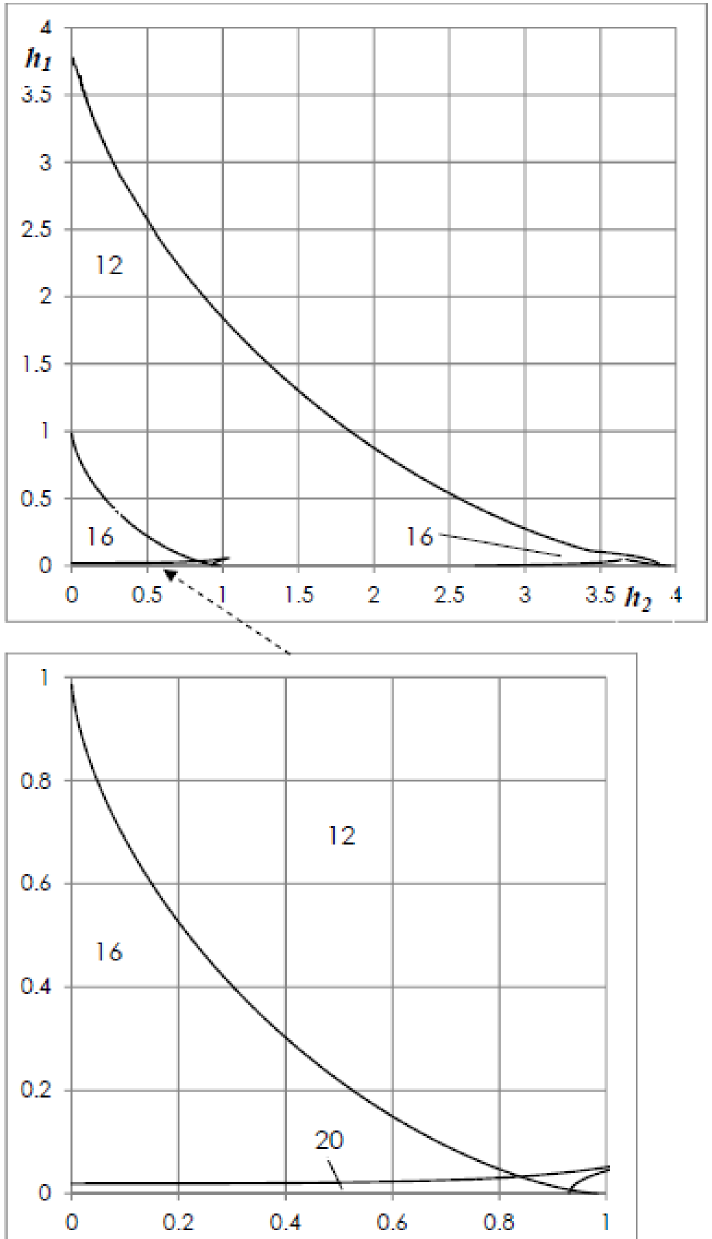

Fig. 49. v=0.99, $h_3 = 0.01$

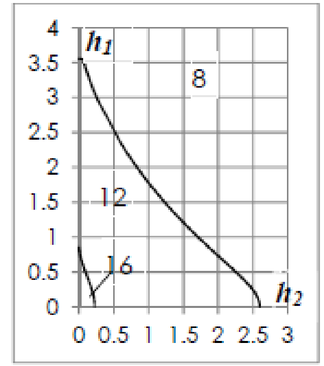

Fig. 50. v=0.99, $h_3 = 0.5$

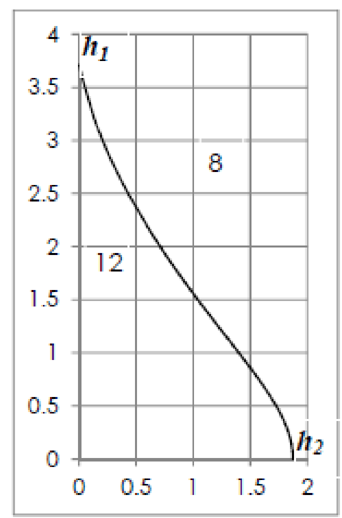

Fig. 51. v=0.99, h_3=1.0

Numerical results *v*=1.0

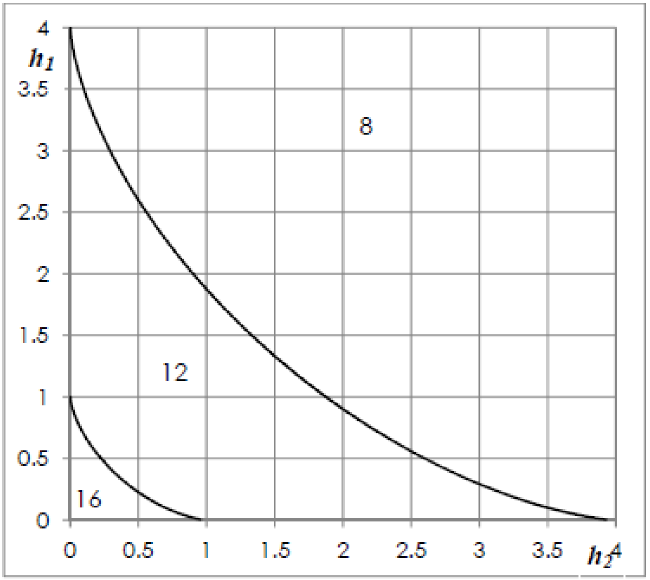

Fig. 52. ν=1.0, $h_3 = 0.005$

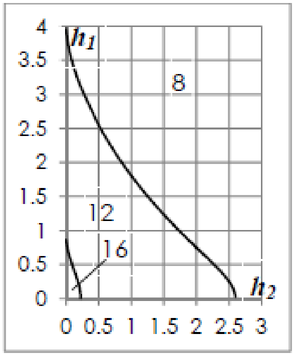

Fig. 53. ν=1.0, h_3 =0.5

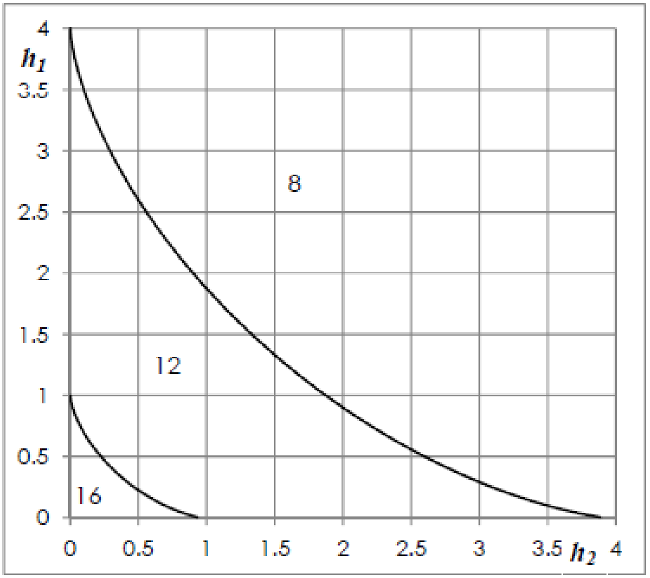

Fig. 54. ν=1.0, $h_3 = 0.01$

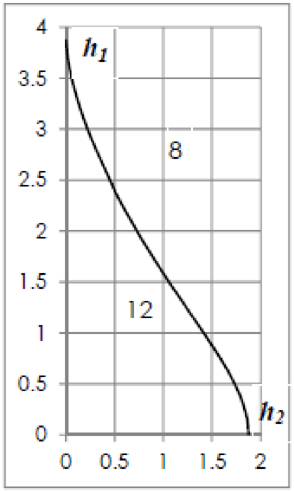

Fig. 55. ν=1.0, $h_3 = 1.0$