Generalized Bruhat decomposition in commutative domains

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Abstract

Deterministic recursive algorithm for the computation of
generalized Bruhat decomposition in commutative domain
are presented.

This method has
the same complexity as matrix multiplication.

$$A = VwU$$

is called the Bruhat decomposition of the matrix A, if V and U are nonsingular upper triangular matrices and w is a matrix of permutation.

The generalized Bruhat decomposition was introduced and developed by D.Grigoriev.

At CASC-2010 there was presented a pivot-free matrix decomposition method in a common case of singular matrices over a field of arbitrary characteristic with the complexity of matrix multiplication.

Now we present the decomposition in domain.

Definition (Bruhat decomposition in domain)

Decomposition of matrix A:

$$A = VwU$$

we call the Bruhat decomposition in the commutative domain R if
a) V and U are upper triangular matrices over R and
b) w is a matrix of permutation, which is multiplied by some diagonal matrix in the field of fractions F over domain R.

Moreover each nonzero element of w has the form $(a^i a^{i-1})^{-1}$, where a^i is some minor of order i of matrix A ($i \leq \mathrm{rank}(A)$).

Example

$$\begin{bmatrix} 1 & -4 & 0 & 1 \\ 4 & 5 & 5 & 3 \\ 1 & 2 & 2 & 2 \\ 3 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -24 & 0 & 12 & 1 \\ 0 & 60 & 15 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & \frac{-1}{144} & 0 \\ 0 & 0 & 0 & \frac{-1}{1440} \\ 0 & \frac{1}{18} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 6 & 6 & 5 \\ 0 & 0 & -24 & -16 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

We construct the decomposition in the form $A = LDU$, where
a) L and U are lower and upper triangular matrices,
b) D is a matrix of permutation, which is multiplied by some diagonal matrix in the field of fractions F and has the same rank as the matrix A.

Then the Bruhat decomposition VwU in the domain R may be easily obtained using the matrices L, D and U.

Definition

$$A = (a_{i,j}) \in R^{n\times n} - \text{matrix of order } n$$

$$\alpha^k_{i,j} - k \times k \text{ minor of matrix } A$$

which disposed in the rows $1, 2, \ldots, k-1, i$
and columns $1, 2, \ldots, k-1, j, \quad 1 \leq i, j, k \leq n$.

$$\alpha^k = \alpha^k_{k,k}, \quad \alpha^0 = 1, \quad \delta_{ij} - \text{ Kronecker delta.}$$

$$\mathcal{A}^k_s = (\alpha^{k+1}_{i,j})$$

- matrix with size $(s-k) \times (s-k)$ of minors
$i, j = k+1, \ldots, s-1, s, \quad 0 \leq k < s \leq n$,

$$A = \mathcal{A}^0_n = (\alpha^1_{i,j})$$

We shall use

Theorem (Sylvester determinant identity)

Let k *and* s *be an integers in the interval* $0 \leq k < s \leq n$. *Then it is true that*

$$\det(\mathcal{A}_s^k) = \alpha^s (\alpha^k)^{s-k-1}. \tag{1}$$

Theorem (LDU decomposition of the minors matrix)

Let $A = (a_{i,j}) \in R^{n\times n}$, $\mathrm{rank}(A) = r$,
$\alpha^i \neq 0$ *for* $i = k, k+1, \ldots, r$, $r \leq s \leq n$, *then*

$$\mathcal{A}_s^k = L_s^k D_s^k U_s^k = (a_{i,j}^j)(\delta_{ij}\alpha^k(\alpha^{i-1}\alpha^i)^{-1})(a_{i,j}^i). \qquad (2)$$

$$L_s^k = (a_{i,j}^j)$$

is a low triangular $(s-k)\times(r-k)$ *matrix,* $k < i \leq s$, $k < j \leq r$,

$$U_s^k = (a_{i,j}^i)$$

is upper triangular $(r-k)\times(s-k)$ *matrix,* $k < i \leq r$, $k < j \leq s$,

$$D_s^k = (\delta_{ij}\alpha^k(\alpha^{i-1}\alpha^i)^{-1})$$

is a diagonal $(r-k)\times(r-k)$ *matrix,* $k < i \leq r$, $k < j \leq r$.

Proof
Equation (2) for $k+1=r$:

$$(a_{i,j}^{k+1}) = (a_{i,k+1}^{k+1})(\delta_{k+1,k+1}a^k(a^k a^{k+1})^{-1})(a_{k+1,j}^{k+1}) \qquad (3)$$

follows from Sylvester determinant identity:

$$a_{i,j}^{k+1}a^{k+1} - a_{i,k+1}^{k+1}a_{k+1,j}^{k+1} = a_{i,j}^{k+2}a^k = a_{i,j}^{r+1}a^{r-1} = 0, \qquad (4)$$

Let for all h, $k < h < r$, the statement (2) be correct for matrices $\mathcal{A}_s^h = (a_{i,j}^{h+1})$. Let

$$a_{i,j}^{k+2} = \sum_{t=k+2}^{min(i,j,r)} a_{i,t}^t \alpha^{k+1}(\alpha^{t-1}\alpha^t)^{-1}a_{t,j}^t.$$

We have to prove the corresponding expression for the elements of the matrix $\mathcal{A}_s^k = (a_{i,j}^{k+1})$. Due to the Sylvester determinant identity (3) we obtain

$$a_{i,j}^{k+1} = a_{i,k+1}^{k+1}(\alpha^{k+1})^{-1}a_{k+1,j}^{k+1} + \alpha^k(\alpha^{k+1})^{-1}a_{i,j}^{k+2} =$$

$$a_{i,k+1}^{k+1}\alpha^k(\alpha^k\alpha^{k+1})^{-1}a_{k+1,j}^{k+1}+$$

$$\alpha^k(\alpha^{k+1})^{-1}\sum_{t=k+2}^{min(i,j,r)} a_{i,t}^t\alpha^{k+1}(\alpha^{t-1}\alpha^t)^{-1}a_{t,j}^t =$$

$$\sum_{t=k+1}^{min(i,j,r)} a_{i,t}^t\alpha^k(\alpha^{t-1}\alpha^t)^{-1}a_{t,j}^t.$$

Consequence (LDU decomposition of matrix A)

Let $A = (a_{i,j}) \in R^{n\times n}$*, be the matrix of rank* r*,* $r \leq n$*,* $\alpha^i \neq 0$ *for* $i = 1, 2, \ldots, r$*, then matrix* A *is equal to the following product of three matrices:*

$$A = L_n^0 D_n^0 U_n^0 = (a_{i,j}^j)(\delta_{ij}(\alpha^{i-1}\alpha^i)^{-1})(a_{i,j}^i). \qquad (4)$$

I_n - the identity matrix
P_n - the matrix with second unit diagonal.

Consequence (Bruhat decomposition of matrix A)

Let matrix $A = (a_{i,j})$ have the rank r, $r \leq n$, and $B = P_n A$. Let $B = LDU$ be the LDU-decomposition of matrix B. Then $V = P_n L P_r$ and U are upper triangular matrices of size $n \times r$ and $r \times n$ correspondingly and

$$A = V(P_r D)U \tag{5}$$

is the Bruhat decomposition of matrix A.

Notations

For any matrix A (or A_q^p) we denote by $A_{j_1,j_2}^{i_1,i_2}$ (or $A_{q;j_1,j_2}^{p;i_1,i_2}$) the block which stands at the intersection of rows $i_1+1,\ldots,i_2$ and columns $j_1+1,\ldots,j_2$ of the matrix. We denote by $A_{i_2}^{i_1}$ the diagonal block $A_{i_1,i_2}^{i_1,i_2}$.

LDU ALGORITHM

Input: $(\mathcal{A}_n^k, \alpha^k)$, $0 \le k < n$.
Output: $\{L_n^k, \{\alpha^{k+1}, \alpha^{k+2}, \ldots, \alpha^n\}, U_n^k, M_n^k, W_n^k\}$,
where $D_n^k = \alpha^k \mathrm{diag}\{\alpha^k \alpha^{k+1}, \ldots, \alpha^{n-1}\alpha^n\}^{-1}$,
$M_n^k = \alpha^k (L_n^k D_n^k)^{-1}, \; W_n^k = \alpha^k (D_n^k U_n^k)^{-1}$.

1. If $k = n-1$, $\mathcal{A}_n^{n-1} = (a^n)$ is a matrix of the first order, then

$$\{a^n, \{a^n\}, a^n, a^{n-1}, a^{n-1}\}, \quad D_n^{n-1} = (\alpha^n)^{-1}.$$

2. If $k = n-2$, $\mathcal{A}_n^{n-2} = \begin{pmatrix} \alpha^{n-1} & \beta \\ \gamma & \delta \end{pmatrix}$ is a matrix of second order, then

$$\begin{bmatrix} \alpha^{n-1} & 0 \\ \gamma & \alpha^n \end{bmatrix} \{\alpha^{n-1}, \alpha^n\} \begin{bmatrix} \alpha^{n-1} & \beta \\ 0 & \alpha^n \end{bmatrix} \begin{bmatrix} \alpha^{n-2} & 0 \\ -\gamma & \alpha^{n-1} \end{bmatrix} \begin{bmatrix} \alpha^{n-2} & -\beta \\ 0 & \alpha^{n-1} \end{bmatrix}$$

where $\alpha^n = (\alpha^{n-2})^{-1} \begin{vmatrix} \alpha^{n-1} & \beta \\ \gamma & \delta \end{vmatrix}$,
$D_n^{n-2} = \alpha^{n-2} \mathrm{diag}\{\alpha^{n-2}\alpha^{n-1}, \alpha^{n-1}\alpha^n\}^{-1}$.

3. If the order of the matrix $\mathcal{A}_n^k$ more than two $(0 \leq k < n-2)$, then we choose s $(k < s < n)$ and divide the matrix into blocks

$$\mathcal{A}_n^k = \begin{pmatrix} \mathcal{A}_s^k & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}. \tag{6}$$

3.1. Recursive step

$$\{L_s^k, \{\alpha^{k+1}, \alpha^{k+2}, \ldots, \alpha^s\}, U_s^k, M_s^k,\ W_s^k\} = \mathbf{LDU}(\mathcal{A}_s^k, \alpha^k)$$

3.2. We compute

$$\widetilde{U} = (\alpha^k)^{-1} M_s^k \mathbf{B}, \quad \widetilde{L} = (\alpha^k)^{-1} \mathbf{C} W_s^k, \tag{7}$$

$$\mathcal{A}_n^s = (\alpha^k)^{-1} \alpha^s (\mathbf{D} - \widetilde{L} D_s^k \widetilde{U}). \tag{8}$$

3.3. Recursive step

$$\{L_n^s, \{\alpha^{s+1}, \alpha^{s+2}, \ldots, \alpha^n\}, U_n^s, M_n^s,\ W_n^s\} = \mathbf{LDU}(\mathcal{A}_n^s, \alpha^s)$$

3.4 *Result:*

$$\{L_n^k, \{\alpha^{k+1}, \alpha^{k+2}, \ldots, \alpha^n\}, U_n^k, M_n^k, W_n^k\},$$

where

$$L_n^k = \begin{pmatrix} L_s^k & 0 \\ \widetilde{L} & L_n^s \end{pmatrix}, \; U_n^k = \begin{pmatrix} U_s^k & \widetilde{U} \\ 0 & U_n^s \end{pmatrix}, \tag{9}$$

$$M_n^k = \begin{pmatrix} M_s^k & 0 \\ -M_n^s \widetilde{L} D_s^k M_s^k / \alpha^k & M_n^s \end{pmatrix}, \tag{10}$$

$$W_n^k = \begin{pmatrix} W_s^k & -W_s^k D_s^k \widetilde{U} W_n^s / \alpha^k \\ 0 & W_n^s \end{pmatrix}. \tag{11}$$

Proof of the correctness of the LDU algorithm

Definition ($\delta_{i,j}^k$ minors and $\mathcal{G}^k$ matrices)

Let $A \in R^{n\times n}$ be a matrix. The determinant of the matrix, obtained from the upper left block $A_{0,k}^{0,k}$ of matrix A by the replacement in matrix A of the column i by the column j is denoted by $\delta_{i,j}^k$. The matrix of such minors is denoted by

$$\mathcal{G}_s^k = (\delta_{i,j}^{k+1}) \quad (12)$$

Theorem (Base minor's identity)

Let $A \in R^{n\times n}$ be a matrix and i, j, s, k, be integers in the intervals: $0 \le k < s \le n,\ 0 < i, j \le n$. Then the following identity is true

$$\alpha^s \alpha_{ij}^{k+1} - \alpha^k a_{ij}^{s+1} = \sum_{p=k+1}^{s} \alpha_{ip}^{k+1} \delta_{pj}^{s}. \quad (13)$$

The minors a_{ij}^{s+1} in the left side of this identity equal zero if $i < s+1$. Therefor this theorem gives the following

Consequence

Let $A \in R^{n\times n}$ be a matrix and s, k, be integers in the intervals: $0 \le k < s \le n$. Then the following identities are true

$$\alpha^s U_{n;s+1,n}^{k;k+1,s} = U_s^k \mathcal{G}_{n;s+1,n}^{k;k+1,s}. \tag{14}$$

$$\alpha^s \mathcal{A}_{n;s+1,n}^{k;k+1,s} = \mathcal{A}_s^k \mathcal{G}_{n;s+1,n}^{k;k+1,s}. \tag{15}$$

The block $\mathcal{A}^{k;k+1,s}_{n;s+1,n}$ of the matrix $\mathcal{A}^k_n$ was denoted by $\mathbf{B}$. Due to Sylvester identity we can write the equation for the adjoint matrix

$$(A^k_s)^* = (A^k_s)^{-1}(\alpha^s)(\alpha^k)^{s-k-1} \tag{16}$$

Let us multiply both sides of equation (15) by adjoint matrix $(A^k_s)^*$ and use the equation (16). Then we get

Consequence

$$(A^k_s)^*\mathbf{B} = (A^k_s)^*\mathcal{A}^{k;k+1,s}_{n;s+1,n} = (\alpha^k)^{s-k-1}\mathcal{G}^{k;k+1,s}_{n;s+1,n}. \tag{17}$$

As well as $L_s^k D_s^k U_s^k = A_s^k$,

$$M_s^k = \alpha^k (L_s^k D_s^k)^{-1} = \alpha^k U_s^k (A_s^k)^{-1} \text{ and } W_s^k = \alpha^k (D_s^k U_s^k)^{-1}. \tag{18}$$

Therefor

$$\widetilde{U} = (\alpha^k)^{-1} M_s^k \mathbf{B} = (\alpha^k)^{-1} U_s^k (A_s^k)^{-1} \mathbf{B} = (\alpha^s)^{-1} (\alpha^k)^{-s+k} U_s^k (A_s^k)^* \mathbf{B}. \tag{19}$$

Equations (19), (17), (14) give the

Consequence

$$\widetilde{U} = U_{n;s+1,n}^{k;k+1,s} \tag{20}$$

In the same way we can prove

Consequence

$$\widetilde{L} = L_{n;k+1,s}^{k;s+1,n}. \tag{21}$$

Now we have to prove the identity (8). Due to the equations (14)-(19) we obtain

$$\widetilde{L}D_s^k\widetilde{U} = (\alpha^k)^{-1}\mathbf{C}W_s^kD_s^k(\alpha^k)^{-1}M_s^k\mathbf{B} =$$

$$(\alpha^k)^{-2}\mathbf{C}(A_s^k)^{-1}\mathbf{B} = (\alpha^k)^{-s+k-1}(\alpha^s)^{-1}\mathbf{C}(A_s^k)^*\mathbf{B} \qquad (22)$$

The identity

$$\mathcal{A}_n^s = (\alpha^k)^{-1}(\alpha^s\mathbf{D} - (\alpha^k)^{-s+k+1}\mathbf{C}(A_s^k)^*\mathbf{B}) \qquad (23)$$

was proved in [4] and [5]. Due to (20) and (21) we obtain the identity (8).

To prove the formula (10) and (11) it is sufficient to verify the identities $M_n^k = \alpha^k(L_n^kD_n^k)^{-1}$ and $W_n^k = \alpha^k(D_n^kU_n^k)^{-1}$ using (9),(10), (11) and definition

$D_n^k = \alpha^k\mathrm{diag}\{\alpha^k\alpha^{k+1},\ldots,\alpha^{n-1}\alpha^n\}^{-1}$.

Theorem

The algorithm has the same complexity as matrix multiplication.

Proof.

Number of matrix multiplications is 7.
Number of recursive calls is 2.
Decomposition of the 2x2 matrix takes 7 multiplicative operations.
So the recurrent equality for complexity:

$$t(n) = 2t(n/2) + 7M(n/2), \quad t(2) = 7.$$

Let $M(n) = \gamma n^{\beta} + o(n^{\beta})$ be the complexity of $n \times n$ matrix multiplication. After summation from $n = 2^k$ to 2^1 we obtain

$$7\gamma(2^0 2^{\beta\cdot(k-1)} + \ldots + 2^{k-2}2^{\beta\cdot 1}) + 2^{k-2}7 = 7\gamma\frac{n^{\beta} - n2^{\beta-1}}{2^{\beta}-2} + \frac{7}{4}n.$$

$$\sim \frac{7\gamma n^{\beta}}{2^{\beta}-2}$$

□

The exact triangular decomposition

Definition

A decomposition of the matrix A of rank r over a domain R

$$A = PLDUQ \tag{24}$$

is called **exact triangular decomposition** if
P and Q are permutation matrces,
L and PLP^T are nonsingular lower triangular matrices over R,
U and Q^TUQ are nonsingular upper triangular matrices over R,
$D = \mathrm{diag}(d_1^{-1}, d_2^{-1}, .., d_r^{-1}, 0, .., 0)$ is a diagonal matrix of rank r,
$d_i \in R\backslash\{0\}$, $i = 1, ..r$.

Designation: $\mathbf{ETD}(A) = (P, L, D, U, Q)$.

Theorem (Main theorem)

Any matrix over a commutative domain has an exact triangular decomposition.

Comment

1) If D matrix is combined with L or U, we get the expression

$$A = PLUQ$$

– the LU-decomposition with permutations of rows and columns.
2) If the factors are grouped in the following way:

$$A = (PLP^T)(PDQ)(Q^TUQ),$$

then we obtain $\mathbf{LDU}$-decomposition.
3) If S is a permutation matrix in which the unit elements are placed on the secondary diagonal, then

$$(SA) = (S\mathbf{L}S)(S^T\mathbf{D})\mathbf{U} = Vw\mathbf{U}$$

– the Bruhat decomposition of the matrix (SA).

Proof

If matrix A is a zero matrix: $\mathbf{ETD}(A) = (I, I, 0, I, I)$.
If A is the first order nonzero matrix: $\mathbf{ETD}(a) = (I, a, a^{-1}, a, I)$.
Let us consider a non-zero matrix of order two. We denote

$$\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}, \varepsilon = \begin{cases} \Delta, \ \Delta \neq 0 \\ 1, \ \Delta = 0. \end{cases} \ \Delta^{-1} = \begin{cases} \frac{1}{\Delta}, \ \Delta \neq 0 \\ 0, \ \Delta = 0. \end{cases}$$

$$\alpha \neq 0 : \mathcal{A} = \begin{pmatrix} \alpha & 0 \\ \gamma & \varepsilon \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \Delta^{-1}\alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \varepsilon \end{pmatrix}.$$

$\alpha = 0, \beta \neq 0 :$

$$\mathcal{A} = \begin{pmatrix} \beta & 0 \\ \delta & \varepsilon \end{pmatrix} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & -\Delta^{-1}\beta^{-1} \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\alpha = 0, \ \gamma \neq 0 :$

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & -\Delta^{-1}\gamma^{-1} \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ 0 & \varepsilon \end{pmatrix}.$$

$\alpha = \beta = \gamma = 0, \delta \neq 0 :$

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are only two different cases for matrices of size 1×2:

If $\alpha \neq 0$, then $\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \alpha \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$.

If $\alpha = 0,\ \beta \neq 0$, then $\begin{pmatrix} 0 & \beta \end{pmatrix} =$
$\begin{pmatrix} \beta \end{pmatrix} \begin{pmatrix} \beta^{-1} & 0 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Two cases for matrices of size 2×1 can be easily obtained by a simple transposition.

These examples allow us to formulate

Sentence

For all matrices $\mathcal{A}$ of size $n \times m$, $n, m < 3$ there exists an exact triangular decomposition.

Property (Property of the factors)

For a matrix $A \in R^{n \times m}$ of rank r, $r < n, r < m$ over a commutative domain R there exists the exact triangular decomposition (24) in which
(α) the matrices L and U are of the form

$$L = \begin{pmatrix} L_1 & 0 \\ L_2 & I_{n-r} \end{pmatrix} U = \begin{pmatrix} U_1 & U_2 \\ 0 & I_{m-r} \end{pmatrix}, \tag{25}$$

(β) the matrices PLP^T and Q^TUQ remain triangular after replacing in the matrices L and Q of unit blocks I_{n-r} and I_{m-r} by arbitrary triangular blocks.

Let matrix $\mathcal{A}$ be of size $N \times M$. Assume that all matrices of size less than $n \times m$ have the exact triangular decomposition.
We split the matrix $\mathcal{A}$ into blocks:

$$\mathcal{A} = \left(\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array} \right),$$

where $\mathbf{A} \in R^{n \times n}$, $n < N$, $n < M$.

(1). Let the block $\mathbf{A}$ have the full rank. There exists exact triangular decomposition of this block: $\mathbf{A} = P_1 L_1 D_1 U_1 Q_1$ then

$$\mathcal{A} = \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ \mathbf{C} Q_1^T U_1^{-1} D_1^{-1} & I \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & \mathbf{D}' \end{bmatrix} \times$$

$$\begin{bmatrix} U_1 & D_1^{-1} L_1^{-1} P_1^T \mathbf{B} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & I \end{bmatrix}.$$

The matrix $\mathbf{D}'$ also has the exact triangular decomposition $\mathbf{D}' = P_2 L_2 D_2 U_2 Q_2$, then

$$\mathcal{A} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ P_2^T \mathbf{C} Q_1^T U_1^{-1} D_1^{-1} & L_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \times$$

$$\begin{bmatrix} U_1 & D_1^{-1} L_1^{-1} P_1^T \mathbf{B} Q_2^T \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

It is easy to see that this decomposition is exact triangular.

(2) Let the block $\mathbf{A}$ has rank r, $r < n$. There exists exact triangular decomposition of this block:

$$\mathbf{A} = P_1 L_1 D_1 U_1 Q_1.$$

Here $U_1 = \begin{pmatrix} U_0 & V_0 \\ 0 & I \end{pmatrix}$, $L_1 = \begin{pmatrix} L_0 & 0 \\ M_0 & I \end{pmatrix}$ and the diagonal matrix $D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$ has a block d_1 of rank r.

Let us denote $(\mathbf{C}_0, \mathbf{C}_1)$= $\mathbf{C} Q_1^T \begin{pmatrix} U_0^{-1} & -V_0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} \mathbf{B}_0 \\ \mathbf{B}_1 \end{pmatrix}$= $\begin{pmatrix} L_0^{-1} & 0 \\ -M_0 & I \end{pmatrix} P_1^T \mathbf{B}$.

Then for the matrix $\mathcal{A}$ we obtain the decomposition:

$$\mathcal{A} = \begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} L_0 & 0 & 0 \\ M_0 & I & 0 \\ \mathbf{C}_0 d_1^{-1} & 0 & I \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & \mathbf{B}_1 \\ 0 & \mathbf{C}_1 & \mathbf{D} \end{pmatrix} \times$$

$$\begin{pmatrix} U_0 & V_0 & d_1^{-1}\mathbf{B}_0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}. \qquad (26)$$

(2.1) Let $\mathbf{B}_1 = 0$ and $\mathbf{C}_1 = 0$ then

$$\begin{pmatrix} 0 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

$\mathbf{D} = P_2 L_2 D_2 U_2 Q_2$. We denote

$$\mathbf{P}_3 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},\ \mathbf{Q}_3 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Then $\mathcal{A} = \mathbf{P}_3 \begin{pmatrix} L_0 & 0 & 0 \\ P_2^T \mathbf{C}_0 d_1^{-1} & L_2 & 0 \\ M_0 & 0 & I \end{pmatrix} \times$

$$\begin{pmatrix} d_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_0 & d_1^{-1} \mathbf{B}_0 Q_2^T & V_0 \\ 0 & U_2 & 0 \\ 0 & 0 & I \end{pmatrix} \mathbf{Q}_3.$$

(2.2) Suppose that at least one of the two blocks of $\mathbf{B}_1$ or $\mathbf{C}_1$ is not zero. Let the exact triangular decomposition exist for these blocks:

$$\mathbf{C}_1 = P_2 L_2 D_2 U_2 Q_2,\ \mathbf{B}_1 = P_3 L_3 D_3 U_3 Q_3.$$

We denote

$$\mathbf{P}_1 = \begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & P_3 & 0 \\ 0 & 0 & P_2 \end{pmatrix},\ \mathbf{Q}_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix},$$

$$\mathbf{Q}_1 = \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix},$$

$\mathbf{P}_3 = \mathbf{P}_1 \mathbf{P}_2$, $\mathbf{Q}_3 = \mathbf{Q}_2 \mathbf{Q}_1$, $\mathbf{D}' = L_2^{-1} P_2^T \mathbf{D} Q_3^T U_3^{-1}$.

Then, basing on the expansion (26) we obtain:

$$\mathcal{A} = \mathbf{P}_3 \left(\begin{array}{ccc} L_0 & 0 & 0 \\ P_3^T M_0 & L_3 & 0 \\ P_2^T \mathbf{C}_0 d_1^{-1} & 0 & L_2 \end{array} \right) \left(\begin{array}{ccc} d_1 & 0 & 0 \\ 0 & 0 & D_3 \\ 0 & D_2 & \mathbf{D}' \end{array} \right) \times$$

$$\left(\begin{array}{ccc} U_0 & V_0 Q_2^T & d_1^{-1} \mathbf{B}_0 Q_3^T \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{array} \right) \mathbf{Q}_3. \qquad (27)$$

We denote d_2 (d_3) nondegenerate blocks of the matrices D_2 (D_3)

$$(V_1, V_4) = V_0 Q_2^T, (V_5, V_6) = d_1^{-1} \mathbf{B}_0 Q_3^T, \begin{pmatrix} M_1 \\ M_4 \end{pmatrix} = P_3^T M_0,$$

$$\begin{pmatrix} M_5 \\ M_6 \end{pmatrix} = P_2^T \mathbf{C}_0 d_1^{-1}$$

$$L_2 = \begin{pmatrix} L_2' & 0 \\ M_2 & I \end{pmatrix}, L_3 = \begin{pmatrix} L_3' & 0 \\ M_3 & I \end{pmatrix}, U_2 = \begin{pmatrix} U_2' & V_2 \\ 0 & I \end{pmatrix},$$

$$U_3 = \begin{pmatrix} U_3' & V_3 \\ 0 & I \end{pmatrix}, \mathbf{D}' = \begin{pmatrix} \mathbf{D}_1' & \mathbf{D}_3' \\ \mathbf{D}_2' & \mathbf{D}_4' \end{pmatrix}.$$

$$M_7 = \mathbf{D}_2' d_3^{-1},\ V_7 = d_2^{-1} \mathbf{D}_1' U_3',\ V_8 = d_2^{-1} (\mathbf{D}_1' V_3 + \mathbf{D}_3').$$

Then (27) can be written as

$$\mathcal{A} = \mathbf{P}_3 \begin{pmatrix} L_0 & 0 & 0 & 0 & 0 \\ M_1 & L_3' & 0 & 0 & 0 \\ M_4 & M_3 & I & 0 & 0 \\ M_5 & 0 & 0 & L_2' & 0 \\ M_6 & 0 & 0 & M_2 & I \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & \mathbf{D}_1' & \mathbf{D}_3' \\ 0 & 0 & 0 & \mathbf{D}_2' & \mathbf{D}_4' \end{pmatrix}$$

$$\begin{pmatrix} U_0 & V_1 & V_4 & V_5 & V_6 \\ 0 & U_2' & V_2 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & U_3' & V_3 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \mathbf{Q}_3 =$$

$$= \mathbf{P}_3 \begin{pmatrix} L_0 & 0 & 0 & 0 & 0 \\ M_1 & L_3' & 0 & 0 & 0 \\ M_4 & M_3 & I & 0 & 0 \\ M_5 & 0 & 0 & L_2' & 0 \\ M_6 & M_7 & 0 & M_2 & I \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_4' \end{pmatrix}$$

$$\begin{pmatrix} U_0 & V_1 & V_4 & V_5 & V_6 \\ 0 & U_2' & V_2 & V_7 & V_8 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & U_3' & V_3 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \mathbf{Q}_3. \tag{28}$$

Let

$$\mathbf{D}'_4 = P_4 L_4 D_4 U_4 Q_4,$$

$\mathbf{P}_4 = \mathrm{diag}(I, I, I, I, P_4)$,
$\mathbf{Q}_4 = \mathrm{diag}(I, I, I, I, Q_4)$,
$\mathbf{P}_5 = \mathbf{P}_3 \mathbf{P}_4$,
$\mathbf{Q}_5 = \mathbf{Q}_4 \mathbf{Q}_3$,

$$(M'_6, M'_7, M'_2) = P_4^T (M_6, M_7, M_2)$$

$$(V'_6, V'_8, V'_3) = (V_6, V_8, V_3) Q_4^T.$$

After substituting (29) into (28) we obtain the decomposition of the matrix $\mathcal{A}$ as

$$\mathcal{A} = \mathbf{P}_5 \begin{pmatrix} L_0 & 0 & 0 & 0 & 0 \\ M_1 & L_3' & 0 & 0 & 0 \\ M_4 & M_3 & I & 0 & 0 \\ M_5 & 0 & 0 & L_2' & 0 \\ M_6' & M_7' & 0 & M_2' & L_4 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_4 \end{pmatrix} \begin{pmatrix} U_0 & V_1 & V_4 & V_5 & V_6' \\ 0 & U_2' & V_2 & V_7 & V_8' \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & U_3' & V_3' \\ 0 & 0 & 0 & 0 & U_4 \end{pmatrix} \mathbf{Q}_5. \tag{30}$$

We rearrange the blocks d_2, d_3 and D_4 to obtain the diagonal matrix $\mathbf{d} = \mathrm{diag}(d_1, d_3, d_2, D_4, 0)$. To do it we use permutation matrices P_6 and Q_6:

$$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, Q_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{d} = \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_4 \end{pmatrix}$$

As a result, we obtain the decomposition:

$$\mathcal{A} = \mathbf{P}_6 \mathbf{L} \mathbf{d} \mathbf{U} \mathbf{Q}_6, \tag{31}$$

with permutation matrices $\mathbf{P}_6 = \mathbf{P}_5 P_6^T$ and $\mathbf{Q}_6 = Q_6^T \mathbf{Q}_5$, diagonal matrix $\mathbf{d}$ and triangular matrices

$$\mathbf{L} = P_6 \begin{pmatrix} L_0 & 0 & 0 & 0 & 0 \\ M_1 & L_3' & 0 & 0 & 0 \\ M_4 & M_3 & I & 0 & 0 \\ M_5 & 0 & 0 & L_2' & 0 \\ M_6' & M_7' & 0 & M_2' & L_4 \end{pmatrix} P_6^T = \begin{pmatrix} L_0 & 0 & 0 & 0 & 0 \\ M_5 & L_2' & 0 & 0 & 0 \\ M_1 & 0 & L_3' & 0 & 0 \\ M_6' & M_7' & M_2' & L_4 & 0 \\ M_4 & 0 & M_3 & 0 & I \end{pmatrix}$$

$$\mathbf{U} = Q_6^T \begin{pmatrix} U_0 & V_1 & V_4 & V_5 & V_6' \\ 0 & U_2' & V_2 & V_7 & V_8' \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & U_3' & V_3' \\ 0 & 0 & 0 & 0 & U_4 \end{pmatrix} Q_6 = \begin{pmatrix} U_0 & V_1 & V_5 & V_6' & V_4 \\ 0 & U_2' & V_7 & V_8' & V_2 \\ 0 & 0 & U_3' & V_3' & 0 \\ 0 & 0 & 0 & U_4 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

We must verify that the matrices $\mathcal{L} = \mathbf{P}_6 \mathbf{L} \mathbf{P}_6^T$ and $\mathcal{Q} = \mathbf{Q}_6^T \mathbf{U} \mathbf{Q}_6$ are triangular, and the matrices $\mathbf{P}, \mathbf{L}, \mathbf{U}, \mathbf{Q}$ satisfy the properties (α) and (β).

It is easy to see that all matrices in sequence

$$\mathcal{L}_1 = P_6 \mathbf{L} P_6^T,\ \mathcal{L}_2 = \mathbf{P}_4 \mathcal{L}_1 \mathbf{P}_4^T,\ \mathcal{L}_3 = \mathbf{P}_2 \mathcal{L}_2 \mathbf{P}_2^T,\ \mathcal{L}_4 = \mathbf{P}_1 \mathcal{L}_3 \mathbf{P}_1^T$$

are triangular and $\mathcal{L}_4 = \mathcal{L}$.

Similarly, all of the matrices in the sequence

$$\mathcal{U}_1 = Q_6^T \mathbf{L} Q_6,\ \mathcal{U}_2 = \mathbf{Q}_4^T \mathcal{U}_1 \mathbf{Q}_4,\ \mathcal{U}_3 = \mathbf{Q}_2^T \mathcal{U}_2 \mathbf{Q}_2,\ \mathcal{U}_4 = \mathbf{Q}_1^T \mathcal{U}_3 \mathbf{Q}_1$$

are triangular and $\mathcal{U}_4 = \mathcal{U}$.

For the matrices $\mathbf{L}$ and $\mathbf{U}$ Properties (α) and (β) are satisfied.

Conclusion

Algorithms for finding the LDU and Bruhat decomposition in commutative domain are described. These algorithms have the same complexity as matrix multiplication.

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