Towards Industrial Application of Approximate Computer Algebra

Tateaki Sasaki (U. Tsukuba)   Daiju Inaba (Assoc. Math. Certif.)   Fujio Kako (Nara Women's U.)

1) **Why Industrial Application ?**
2) **Series Expansion** at **Critical** point
3) Very Useful: **approx. SqFr decomp.**
4) Also Useful: **"effective FLOAT #s"**
5) Seems often: **approx. Factorization**

Why Industrial Application ?

- Approx. Comp. Algebra has been
  developed for Industrial Application

- However, a few applications so far
  (many Mathematical Applications)

- As Founder of Approx. Comp. Alg.,
  must Open wide Indust. Application

Multivariate Algebraic Function

$$F(X, \boldsymbol{u}) \equiv F(X, u_1, \cdot\cdot, u_\ell) = 0 \;\Rightarrow\; X = \phi(\boldsymbol{u})$$

Expand $\phi(\boldsymbol{u})$ **at** $\boldsymbol{u} = \boldsymbol{s} \in \mathbb{C}^\ell$, **where**

$$F(X, \boldsymbol{s}) = \frac{\partial F}{\partial X}(X, \boldsymbol{s}) = 0 \quad (\boldsymbol{s} : \text{critical point})$$

Univariate $(\ell = 1)$ **:** $\phi(u)$

$\Rightarrow$ **Puiseux series** (well known)

Multivariate $(\ell \geq 2)$ **:** $\phi(u_1, \cdot\cdot, u_\ell)$

$\Rightarrow$ **Hensel series** (Sasaki et al., 1993)

(computed by Ext. Hensel Construction)

Hensel Series : an Example

$$F(x,u,v) = x^2 - 2(u+v)\,x + (u^2+v^2) - (u^3-v^3)$$

$$\phi_{\pm}^{(2)}(u,v,t) = \boldsymbol{\alpha_{\pm} \pm t\,\frac{u^3-v^3}{2\sqrt{2uv}} \mp t^2\frac{(u^3-v^3)^2}{16uv\sqrt{2uv}}}$$

here, $\boldsymbol{\alpha_{\pm} = u+v \pm \sqrt{2uv}}$

$\boldsymbol{t}$ = **expansion order**

$$\Downarrow$$

different from MultiVar. Puiseux series

compact, **simple** method, **useful?**

Target Calculation

Hensel Series expan. of MultiVar. **Eigenval.** of $\boldsymbol{A}$ **:** a matrix controlling the **Aircraft** Motion

State vector **:** $\boldsymbol{U} = (u_x, u_y, u_z, \phi, \theta, \psi, \omega_\phi, \omega_\theta, \omega_\psi)^{\mathrm{t}}$

Eq. of motion **:** $\boldsymbol{\frac{d}{dt}U = AU + b}$ (linear model)

$$\Downarrow$$

$$\boldsymbol{U = e^{A(t-t_0)}U_0 + \int_{t_0}^{t} e^{A(t-t')}b(t')dt'}$$

$\boldsymbol{A} = A_0 + A_F$ **:** Force due to Aircraft structure

$\boldsymbol{b} = \boldsymbol{b}_0 + \boldsymbol{b}_F$ **:** Force due to Pilot controlling

$$\qquad \boldsymbol{u_x} \quad \boldsymbol{u_y} \quad \boldsymbol{u_z} \quad \boldsymbol{\phi} \quad \boldsymbol{\theta} \quad \boldsymbol{\psi} \quad \boldsymbol{\omega_\phi} \quad \boldsymbol{\omega_\theta} \quad \boldsymbol{\omega_\psi}$$

$$A_0 = \begin{pmatrix} -2D & 0 & 0 & 0 & -\alpha_{m0}L & 0 & 0 & 0 & 0 \\ 0 & -D-P_{\mathrm{v}} & 0 & -\alpha_{m0}L & 0 & T+P_{\mathrm{v}} & 0 & 0 & 0 \\ 2\alpha_{m0}L & 0 & -D-L-P_{\mathrm{h}} & 0 & L+T+P_{\mathrm{h}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \alpha_{m0}\Lambda+Q_{\mathrm{vv}} & 0 & 0 & 0 & -\alpha_{m0}\Lambda-Q_{\mathrm{vv}} & 0 & 0 & -\alpha_{m0}Q_{\mathrm{o}} \\ 0 & 0 & Q_{\mathrm{h}} & 0 & -Q_{\mathrm{h}} & 0 & 0 & 0 & 0 \\ 0 & Q_{\mathrm{v}} & 0 & 0 & 0 & -Q_{\mathrm{v}} & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{b}_0 = (T_d\ ,\ 0\ ,\ L_d\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ Q_{\mathrm{ai}}\ ,\ Q_{\mathrm{el}}\ ,\ Q_{\mathrm{ru}})^t$$

$$\boldsymbol{\alpha_{m0}} = \alpha_m + \alpha, \qquad \alpha\ \mathbf{:}\ \text{attack angle } (\propto\ \text{lift})$$

$$\boldsymbol{\phi} \quad \boldsymbol{\theta} \quad \boldsymbol{\psi}$$

$$A_F = \begin{pmatrix} 0 & 0 & 0 & 0 & -f_+F_{ac} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f_+F_{ac} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_+F_{as} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{b}_F = \begin{pmatrix} -f_+F_{as} \\ 0 \\ f_+F_{ac} \\ 0 \\ 0 \\ 0 \\ -f_-H_{\phi ac} \\ -f_+H_{\theta ac} + (2+e_+)T_\theta \\ -f_-H_{\psi as}+e_-T_\psi \end{pmatrix}$$

$\boldsymbol{e}_+$ **:** engine power, $\boldsymbol{f}_+$ **:** flap extension

$$\boxed{\textbf{EigenPol}\quad C(X) = |X I_9 - (A_0 + A_F)|/X}$$

$$C(X) = X^8 + C_7 X^7 + \cdots + \boldsymbol{C_2} X^2 + \boldsymbol{C_1} X + \boldsymbol{C_0}$$

$$\left\{\begin{aligned}
\boldsymbol{C_0} &= 2\times\alpha_{m0} Q_{\mathrm{h}} Q_{\mathrm{v}} Q_{\mathrm{o}} \times (\alpha_{m0} L + f_+ F_{ac}) \\
&\quad \times \{(2T_0 + T_0 e_+ - D) D - \alpha_{m0} L(\alpha_{m0} L + f_+ F_{ac}) - D f_+ F_{ac}\}, \\
\boldsymbol{C_1} &= -\alpha_{m0} Q_{\mathrm{h}} \times [16 \text{ terms}] + 2\times D Q_{\mathrm{h}} \times (2T_0 + T_0 e_+ - D - f_+ F_{as}) \\
&\qquad\qquad\qquad\qquad \times \{(2T_0 + T_0 e_+ - D) Q_{\mathrm{v}} - f_+ Q_{\mathrm{vv}} F_{ac}\}, \\
\boldsymbol{C_2} &= \alpha_{m0} \times [22 \text{ terms}] \\
&\quad + Q_{\mathrm{h}} Q_{\mathrm{v}} \times \{T_0^2 (2 + e_+)^2 - D(12 T_0 + 6 T_0 e_+ - 5D)\} \\
&\quad + f_+ F_{as} Q_{\mathrm{h}} Q_{\mathrm{v}} \times (-2T_0 - T_0 e_+ + 3D) \\
&\quad + f_+ F_{ac} Q_{\mathrm{h}} Q_{\mathrm{vv}} \times (-2T_0 - T_0 e_+ + 3D + f_+ F_{as}),
\end{aligned}\right.$$

$$\bar{C}(X, \alpha_{m0}, e_+, f_+) \Leftarrow \text{substitute values into } C(X)$$

Compute **Hensel Series** of $\boldsymbol{\bar{C}(X, \alpha_{m0}, e_+, f_+)}$

Approx. SqFr Decomp. is Nice

$F(X)$ **:** m **close roots** at $X = r$

$$\Updownarrow$$

$$F(X) = (X - \boldsymbol{r})^{\boldsymbol{m}} H(X) + D(X)$$

cluster size $\delta \;\Leftarrow\; \|D\| = O(\boldsymbol{\delta^2})$

$$\Downarrow$$

$$\textbf{appgcd}\left(\frac{d^{m-2}F(X)}{dX^{m-2}}, \frac{d^{m-1}F(X)}{dX^{m-1}}\right) = X - r$$

cluster **center**= $\boldsymbol{r}$, **size**= $\boldsymbol{\delta}$, #(roots)= $\boldsymbol{m}$

$\Longleftarrow$ all by **simple computation**

Effective FLOAT, What ?

(to detect cancellation errors automatically)
(Kako-Sasaki (1997))

$\#\mathrm{e}[\boldsymbol{f},\ \boldsymbol{e}]$ $\Leftarrow$ representation of **efloat**

error-part $\approx$ cancellation error

value-part $=$ same as usual **float**

initial set : $\quad e \ := \ \mathbf{5} \cdot |f| \times \varepsilon_{\mathrm{M}}$

if $|\boldsymbol{f}| < \boldsymbol{e}$ **:** $\quad$ **set** $\#\mathrm{e}[f, e] := \mathbf{0}$

Step-1 : Determine a Critical Point

$F(x, \boldsymbol{u_c})$ has **multiple roots** $\Longleftrightarrow$
$F(x, \boldsymbol{u_0})$ has **close roots** at $\boldsymbol{u_0} \approx \boldsymbol{u_c}$

Search for a Critical point near $\boldsymbol{u}_0 \in \mathbb{C}^\ell$

$\Rightarrow$ **Approx SqFr decomp.** of $F(X, \boldsymbol{u}_0)$

$$F(X, \boldsymbol{u}_0) = (X - x_0)^m H(X) + \delta F(X)$$

$\Rightarrow$ Determine $\boldsymbol{u_c}$ iteratively from $(x_0, \boldsymbol{u}_0)$
(currently, find critical point at $X = 0$)

Approx. SqFr Decomp. of $\bar{C}(X, \mathbf{0})$

$$(X + 0.0019239\cdots)^{\mathbf{4}} \times (X^4 + 1.73\cdots X^3 + \cdots + 182.\cdots)$$
$$\text{(Tolerance } 0.01793\cdots\text{)},$$
$$(X + 6.5961\cdots\text{e–}5)^{\mathbf{3}} \times (X^5 + 1.74\cdots X^4 + \cdots + 1.41\cdots)$$
$$\text{(Tolerance } 3.63\cdots\text{e–}6\text{)},$$
$$(X - 9.9999\cdots\text{e–}5)^{\mathbf{2}} \times (X^6 + 1.74\cdots X^5 + \cdots + 1.41\cdots)$$
$$\text{(Tolerance } 0.00000\cdots\text{)}.$$

2-nd Decomp $\Rightarrow$ **3-roots** cluster of size $O(0.002)$
3-rd Decomp $\Rightarrow$ inside it **:** **2-roots** small cluster

$\Rightarrow$ Determine $\boldsymbol{u_c}$ from terms of tdeg ≤ 3 of C_0, C_1

Step-2 : Separate the Critical Factor

(after Moving the **Origin** to the **Exp. Point**)

$$\bar{C}(X,\boldsymbol{u}) \equiv \widehat{S}(X,\boldsymbol{u})\widetilde{C}(X,\boldsymbol{u}) \pmod{\boldsymbol{u}^{k+1}}$$

$$\widehat{S}(X,\boldsymbol{0}) = 0, \quad \widehat{S}(X,\boldsymbol{u}) \Rightarrow \textbf{Hensel} \text{ series}$$

$$\widetilde{C}(X,\boldsymbol{0}) \neq 0, \quad \widetilde{C}(X,\boldsymbol{u}) \Rightarrow \textbf{Taylor} \text{ series}$$

Q: Is factor-separation **accurate** ?

Sasaki-Yamaguchi (1998) says

$\Rightarrow$ **NO** if initial factor has close roots

$\Rightarrow$ First, separate **4-close** roots

If initial factor has close-roots $\cdots$

$$
\begin{aligned}
F(X,\boldsymbol{u}) &\equiv G_0(X)\,H_0(X) && \pmod{\boldsymbol{u}} \\
&\equiv G_k(X,\boldsymbol{u})\,H_k(X,\boldsymbol{u}) && \pmod{\boldsymbol{u}^{k+1}}
\end{aligned}
$$

Hensel Construction is done using $\boldsymbol{A_i(X), B_i(X)}$ **:**

$$A_i(X)G_0(X) + B_i(X)G_0(X) = X^i \quad (i = 0, 1, 2, \ldots)$$

Let's calculate $A_i(X), B_i(X)$ in two ways

FLOAT : **usual** double float number

eFLOAT : double **effective FLOAT**

Simple Experiment

$$G_0(X) = (X - 0.0001) \cdot (3X^2 + X - 2)/3.0$$
$$H_0(X) = (X^2 + 0.0002\,X) \cdot (7X^2 - 5X + 3)/7.0$$

i	Calc. by **FLOAT**
1	$+\mathbf{2.24e{-}8}\ \boldsymbol{X}^{5} + \cdots + 0.9999999999998\,X$
3	$-\mathbf{7.75e{-}9}\ \boldsymbol{X}^{6} + \cdots + 1.000000001092\,X^3 + \cdots$
5	$-\mathbf{1.66e{-}9}\ \boldsymbol{X}^{6} + 0.9999999967943\,X^5 + \cdots$

i	Calc. by **eFLOAT**
1	$\#e(0.9999999999998, 5.43e{-}12)\ X$
3	$\#e(1.000000001092\ \ , 0.000121)\ X^3$
5	$\#e(0.9999999967943, 0.000258)\ X^5$

Approx. Comp. Algebra is Dangerous

- ♢ Big **Cancellation Error** will occur
  above example : $O(10^6 \sim 10^7)$
- ♢ Even **Fully-erroneous** term appears
  (Term with NO significant bits)
- ♢ Industrial : **Big**, **Tiny** terms co-exist
  keep significant **tiny terms** but
  discard **fully-erroneous** terms

eFLOAT is a **simple** and **effective** tool

Step-3 : Ext. Hensel Const. (EHC)

$\boldsymbol{F}_{\text{New}}$: **Newton Polynom** of $F(X, \boldsymbol{u})$

- Key concept of Ext. Hensel Const.
- factors of F_{New} : Starting of EHC

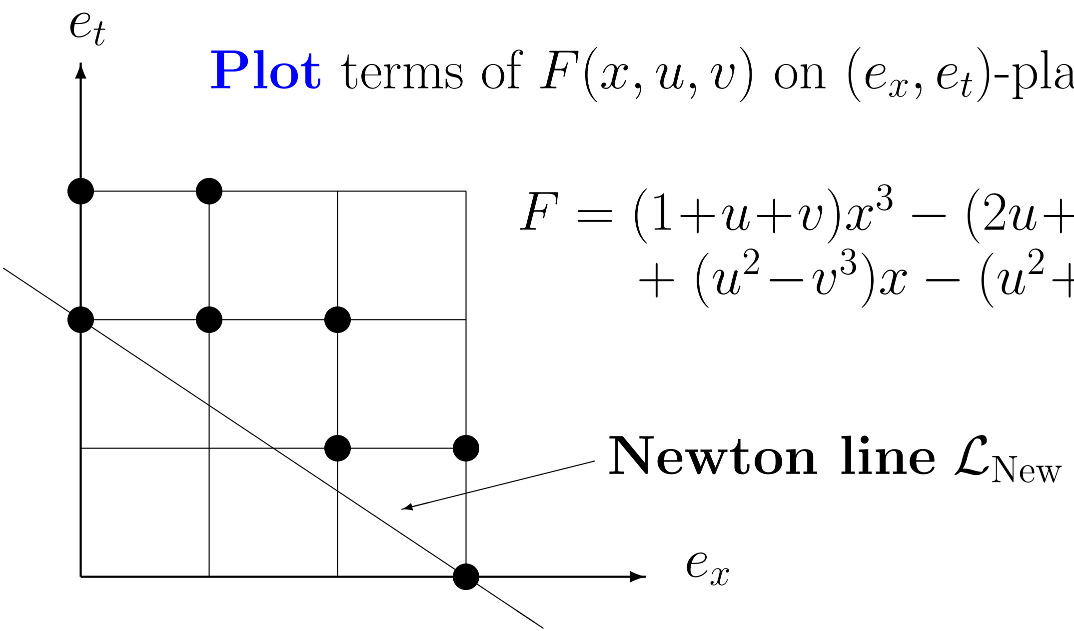
Key: **Irred. factorization** of F_{New}

- F_{New} is **multiVar** (uniVar in HC)
- irreducible $\Rightarrow$ introduce Alg. Funcs.
  reduction by Def.-polynom is heavy
- over $\mathbb{F}$: **Approx. Factorization**

Construction of Newton Polynom

Plot terms of $F(x, u, v)$ on (e_x, e_t)-plane

$$F = (1+u+v)x^3 - (2u+v^2)x^2 + (u^2-v^3)x - (u^2+2v^3)$$



Newton Polynom : $\boldsymbol{F_{\text{New}} = x^3 - u^2}$

Approx. Factorization of F_{New}

(at **critical-point** of **order 2**)

$$
\begin{aligned}
X^2 \; + \; X^1 (- \, 46.00100771767 \, \alpha_{m0} & - 0.2536988065195 \, e_+ \\
& + 0.8498910018404 \, f_+) \\
+ \; X^0 (- \, 5.664934365224 \, \alpha_{m0}^2 & + 11.70156007615 \alpha_{m0} \, e_+ \\
& - 39.20022625510 \, \alpha_{m0} f_+) \\
= \; (X - 46.12382784406 \, \alpha_{m0}) & \\
\times \; (X + 0.1228201263873 \, \alpha_{m0} & - 0.2536988065195 \, e_+ \\
& + 0.8498910018404 \, f_+) \\
& \text{(Tolerance } \mathbf{1.07e{-}12}\text{).}
\end{aligned}
$$

(at **critical-point** of **order 3**)

$$
\begin{aligned}
X^3 + \quad & X^2 \,(+\, 0.7532833020638\, \alpha_{m0} - 0.5128205128205\, e_+ \\
& \qquad\qquad\qquad\qquad + 2.434021263290\, f_+) \\
+ \quad & X^1 \,(-\, 15.96482577252\, \alpha_{m0}^2 - 0.1931495646317\, \alpha_{m0} e_+ \\
& \quad - 32.73394830102\, \alpha_{m0} f_+ + 0.06574621959237\, e_+^2 \\
& \quad + 0.06574621959237\, e_+^2 - 0.6241080162281\, e_+ f_+ \\
& \qquad\qquad\qquad\qquad + 1.352924904989\, f_+^2) \\
+ \quad & X^0 \,(+\, 4.093545069876\, \alpha_{m0}^2 e_+ - 13.71337598409\, \alpha_{m0}^2 f_+ \\
& \quad + 8.559230600651\, \alpha_{m0} e_+ f_+ - 28.67342251218\, \alpha_{m0} f_+^2) \\
= \quad & (X - 0.2564102564101\, e_+ + 0.8589743589743\, f_+) \\
\times \quad & (X^2 + 0.7532833020638 X \alpha_{m0} - 0.2564102564104 X e_+ \\
& \qquad\qquad\qquad\qquad + 1.575046904316 X f_+ \\
& \quad - 15.96482577252\, \alpha_{m0}^2 - 33.38099934254\, \alpha_{m0} f_+)
\end{aligned}
$$

(Tolerance **1.82e−13**).

Part of Hensel Series Computed

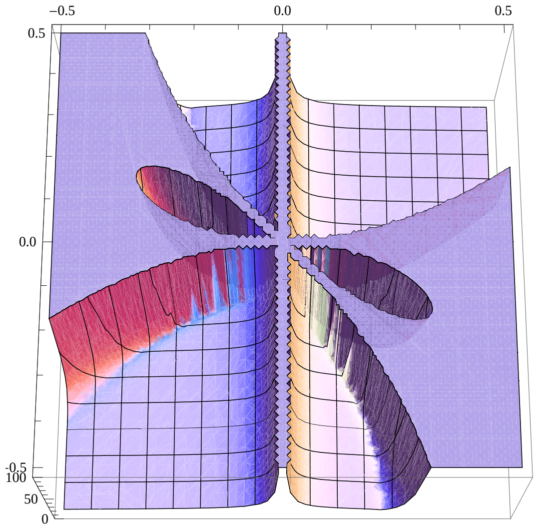
$$
\begin{aligned}
G_2^{(1)} \; = \; & X + 0.1228201263873\alpha_{m0} - 0.2536988065195 e_+ \\
& \qquad + 0.8498910018404 f_+ \\
& + (182.7\alpha_{m0}^3 - 2.461\alpha_{m0}^2 e_+ + 388.8\alpha_{m0}^2 f_+ \\
& \quad - \mathbf{4.918e{-}11}\, \alpha_{m0}^2 + 7.401\mathrm{e}{-}3\, \alpha_{m0} e_+^2 \\
& \quad - 2.156\alpha_{m0} e_+ f_+ - \mathbf{1.781e{-}12}\, \alpha_{m0} e_+ \\
& \quad + 7.109\alpha_{m0} f_+^2 - \mathbf{2.943e{-}16}\, \alpha_{m0} f_+ - 1.234\mathrm{e}{-}5\, e_+^3 \\
& \quad + 1.240\mathrm{e}{-}4\, e_+^2 f_+ - 4.155\mathrm{e}{-}4\, e_+ f_+^2 + 4.640\mathrm{e}{-}4\, f_+^3) \\
& \quad / \; (46.25\alpha_{m0} - 0.2537 e_+ + 0.8499 f_+).
\end{aligned}
$$

very **tiny terms** will be **Fully-erroneous**
(computed by **Mathematica** with **FLOAT**)

Summary and Comments

- in **Industry** : **Use of FLOATs** is Natural
  (current : convert into **RATionals**)
  $\Rightarrow$ People want to compute with FLOATs
- Comput. with FLOATs is very Dangerous
  $\Rightarrow$ **eFLOAT** is a simple and useful TOOL
- **Approx. SqFr Decomp.** seems quit useful
- **Absolute Factorization** is often in Theory
  $\Rightarrow$ replaced by **Approx. Factorization**
- **Hensel Series** seems useful in Industry
  $\Rightarrow$ should apply to $\phi(u)$ **diverging at** u_s

$$[(u+v)X-1](\boldsymbol{uX-1})(vX+1)+(u^4+v^4)X^2+u^3v^3$$



$$\boldsymbol{\phi_2^{(10)}(u,v)} \Leftrightarrow (\boldsymbol{uX-1}) : (-0.5 \le u, v \le 0.5)$$

THANK YOU for your ATTENSION

Please See **“another application”** at SYNASC 2013