A Note on Sekigawa's Zero Separation Bound

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Numerical Computational Geometry

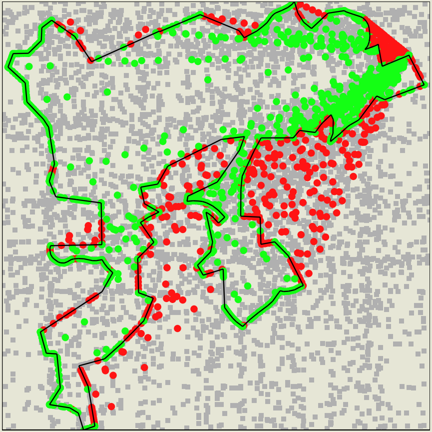
Mairson, H., Stolfi, J.: Reporting and Counting Intersections Between Two Sets of Line Segments. (1988):

> *As is the rule in computational geometry problems with discrete output, we assume all the computations are performed with exact (infinite-precision) arithmetic. Without this assumption it is virtually impossible to prove the correctness of any geometric algorithms.*

Theory:
exact real arithmetic

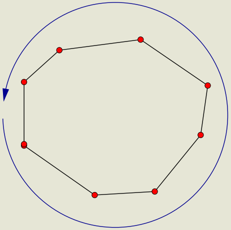
Practice:
inherently imprecise
floating-point arithmetic

- false positives
- false negatives



S. Schirra: How Reliable Are Practical Point-in-Polygon Strategies?, ESA 2008: 744-755

Programs may crash, loop, or compute garbage:



`segmentation fault`

Correct Decisions Computation

correct decisions

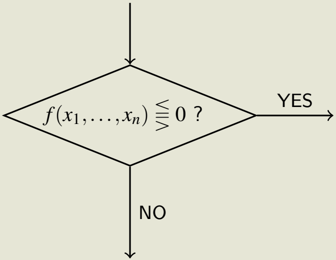
$\Downarrow$

correct combinatorics

numerical data might be inaccurate

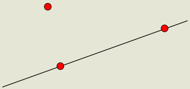
Yap, C.: Towards Exact Geometric Computation. Comp. Geom. 7, 3–23 (1997)

Correct Signs Computation



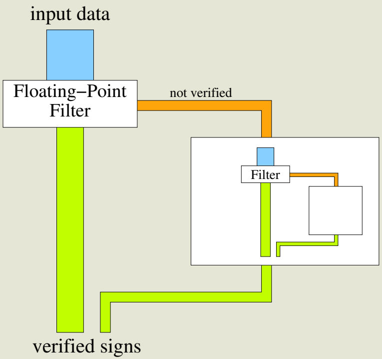
$$f(x_1,\ldots,x_n) \overset{<}{\underset{>}{=}} 0 \text{ ?}$$
$$\Downarrow$$
$$\mathrm{sign}(\, f(x_1,\ldots,x_n) \,)$$

Example: $\textsc{LeftTurn}((p_x,p_y),(q_x,q_y),(r_x,r_y))$:
$((q_x-p_x)(r_y-p_y)-(q_y-p_y)(r_x-p_x)) < 0$?



Yap, C.: Towards Exact Geometric Computation. Comp. Geom. 7, 3–23 (1997)

Floating-Point Filter



Fortune, S., Van Wyk, C.J.: Efficient Exact Arithmetic for Computational Geometry. In: 9th ACM Symposium on Computational Geometry, pp. 163–172. ACM (1993)

ξ = exact value of expression E

CascadedFiltering

compute floating-point approximation $\tilde{\xi}$ for E
compute an error bound $\Delta_{\text{error}} \geq |\tilde{\xi} - \xi|$
while $(|\tilde{\xi}| < \Delta_{\text{error}})$
    do compute a better floating-point approximation $\tilde{\xi}$
    using higher precision (software) floating-point
    arithmetic and a corresponding error bound Δ_{error}
    or compute a better error bound Δ_{error}
    for the already existing approximation $\tilde{\xi}$
return $\text{sign}(\tilde{\xi})$

How to detect $\xi = 0$?

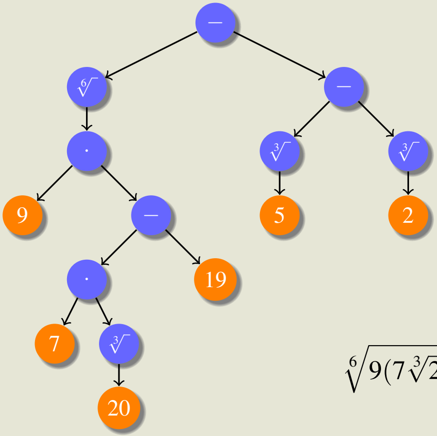
Zero Separation Bound

Given an arithmetic expression E over a set of allowed operations and operands, a constructive zero separation bound comprises rules to derive a value $\mathrm{sep}(E)$ which is a lower bound on the absolute value $|\xi|$ of E, unless $\xi = 0$.

$$\xi \neq 0 \quad \Rightarrow \quad |\xi| \geq \mathrm{sep}(E)$$

while $(|\tilde{\xi}| < \Delta_{\mathrm{error}})$ and $(|\tilde{\xi}| + \Delta_{\mathrm{error}} \geq \mathrm{sep}(E))$
        do ...
            ...
if $(|\tilde{\xi}| \leq \Delta_{\mathrm{error}})$
    then return $\mathrm{sign}(\tilde{\xi})$
    else return 0

Expression DAGs



$$\sqrt[6]{9(7\sqrt[3]{20}-19)}-\sqrt[3]{5}-\sqrt[3]{2}$$

Some Algebraic Background

$$P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 = a_d \prod_{i=1}^{d} (X - \alpha_i) \in \mathbb{Z}[X]$$

$$\text{length}(P): \quad \sum_{i=0}^{d} |a_i|$$

$$\text{height}(P): \quad \max(|a_0|, |a_1|, \ldots, |a_d|)$$

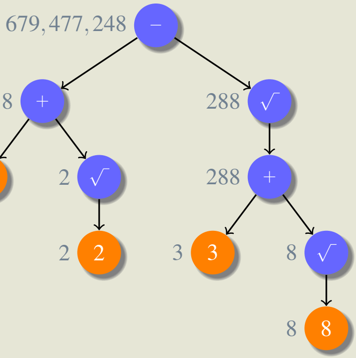
$$\text{measure}(P): \quad |a_d| \prod_{i=1}^{d} \max(1, |\alpha_i|)$$

Degree-Measure Bound

E	$\hat{M}(E)$
integer N	$\lvert N \rvert$
$A \cdot B$	$\hat{M}(A)^{D(B)} \cdot \hat{M}(B)^{D(A)}$
$A \pm B$	$\hat{M}(A)^{D(B)} \cdot \hat{M}(B)^{D(A)} \cdot 2^{D(E)}$
$\sqrt[k]{A}$	$\hat{M}(A)$

Then $\hat{M}(E)^{-1}$ is a separation bound.

$$1+\sqrt{2}-\sqrt{3+\sqrt{8}}$$



BFMS Bound

E	$U(E)$
integer N	$\lvert N\rvert$
$A \pm B$	$U(A) + U(B)$
$A \cdot B$	$U(A) \cdot U(B)$
$\sqrt[k]{A}$	$\sqrt[k]{U(A)}$

Then $U(E)^{-(D(E)-1)}$ is a separation bound.

Sekigawa's Bound

E	$M(E)$
integer N	$\lvert N \rvert$
$A \cdot B$	$M(A)^{D(B)} \cdot M(B)^{D(A)}$
$A \pm B$	$(*)$
$\sqrt[k]{A}$	$M(A)$

where $(*)$ is the product of the $D(E)$ largest values of

$$M(A)+M(B), \underbrace{M(A)+1, \ldots, M(A)+1}_{D(B)-1}, \underbrace{M(B)+1, \ldots, M(B)+1}_{D(A)-1}, \quad \underbrace{2, \ldots, 2}_{(D(A)-1)(D(B)-1)}$$

Then $M(E)^{-1}$ is a separation bound.

Comparison of Zero Separation Bounds

A separation bound sep dominates another bound sep' for a class of arithmetic expressions $\mathscr{E}$ if $\mathrm{sep}(E) \geq \mathrm{sep}'(E)$ for all E in $\mathscr{E}$.

$\mathcal{A}$-Bound $\succeq$ $\mathcal{B}$-Bound

$\mathcal{A}$-Bound dominates $\mathcal{B}$-Bound for division-free radical expressions,
i..e, arithmetic expressions with operations $+, -, \cdot$ and $\sqrt[k]{\ }$
and integer operands

Known Domination Results

Lemma:

$$\begin{aligned} \textit{Degree-Measure Bound} &\succeq \textit{Degree-Height Bound} \\ \textit{Degree-Measure Bound} &\succeq \textit{Degree-Length Bound} \\ \textit{BFMS Bound} &\succeq \textit{Degree-Measure Bound} \\ \textit{BFMS Bound} &\succeq \textit{Scheinerman's Bound} \end{aligned}$$

Lemma:

$$\textit{Sekigawa's Bound} \succeq \textit{Degree-Measure Bound}$$

Sekigawa vs. BFMS

Lemma: *BFMS Bound* $\succeq$ *Sekigawa's Bound*

We prove

$$U(E)^{D(E)} \leq M(E)$$

by structural induction:

Basis:

The claim holds for the base case where E is an integer N, since the rules are identical and $D(E) = 1$.

Induction Hypothesis:

$$U(A)^{D(A)} \leq M(A) \qquad \text{and} \qquad U(B)^{D(B)} \leq M(B).$$

Inductive Steps:

$$E = \sqrt[k]{A} \qquad E = A \cdot B \qquad E = A \pm B$$

$$E = \sqrt[k]{A}$$

$M(E) = M(A)$ and $U(E) = \sqrt[k]{U(A)}$ and $D(E) = k \cdot D(A)$

$$\begin{aligned} U(E)^{D(E)} &= (\sqrt[k]{U(A)})^{k \cdot D(A)} \\ &= U(A)^{D(A)} \\ &\leq M(A) \qquad \text{by I.H.} \\ &= M(E) \end{aligned}$$

$$E = A \cdot B$$

$$M(E) = M(A)^{D(B)} M(B)^{D(A)}$$
$$U(E) = U(A) \cdot U(B)$$
$$D(E) \leq D(A) D(B)$$

$$\begin{aligned}
U(E)^{D(E)} &= (U(A) \cdot U(B))^{D(E)} \\
&\leq (U(A) \cdot U(B))^{D(A) \cdot D(B)} \\
&= U(A)^{D(A) \cdot D(B)} \cdot U(B)^{D(B) \cdot D(A)} \\
&\leq M(A)^{D(B)} \cdot M(B)^{D(A)} \qquad \text{by I.H.} \\
&= M(E)
\end{aligned}$$

$E = A \pm B$

Let us first assume $D(E) = D(A) \cdot D(B)$. Then $M(E) =$

$$(M(A) + M(B)) \cdot (M(A)+1)^{D(B)-1} \cdot (M(B)+1)^{D(A)-1} \cdot 2^{(D(A)-1)(D(B)-1)}$$

$$U(E) = U(A) + U(B)$$

$$\begin{aligned} U(E)^{D(E)} &= (U(A)+U(B))^{D(E)} \\ &\leq \left(\sqrt[D(A)]{M(A)} + \sqrt[D(B)]{M(B)} \right)^{D(E)} \qquad \text{by I.H.} \end{aligned}$$

To complete the inductive step we use

Lemma: Let S be a set of pairs (i,j), $1 \le i \le m,\ 1 \le j \le n$, and let

$$F(x_1,\ldots,x_m,y_1,\ldots,y_n) = \prod_{(i,j)\in S}(x_i+y_j).$$

For constants $\mathbf{a},\mathbf{b} \ge 1$, the maximum value of the continous function F on the compact set

$$\mathscr{D} = \left\{ (x_1,\ldots,x_m,y_1,\ldots,y_n) \in \mathbb{R}^{m+n} \;\middle|\; \begin{array}{l} \prod_{i=1}^{m} x_i = \mathbf{a},\ x_i \ge 1, i = 1,\ldots,m \\ \prod_{j=1}^{m} y_j = \mathbf{b},\ y_j \ge 1, j = 1,\ldots,n \end{array} \right\}$$

is not greater than the product of the $|S|$ largest numbers among the following mn numbers:

$$\mathbf{a}+\mathbf{b}, \underbrace{\mathbf{a}+1,\ldots,\mathbf{a}+1}_{n-1}, \underbrace{\mathbf{b}+1,\ldots,\mathbf{b}+1}_{m-1}, \underbrace{2,\ldots,2}_{(n-1)(m-1)}$$

Apply this lemma with $m = D(A)$, $x_i = \sqrt[D(A)]{M(A)}$ for $i = 1, \ldots, m$,
$\mathbf{a} = M(A)$, $n = D(B)$, $y_j = \sqrt[D(B)]{M(B)}$ for $j = 1, \ldots, n$, and $\mathbf{b} = M(B)$:

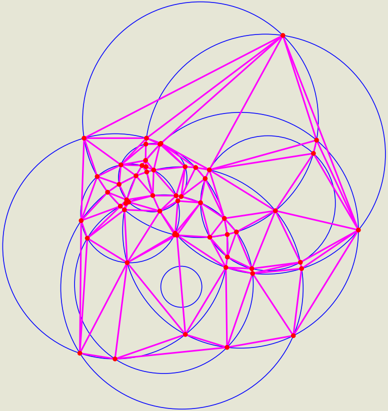
$$\begin{aligned} \left(\sqrt[D(A)]{M(A)} + \sqrt[D(B)]{M(B)} \right)^{D(E)} &= \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) \\ &\leq (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+1)^{n-1} \cdot (\mathbf{b}+1)^{m-1} \cdot 2^{(m-1)(n-1)} \\ &= M(E) \end{aligned}$$

If $D(E) < D(A)D(B)$, apply the lemma with $|S| = D(E) < nm$. ■

Remarks

- The BFMS bound also dominates zero separation bounds derived from the polynomial system bounds by Canny and by Emiris, Mourrain, and Tsigaridas.
- Sekigawa does not provide rules for divisions. Corresponding rules can be added.
- For radical expressions with divisions, BFMSS, Li-Yap, and Degree-Measure Bound are incomparable.

Need for $\sqrt{\ }$



Delaunay Triangulation of Intersection Points of Circles

C++ Number Types

New York University (Chee Yap et al.)
`CORE::Expr`

Max Planck Institute for Computer Science (Kurt Mehlhorn et al.)
`leda::real`

Otto von Guericke University Magdeburg (w. Marc Mörig)
`RealAlgebraic`