APPLICATIONS OF SYMBOLIC CALCULATIONS AND POLYNOMIAL INVARIANTS TO THE CLASSIFICATION OF SINGULARITIES OF DIFFERENTIAL SYSTEMS

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GOAL OF THE LECTURE:

To show how symbolic calculations and polynomial invariants are essential tools in classification problems of planar polynomial differential systems.

More specifically we show here how they are instrumental in obtaining the bifurcation diagram of the global configurations of singularities, of quadratic differential systems having a unique simple finite singularity.

Work by the authors together with J.C. Artés and J. Llibre on the general case of quadratic differential systems having finite singularities of total multiplicity $m_f \leq 4$ using polynomial invariants, is in progress.

Our results are in invariant form and hence they can be applied for any family of quadratic systems, given in any normal form. Determining the configurations of singularities for any family of quadratic systems, becomes thus a simple task using computer symbolic calculations.

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x, y),$$

$$\frac{dy}{dt} = q(x, y),$$

with $p, q \in \mathbb{R}[x, y]$.

We call *degree* of such a system the integer

$$m = \max(\deg p, \deg q).$$

We call *quadratic* such a differential system with $m = 2$.

A singular point of such a system is a point (x_0, y_0) such that

$$p(x_0, y_0) = 0,\ q(x_0, y_0) = 0.$$

We denote by **QS** the whole class of real quadratic differential systems.

Quadratic Differential Systems occur very often in many areas of Applied Mathematics in:

POPULATION DYNAMICS,

CHEMISTRY,

ELECTRIC CIRCUITS,

NEURAL NETWORKS,

LASER PHYSICS,

HYDRODYNAMICS,

ASTROPHYSICS, ETC.

Apart from the linear systems, the quadratic systems are the simplest ones among the polynomial systems. Yet several problem on the class **QS**, formulated more than a century ago, are still open for this class.

There are three reasons for this situation:

I) the elusive nature of **limit cycles**.
These periodic solutions are hard to pin down.

II) The rather **large number** of parameters involved.
QS depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on **5 parameters**.

So for the bifurcation diagram of this class, we need to work in a five dimensional space which is not $\mathbb{R}^5$ but a much more complicated topological space, quotient of R^{12} by this group action.

III) To gain global insight into **QS** one needs to perform a **large number of ample calculations**.

For example the bifurcation points for singularities are located on algebraic hypersurfaces which could be of high degree. We have several such algebraic surfaces and we need to find their intersection points, their singularities, all leading to problems for symbolic calculations, some quite hard to solve even for quadratic systems.

Another example where ample calculations are needed is in

The Problem of the Center

stated by Poincaré in 1885. We recall that a **center** is an isolated singular point of a system surrounded by closed phase curves.

One way in which we can state this problem is the following:

GIVE AN ALGORITHM TO DECIDE WETHER A SINGULAR POINT OF A POLYNOMIAL DIFFERENTIAL SYSTEM IS A CENTER.

Another way in which we can state this problem is:

FIND NECESSARY AND SUFFICIENT CONDITIONS FOR THE COEFFICIENTS OF A SYSTEM TO HAVE A CENTER AT A SINGULAR POINT.

Poincaré considered the case of a singular point which has purely imaginary eigenvalues. Such a system can be written as:

$$\frac{dx}{dt} = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + ... + a_{0n}y^n,$$

$$\frac{dy}{dt} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + ... + b_{0n}y^n,$$

$p,\ q \in \mathbb{R}[x, y]$

It is known that for such a system the origin is either a **center** or a **focus**.

We call **strong focus** a focus with non–zero trace of the linearization matrix at this point. Such a focus will be considered to have the **order zero**. A focus with trace zero is called a **weak focus**.

In this case it is easily shown that there exists a formal power series F with coefficients in $\mathbb{Q}[a_{20}, \ldots, b_{0,n}]$ such that

$$dF/dt = \Sigma_{i=1}^{\infty} V_i (x^2 + y^2)^{i+1}.$$

To distinguish among the foci of various orders we use the values of V_i's.

A weak focus is of the **order** i if for all $j < i$ we have $V_j = 0$ **and** $V_i \neq 0$.

Theorem (Poincaré 1885) Such a system has a center at $(0,0)$ if and only if it has a local analytic, nonconstant first integral

$$F(x,y) = c_{10}x + c_{01}y + c_{20}x^2 + ... + c_{0n}y^n + ...,$$

where $F : N \to \mathbb{R}$, N is a neighborhood of $(0,0)$ and $\frac{dF}{dt} = 0$, i.e.

$$\frac{\partial F}{\partial x} p(x,y) + \frac{\partial F}{\partial y} q(x,y) = 0$$

for every point $(x,y) \in N$.

The problem of the center was solved for **quadratic differential systems**.

The problem of the center is still open for **cubic differential systems** because our computers are still not sufficiently powerful to perform these massive computations necessary for this problem.

We now have several tools for performing such calculations, among them programs like

Mathematica,
Maple,
Reduce,
CoCoA,
MaCauley 2.

We also have **the program P4** which combines both symbolic and numerical calculations for drawing phase portraits of individual planar polynomial differential systems.

NEW RESULTS obtained with the help of **symbolic and numerical calculations**:

The bifurcation diagrams were done for the following Subclasses of **QS** using Mathematica and P4, by Artés, Llibre and Schlomiuk:

1) The class **QW3** of quadratic differential systems with a singular point which is a weak focus of order 3

2) The class **QW2** of quadratic differential systems with a singular point which is a weak focus of order 2

3) The class **QW1IL** of quadratic differential systems with an invariant straight line and a weak focus of order 1

The subclasses **QW3**, **QW2** and **QW1IL** of **QS**, modulo the group action of affine transformation and time rescaling are **3-dimensional** for **QW2** and **QW1IL** and **2-dimensional** for **QW3**.

The low dimension of these classes allowed us to study them using interdisciplinary methods, among them symbolic calculations.

The subclass **QW1** of **QS**, formed by quadratic systems possessing a weak focus of order one, modulo the group action of affine transformation and time rescaling is **4-dimensional**.

The study of this class is difficult and one of the difficulties is the higher dimension, modulo the group action, of this class.

For the moment the global studies of **4-dimensional** subclasses of **QS**, modulo the group action, remain a challenge.

ANOTHER LINE OF WORK:

Focus on specific **GLOBAL** FEATURES of the systems in
THE WHOLE QUADRATIC CLASS,
in particular on the global study of singularities and their bifurcation diagram.

The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying on the sphere or on the Poincaré disk the planar polynomial differential systems.

It is now possible, with the help of symbolic calculations to do a complete study of the global configurations of singularities, finite and infinite of the whole class **QS**.

Indeed, the whole bifurcation diagram of the global configurations of singularities, finite and infinite, in quadratic vector fields and more generally in polynomial vector fields can be obtained by using only **algebraic** means, among them, **the algebraic tool of polynomial invariants computed using symbolic calculations**.

We show here how is this done for a particular case:

for the family of quadratic differential systems having a **single** finite singularity which in addition is **simple**.

The general case is work in progress with J.C. Artés and J. Llibre.

Invariant polynomials

Consider real quadratic systems of the form:

$$\begin{aligned}\frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y),\\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y)\end{aligned} \tag{1}$$

with homogeneous polynomials p_i and q_i ($i = 0,1,2$) of degree i in x, y:

$$p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y,$$

$$p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2$$

$$q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y,$$

$$q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.$$

Let $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems in **QS** and denote the corresponding polynomial rings by $\mathbb{R}[a, x, y] = \mathbb{R}[a_{00}, \ldots, b_{02}, x, y]$.

The group *Aff*$(2, \mathbb{R})$ of the affine transformations on the plane acts on the set **QS**. For every subgroup $G \subseteq$ *Aff*$(2, \mathbb{R})$ we have an induced action of G on $\mathbf{QS}$. We can identify the set $\mathbf{QS}$ with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each such system the 12-tuple $a = (a_{00}, a_{10}, \ldots, b_{02})$ of its coefficients.

The action of *Aff*$(2, \mathbb{R})$ on $\mathbf{QS}$ yields an action of this group on (the corresponding subset of) $\mathbb{R}^{12}$. This action indeed exists on $\mathbb{R}^{12}$. For every $\mathfrak{g} \in$ *Aff*$(2, \mathbb{R})$ let $r_{\mathfrak{g}} : \mathbb{R}^{12} \longrightarrow \mathbb{R}^{12}$, $r_{\mathfrak{g}}(a) = \tilde{a}$ where $\tilde{a}$ is the 12-tuple of coefficients of the transformed system $\tilde{S}$.

A polynomial $U(a\,,x,y)\in\mathbb{R}[a,x,y]$ is called a *comitant* of systems (1) with respect to a subgroup G of $Aff(2,\mathbb{R})$, if there exists $\chi\in\mathbb{Z}$ such that for every $(\mathfrak{g},\,\boldsymbol{a})\in G\times\mathbb{R}^{12}$ and for every $(x,y)\in\mathbb{R}^2$ the following relation holds:

$$U(r_{\mathfrak{g}}(\boldsymbol{a}),\ \mathfrak{g}(x,y)\,)\equiv\ (\det\mathfrak{g})^{-\chi}\,U(\boldsymbol{a},x,y),$$

where $\det\mathfrak{g}$ is the determinant of the linear matrix of the transformation $\mathfrak{g}\in$ *Aff*$(2,\mathbb{R})$. If the polynomial U does not explicitly depend on x and y then it is called **invariant**. The number $\chi\in\mathbb{Z}$ is called the **weight** of the comitant $U(a,x,y)$.

Let us consider the polynomials

$$C_i(a,x,y) = yp_i(a,x,y) - xq_i(a,x,y), \ i = 0,1,2,$$

$$D_i(a,x,y) = \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y), \ i = 1,2.$$

It was proved by Sibirschi that the polynomials

$$\Big\{ \ C_0(a,x,y), \quad C_1(a,x,y), \quad C_2(a,x,y)$$

$$D_1(a), \quad D_2(a,x,y) \ \Big\}$$

are GL-comitants of these systems.

Let $f,\ g \in\ R[a,x,y]$ and we denote

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{2}$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the *transvectant of index* k of (f,g).

Theorem 1. *(Vulpe) Any* GL*-comitant of quadratic systems can be constructed from the elements of the set* $C_i, i = 1,2,3, D_i, i = 1,2$ *by using the operations:* $+$, $-$, $\times$, *and by applying the differential operation* $(f,g)^{(k)}$.

Consider the differential operator

$$\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$$

acting on $\mathbb{R}[a, x, y]$ where

$$\mathbf{L}_1 = 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},$$

$$\mathbf{L}_2 = 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.$$

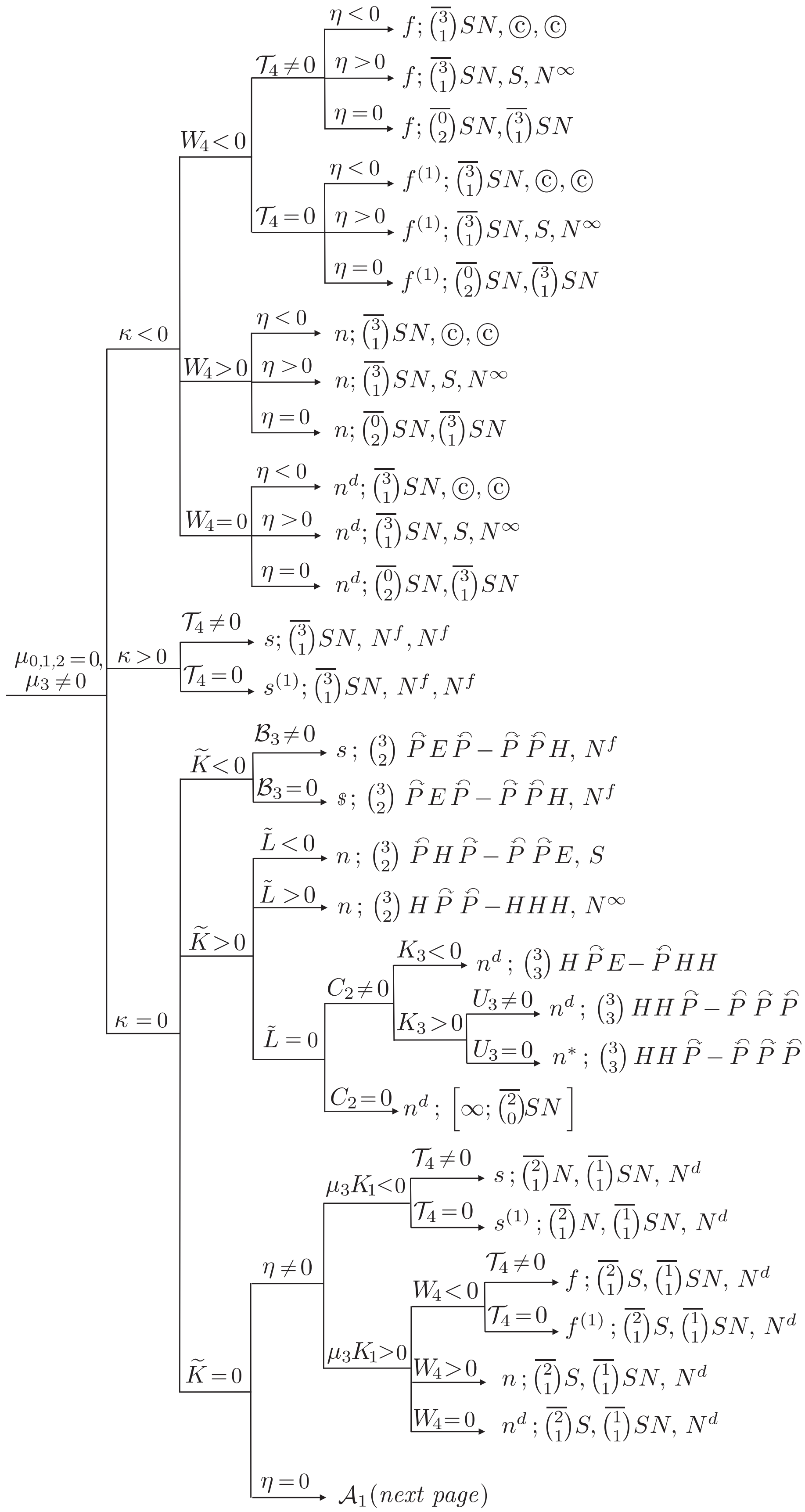
Using this operator and the affine invariant $\mu_0 = \text{Res}_x\Big(p_2(a,x,y), q_2(a,x,y)\Big)/y^4$ we construct the following polynomials $\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)$, $i = 1,..,4$ where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$.

The other invariants and comitants of differential equations used for proving our main results are obtained by using these operations and following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples.

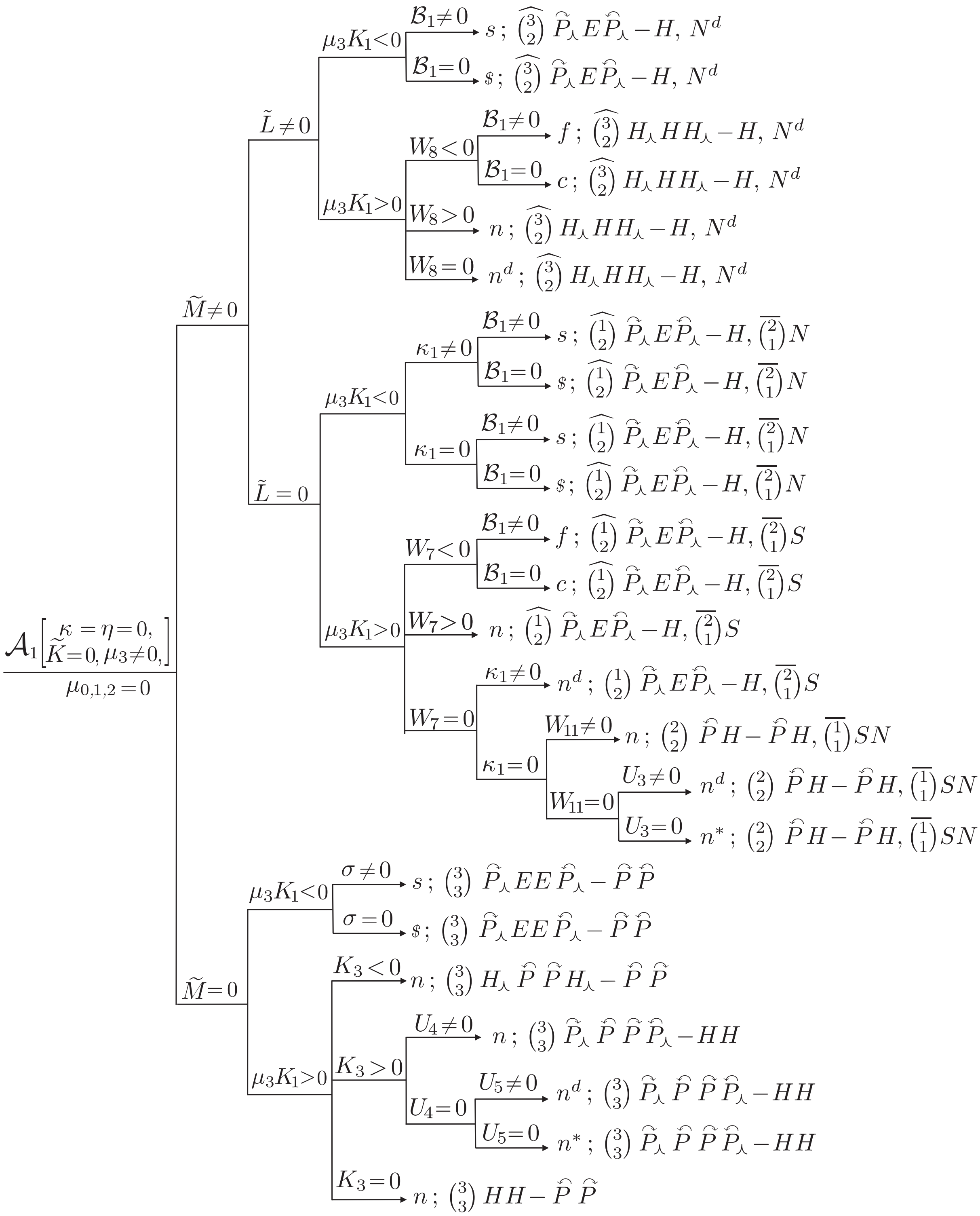
Main Theorem. (A) *The configurations of singularities, finite and infinite, of all quadratic vector fields with a single finite singularity which is simple (the total multiplicity of finite singularities is $m_f = 1$) are classified in the following* Diagram *according to the geometric equivalence relation. We have 52 geometrically distinct global configurations of singularities.*

(B) *Necessary and sufficient conditions for each one of the 52 different equivalence classes can be assembled from these diagram in terms of 25 invariant polynomials with respect to the action of the affine group and time rescaling.*

(C) *The* Diagram *actually contains the global bifurcation diagram in the 12-dimensional space of parameters, of the global configurations of singularities, finite and infinite, of this family of quadratic differential systems.*



$\mu_{0,1,2}=0$, $\mu_3\neq 0$
$\kappa<0$
$W_4<0$
$\mathcal{T}_4\neq 0$
$\eta<0$ $f;\ \overline{\binom{3}{1}}SN,$ ⓒ, ⓒ
$\eta>0$ $f;\ \overline{\binom{3}{1}}SN, S, N^{\infty}$
$\eta=0$ $f;\ \overline{\binom{0}{2}}SN, \overline{\binom{3}{1}}SN$
$\mathcal{T}_4=0$
$\eta<0$ $f^{(1)};\ \overline{\binom{3}{1}}SN,$ ⓒ, ⓒ
$\eta>0$ $f^{(1)};\ \overline{\binom{3}{1}}SN, S, N^{\infty}$
$\eta=0$ $f^{(1)};\ \overline{\binom{0}{2}}SN, \overline{\binom{3}{1}}SN$
$W_4>0$
$\eta<0$ $n;\ \overline{\binom{3}{1}}SN,$ ⓒ, ⓒ
$\eta>0$ $n;\ \overline{\binom{3}{1}}SN, S, N^{\infty}$
$\eta=0$ $n;\ \overline{\binom{0}{2}}SN, \overline{\binom{3}{1}}SN$
$W_4=0$
$\eta<0$ $n^d;\ \overline{\binom{3}{1}}SN,$ ⓒ, ⓒ
$\eta>0$ $n^d;\ \overline{\binom{3}{1}}SN, S, N^{\infty}$
$\eta=0$ $n^d;\ \overline{\binom{0}{2}}SN, \overline{\binom{3}{1}}SN$
$\kappa>0$
$\mathcal{T}_4\neq 0$ $s;\ \overline{\binom{3}{1}}SN,\ N^f, N^f$
$\mathcal{T}_4=0$ $s^{(1)};\ \overline{\binom{3}{1}}SN,\ N^f, N^f$
$\kappa=0$
$\widetilde{K}<0$
$\mathcal{B}_3\neq 0$ $s;\ \binom{3}{2}\ \widehat{P}E\widehat{P}-\widehat{P}\,\widehat{P}H,\ N^f$
$\mathcal{B}_3=0$ $\$;\ \binom{3}{2}\ \widehat{P}E\widehat{P}-\widehat{P}\,\widehat{P}H,\ N^f$
$\widetilde{K}>0$
$\tilde{L}<0$ $n;\ \binom{3}{2}\ \widehat{P}H\widehat{P}-\widehat{P}\,\widehat{P}E,\ S$
$\tilde{L}>0$ $n;\ \binom{3}{2}\ H\widehat{P}\,\widehat{P}-HHH,\ N^{\infty}$
$\tilde{L}=0$
$C_2\neq 0$
$K_3<0$ $n^d;\ \binom{3}{3}\ H\widehat{P}E-\widehat{P}HH$
$K_3>0$
$U_3\neq 0$ $n^d;\ \binom{3}{3}\ HH\widehat{P}-\widehat{P}\,\widehat{P}\,\widehat{P}$
$U_3=0$ $n^*;\ \binom{3}{3}\ HH\widehat{P}-\widehat{P}\,\widehat{P}\,\widehat{P}$
$C_2=0$ $n^d;\ \left[\infty;\overline{\binom{2}{0}}SN\right]$
$\widetilde{K}=0$
$\eta\neq 0$
$\mu_3K_1<0$
$\mathcal{T}_4\neq 0$ $s;\ \overline{\binom{2}{1}}N, \overline{\binom{1}{1}}SN,\ N^d$
$\mathcal{T}_4=0$ $s^{(1)};\ \overline{\binom{2}{1}}N, \overline{\binom{1}{1}}SN,\ N^d$
$\mu_3K_1>0$
$W_4<0$
$\mathcal{T}_4\neq 0$ $f;\ \overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$
$\mathcal{T}_4=0$ $f^{(1)};\ \overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$
$W_4>0$ $n;\ \overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$
$W_4=0$ $n^d;\ \overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$
$\eta=0$ $\mathcal{A}_1$(next page)



$\mathcal{A}_1\left[\begin{smallmatrix}\kappa=\eta=0,\\ \widetilde{K}=0,\mu_3\neq 0,\end{smallmatrix}\right]$
$\mu_{0,1,2}=0$
$\widetilde{M}\neq 0$
$\tilde{L}\neq 0$
$\mu_3K_1<0$
$\mathcal{B}_1\neq 0$
$s\,;\ \widehat{\binom{3}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ N^d$
$\mathcal{B}_1=0$
$\$;\ \widehat{\binom{3}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ N^d$
$\mu_3K_1>0$
$W_8<0$
$\mathcal{B}_1\neq 0$
$f\,;\ \widehat{\binom{3}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^d$
$\mathcal{B}_1=0$
$c\,;\ \widehat{\binom{3}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^d$
$W_8>0$
$n\,;\ \widehat{\binom{3}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^d$
$W_8=0$
$n^d\,;\ \widehat{\binom{3}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^d$
$\tilde{L}=0$
$\mu_3K_1<0$
$\kappa_1\neq 0$
$\mathcal{B}_1\neq 0$
$s\,;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}N$
$\mathcal{B}_1=0$
$\$;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}N$
$\kappa_1=0$
$\mathcal{B}_1\neq 0$
$s\,;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}N$
$\mathcal{B}_1=0$
$\$;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}N$
$\mu_3K_1>0$
$W_7<0$
$\mathcal{B}_1\neq 0$
$f\,;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}S$
$\mathcal{B}_1=0$
$c\,;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}S$
$W_7>0$
$n\,;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}S$
$W_7=0$
$\kappa_1\neq 0$
$n^d\,;\ \binom{1}{2}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ \overline{\binom{2}{1}}S$
$\kappa_1=0$
$W_{11}\neq 0$
$n\,;\ \binom{2}{2}\ \widehat{P}H - \widehat{P}H,\ \overline{\binom{1}{1}}SN$
$W_{11}=0$
$U_3\neq 0$
$n^d\,;\ \binom{2}{2}\ \widehat{P}H - \widehat{P}H,\ \overline{\binom{1}{1}}SN$
$U_3=0$
$n^*\,;\ \binom{2}{2}\ \widehat{P}H - \widehat{P}H,\ \overline{\binom{1}{1}}SN$
$\widetilde{M}=0$
$\mu_3K_1<0$
$\sigma\neq 0$
$s\,;\ \binom{3}{3}\ \widehat{P}_\curlywedge EE\widehat{P}_\curlywedge - \widehat{P}\,\widehat{P}$
$\sigma=0$
$\$;\ \binom{3}{3}\ \widehat{P}_\curlywedge EE\widehat{P}_\curlywedge - \widehat{P}\,\widehat{P}$
$\mu_3K_1>0$
$K_3<0$
$n\,;\ \binom{3}{3}\ H_\curlywedge\widehat{P}\,\widehat{P}H_\curlywedge - \widehat{P}\,\widehat{P}$
$K_3>0$
$U_4\neq 0$
$n\,;\ \binom{3}{3}\ \widehat{P}_\curlywedge\widehat{P}\,\widehat{P}\widehat{P}_\curlywedge - HH$
$U_4=0$
$U_5\neq 0$
$n^d\,;\ \binom{3}{3}\ \widehat{P}_\curlywedge\widehat{P}\,\widehat{P}\widehat{P}_\curlywedge - HH$
$U_5=0$
$n^*\,;\ \binom{3}{3}\ \widehat{P}_\curlywedge\widehat{P}\,\widehat{P}\widehat{P}_\curlywedge - HH$
$K_3=0$
$n\,;\ \binom{3}{3}\ HH - \widehat{P}\,\widehat{P}$

The computation of polynomial invariants is done using symbolic calculations.

This bifurcation diagram is expressed in terms of polynomial invariants. The results can therefore be applied **to any family of quadratic systems, given in any normal form**.

This diagram gives us also an **algorithm** for computing **the global configurations of singularities** for any quadratic system given in any possible normal form. We only need to compute the indicated polynomials by following step by step the Diagram.

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Here are some definitions of concepts used in this work and some notations.
In a classification we have two objects: a set X and an equivalence relation on X.

In this work X is the set of all global configurations of singularities of systems in **QS** and the equivalence relation on X is the **geometric equivalence relation** which is finer than the topological equivalence relation.

The topological equivalence relation does not distinguish between foci and nodes, or between foci of different orders.

The **geometric equivalence relation** distinguishes between foci and nodes, between the strong and weak foci, between strong and weak saddles, between foci of different orders, between saddles of different orders, between integrable saddles and weak saddles of positive orders, and between the different kinds of nodes.

Such distinctions are important in the production of limit cycles. Indeed, for example the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

Topological versus geometrical equivalence relations:

Two singularities are **topologically equivalent** if they possess neighborhoods N_1 and N_2 for which there is a homeomorphism $\phi : N_1 \to N_2$ which carries oriented phase curves in N_1 to oriented phase curves in N_2 preserving the orientation.

The notion of **geometric equivalence relation** is completely defined in terms of an algebraic nature.

Algebraic information may not be significant for the local (topological) phase portrait around a singularity. For example, topologically there is **no distinction** between a **focus** and a **node** or between a **weak** and a **strong focus**. But algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities.

We use the following terminology for singularities:

> We call **elemental** a singular point with its both eigenvalues not zero;
>
> We call **semi–elemental** a singular point with exactly one of its eigenvalues equal to zero;
>
> We call **nilpotent** a singular point with both its eigenvalues zero but with its Jacobian matrix at that point not identically zero;
>
> We call **intricate** a singular point with its Jacobian matrix identically zero.

In the literature, **intricate** singularities are usually called **linearly zero**.

Two singularities p_1 and p_2 of two polynomial vector fields are **locally geometrically equivalent** if and only if they are topologically equivalent, they have the same multiplicity and one of the following conditions is satisfied:

- p_1 and p_2 are order equivalent foci (or saddles).
  (Two foci (or saddles) are **order equivalent** if their corresponding orders coincide);

- p_1 and p_2 are tangent equivalent simple nodes
  Two simple finite nodes, with the respective eigenvalues λ_1, λ_2 and σ_1, σ_2, are **tangent equivalent** if and only if they satisfy one of the following three conditions: a) $(\lambda_1 - \lambda_2)(\sigma_1 - \sigma_2) \neq 0$; b) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and both linearization matrices at the two singularities are diagonal; c)

$\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and the corresponding linearization matrices are not diagonal.);

- p_1 and p_2 are both centers;

- p_1 and p_2 are both semi–elemental singularities;

- p_1 and p_2 are blow–up equivalent nilpotent or intricate singularities.

We say that two infinite simple nodes P_1 and P_2 are **tangent equivalent** if and only if their corresponding singularities on the sphere are tangent equivalent and in addition, in case they are generic nodes, we have $(|\lambda_1| - |\lambda_2|)(|\sigma_1| - |\sigma_2|) > 0$ where λ_1 and σ_1 are the eigenvalues of the eigenvectors tangent to the line at infinity.

Multiplicity of singularities

Roughly speaking a singular point p of an analytic differential system χ is a **multiple singularity of multiplicity** m if p generates m singularities, as close to p as we wish, under analytic perturbations of this system and m is the maximal such number.

In polynomial differential systems of fixed degree n we have several possibilities for obtaining multiple singularities. **i)** A finite singular point splits into several finite singularities in n-degree polynomial perturbations. **ii)** An infinite singular point splits into some finite and some infinite singularities in n-degree polynomial perturbations. **iii)** An infinite singularity splits only in infinite singular points of the systems in n-degree perturbations.

To all these cases we can give a precise mathematical meaning using the notion of intersection multiplicity at a point p of two algebraic curves based on work of D.S (1997), and D.S and Pal (2001).

We denote the finite singularities with lower case letters and the infinite ones with capital letters placing first the finite ones, then the infinite ones, separating them by a semicolon‘;’.

Elemental points: We use the letters ‘s’,‘S’ for “saddles”; ‘n’, ‘N’ for “nodes”; ‘f’ for “foci”; ‘c’ for “centers” and © (respectively ©) for complex finite (respectively infinite) singularities. We distinguish the finite nodes as follows:

- ‘n’ for a node with two distinct eigenvalues (generic node);

- ‘n^d’ (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;

- 'n^*' (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. We will denote them as 'N^∞' and 'N^f' respectively.

When the trace of the Jacobian matrix of elemental saddle and focus is zero, in the quadratic case, one may have up to 3 finite orders. We denote them by '$s^{(i)}$' and '$f^{(i)}$' where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by 'c' and saddles of infinite order (integrable saddles) which we denote by '\$'.

To define the notion of **geometric configuration** of singularities we distinguish two cases:

Case 1) in which we have a finite number of infinite singular points. Then we call **configuration of singularities**, finite and infinite, the set of all these singularities each endowed with its own multiplicity together with their local phase portraits endowed with **additional geometric structure** involving the concepts of **tangent, order and blow–up equivalences**.

Let χ_1 and χ_2 be two polynomial vector fields each having a finite number of singularities. We say that χ_1 and χ_2 have **geometric equivalent configurations of singularities** if and only if we have a bijection ϑ carrying the singularities of χ_1 to singularities of χ_2 and for every singularity p of χ_1, $\vartheta(p)$ is **geometrically equivalent** with p.

2) If the line at infinity $Z = 0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the system is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z = 0$.

In case 2) We call **configuration of singularities**, finite and infinite, the union of the set of all points at infinity (they are all singularities) with the set of finite singularities, taking care of singling out the singularities of the "reduced" system at infinity, taken together with the local phase portraits of finite singularities endowed with additional geometric structure as above and of the infinite singularities of the reduced system.