Singularities of Implicit Differential Equations and Static Bifurcations

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- *fibred manifold:* $\pi : \mathcal{E} \to \mathcal{T}$ with $\dim \mathcal{T} = 1$
  - trivial case: $\mathcal{E} = \mathcal{T} \times \mathcal{U}$, $\pi = \mathrm{pr}_1$
  - local coordinates: $(t, \mathbf{u})$
    (*independent variable* t, *dependent variables* $\mathbf{u}$)
- *section:* smooth map $\sigma : \mathcal{T} \to \mathcal{E}$ with $\pi \circ \sigma = \mathrm{id}$
  (locally: $\sigma(t) = (t, \mathbf{s}(t))$ with function $\mathbf{s} : \mathcal{T} \to \mathcal{U}$)
- q*-jet* : class of all sections with same Taylor polynomial of degree q
- *jet bundle* $\mathcal{J}_q\pi$: set of all q-jets $[\sigma]_t^{(q)}$
  - local coordinates: $(t, \mathbf{u}^{(q)})$ (derivatives up to order q)
  - natural hierarchy with projections

$$\pi^q_r : \mathcal{J}_q\pi \longrightarrow \mathcal{J}_r\pi \qquad 0 \leq r < q$$

$$\pi^q : \mathcal{J}_q\pi \longrightarrow \mathcal{T}$$

- *ordinary differential equation of order* q $\rightsquigarrow$ submanifold $\mathcal{R}_q \subseteq \mathcal{J}_q\pi$ such that $\operatorname{im} \pi^q|_{\mathcal{R}_q}$ dense in $\mathcal{T}$
  - ☐ no conditions on independent variable
  - ☐ no distinction *scalar equation* or *system*
  - ☐ basic assumption: equation *formally integrable*
- *prolongation* of section $\sigma : \mathcal{T} \to \mathcal{E}$ $\rightsquigarrow$ section $j_q\sigma : \mathcal{T} \to \mathcal{J}_q\pi$

$$j_q\sigma(t) = \big(t, \mathbf{s}(t), \dot{\mathbf{s}}(t), \dots, \mathbf{s}^{(q)}(t)\big)$$

- *classical solution*: section $\sigma : \mathcal{T} \to \mathcal{E}$ such that $\operatorname{im}(j_q\sigma) \subseteq \mathcal{R}_q$

Def: *contact distribution* $\mathcal{C}_q \subset T(\mathcal{J}_q\pi)$ generated by vector fields

$$C^{(q)}_{\mathrm{trans}} = \partial_t + \sum_{\alpha=1}^{m}\sum_{j=0}^{q-1} u^\alpha_{j+1}\partial_{u^\alpha_j}$$

$$C^{(q)}_{\alpha} = \partial_{u^\alpha_q} \qquad 1 \leq \alpha \leq m$$

Prop: section $\gamma : \mathcal{T} \to \mathcal{J}_q\pi$ of the form $\gamma = j_q\sigma$ $\Longleftrightarrow$ $T\mathrm{im}(\gamma) \subset \mathcal{C}_q$

Proof: *chain rule!*

Consider prolonged solution $j_q\sigma$ of equation $\mathcal{R}_q \subseteq \mathcal{J}_q\pi$:

- *integral elements* $\rightsquigarrow$ $T_\rho\bigl(\mathrm{im}(j_q\sigma)\bigr)$ for $\rho \in \mathrm{im}(j_q\sigma)$
- solution $\Longrightarrow$ $T_\rho\bigl(\mathrm{im}(j_q\sigma)\bigr) \subseteq T_\rho\mathcal{R}_q$
- prolonged section $\Longrightarrow$ $T_\rho\bigl(\mathrm{im}(j_q\sigma)\bigr) \subseteq \mathcal{C}_q|_\rho$

Def: *Vessiot space* at point $\rho \in \mathcal{R}_q$

$$\mathcal{V}_\rho[\mathcal{R}_q] = T_\rho\mathcal{R}_q \cap \mathcal{C}_q|_\rho$$

- generally: $\dim \mathcal{V}_\rho[\mathcal{R}_q]$ depends on ρ $\rightsquigarrow$ regular distribution only on open subset of $\mathcal{R}_q$
- computing Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ corresponds to “projective” form of prolonging from $\mathcal{R}_q$ to $\mathcal{R}_{q+1}$
- computation requires only linear algebra

Consider *square* first-order ordinary differential equation $\mathcal{R}_1 \subset \mathcal{J}_1\pi$ with local representation $\mathbf{\Phi}(t, \mathbf{u}, \dot{\mathbf{u}}) = 0$ where $\mathbf{\Phi} : \mathcal{J}_1\pi \to \mathbb{R}^m$

- define $m \times m$ matrix A and m-dimensional vector $\mathbf{d}$

$$A = \mathbf{C}^{(1)}\mathbf{\Phi} = \frac{\partial \mathbf{\Phi}}{\partial \dot{\mathbf{u}}} \qquad \mathbf{d} = C^{(1)}_{\text{trans}}\mathbf{\Phi} = \frac{\partial \mathbf{\Phi}}{\partial t} + \frac{\partial \mathbf{\Phi}}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}}$$

- assume A almost everywhere non-singular (i. e. given equation *not* underdetermined)
- compute determinant $\delta = \det A$ and adjugate $C = \operatorname{adj} A$

Lemma: $\mathcal{V}[\mathcal{R}_1]$ almost everywhere generated by single vector field

$$X = \delta C^{(1)}_{\text{trans}} - (C\mathbf{d})^T \mathbf{C}^{(1)}$$

Def: ordinary differential equation $\mathcal{R}_q \subseteq J_q\pi$

generalised solution $\rightsquigarrow$ integral curve $\mathcal{N} \subseteq \mathcal{R}_q$ of $\mathcal{V}[\mathcal{R}_q]$

geometric solution $\rightsquigarrow$ projection $\pi_0^q(\mathcal{N})$ of generalised solution $\mathcal{N}$

- geometric solution in general *not* image of a section (thus *no* interpretation as a function!)
- geometric solution $\pi_0^q(\mathcal{N})$ is classical solution $\Longleftrightarrow$ $\mathcal{N}$ everywhere transversal to π^q
- geometric solutions allow for for modelling of *multi-valued* solutions (e.g. “breaking waves”)

Formally integrable ordinary differential equation: $\mathcal{R}_q \subset \mathcal{J}_q\pi$

local description: $\mathbf{\Phi}(t, \mathbf{u}^{(q)}) = 0 \quad (\dim \mathbf{u} = m)$

$\mathcal{R}_q$ of *finite type* $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Def: $\rho \in \mathcal{R}_q$ *geometric singularity* $\rightsquigarrow$ ρ critical point of $\pi^q_0|_{\mathcal{R}_q}$

point $\rho \in \mathcal{R}_q$ is called

- *regular* $\rightsquigarrow$ $\mathcal{V}_\rho[\mathcal{R}_q]$ 1-dimensional and transversal to π^q
- *regular singular* $\rightsquigarrow$ $\mathcal{V}_\rho[\mathcal{R}_q]$ 1-dimensional and *not* transversal to π^q
- *irregular singular* (s-*singular*) $\rightsquigarrow$ $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1 + s$ with $s > 0$

(regular) singularities are also called *impasse points*

Formally integrable ordinary differential equation: $\mathcal{R}_q \subset \mathcal{J}_q\pi$

local description: $\mathbf{\Phi}(t, \mathbf{u}^{(q)}) = 0 \quad (\dim \mathbf{u} = m)$

$\mathcal{R}_q$ of *finite type* $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Prop: point $\rho \in \mathcal{R}_q$

- ρ regular $\Longleftrightarrow$ $\operatorname{rank} \left(\mathbf{C}^{(q)}\mathbf{\Phi}\right)_\rho = m$
- ρ regular singular $\Longleftrightarrow$ ρ not regular and

$$\operatorname{rank} \left(\mathbf{C}^{(q)}\mathbf{\Phi} \mid C^{(q)}_{\text{trans}}\mathbf{\Phi}\right)_\rho = m$$

Formally integrable ordinary differential equation: $\mathcal{R}_q \subset \mathcal{J}_q\pi$

local description: $\mathbf{\Phi}(t, \mathbf{u}^{(q)}) = 0 \quad (\dim \mathbf{u} = m)$

$\mathcal{R}_q$ of *finite type* $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Thm: assume $\mathcal{R}_q$ has *no* irregular singularities

- $\rho \in \mathcal{R}_q$ regular point $\Longrightarrow$
  - **(i)** unique classical solution σ exists with $\rho \in \operatorname{im} j_q\sigma$
  - **(ii)** solution σ can be extended in any direction until $j_q\sigma$ reaches either boundary of $\mathcal{R}_q$ or a regular singularity
- $\rho \in \mathcal{R}_q$ regular singularity $\Longrightarrow$ dichotomy
  - **(i)** either *two* classical solutions σ_1, σ_2 exist with $\rho \in \operatorname{im} j_q\sigma_i$ (both ending or both starting in ρ)
  - **(ii)** or *one* classical solution σ exists with $\rho \in \operatorname{im} j_q\sigma$ whose derivative of order $q+1$ blows up at $x = \pi^q(\rho)$

Formally integrable ordinary differential equation: $\mathcal{R}_q \subset \mathcal{J}_q\pi$

local description: $\mathbf{\Phi}(t, \mathbf{u}^{(q)}) = 0 \quad (\dim \mathbf{u} = m)$

$\mathcal{R}_q$ of *finite type* $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Proof: $\mathcal{V}[\mathcal{R}_q]$ locally generated by vector field X

ρ regular singularity $\Longrightarrow$ X vertical wrt π^q

dichotomy $\rightsquigarrow$ ∂_t-component of X does or does not change sign at ρ

Formally integrable ordinary differential equation: $\mathcal{R}_q \subset \mathcal{J}_q\pi$

local description: $\mathbf{\Phi}(t, \mathbf{u}^{(q)}) = 0$ $(\dim \mathbf{u} = m)$

$\mathcal{R}_q$ of *finite type* $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

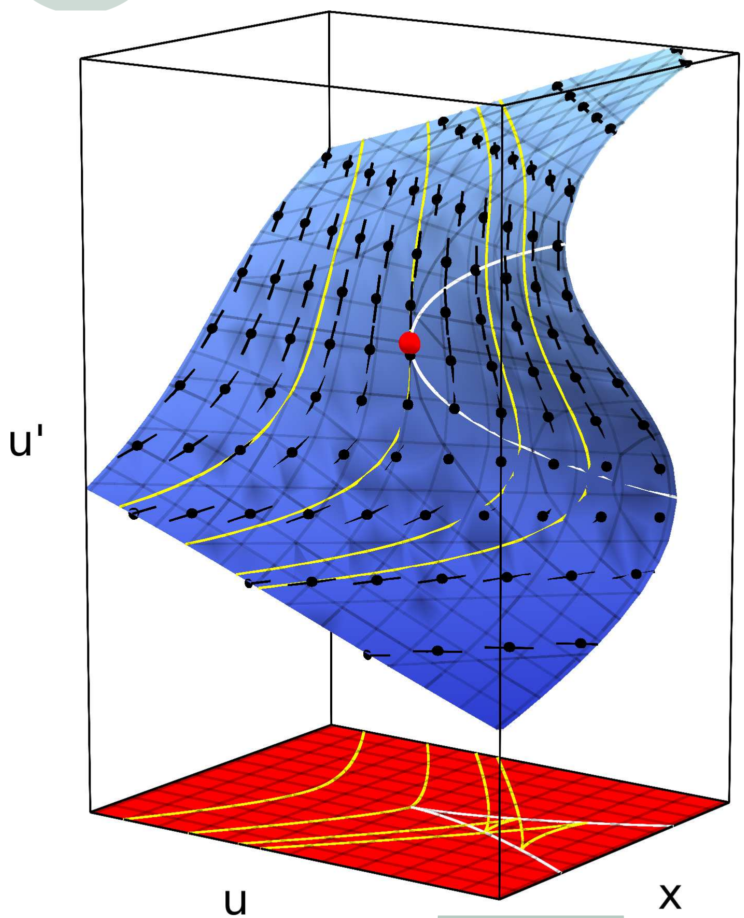
let $\rho \in \mathcal{R}_q$ be an *irregular* singularities

- consider simply connected open set $\mathcal{U} \subset \mathcal{R}_q$ without any irregular singularities such that $\rho \in \overline{\mathcal{U}}$
- in $\mathcal{U}$ Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ generated by single vector field X

Thm: any smooth extension of X vanishes at ρ

Consequence: behaviour of solutions in neighbourhood of isolated irregular singularity ρ determined by *eigenstructure* of $\mathrm{Jac}_\rho\, X$

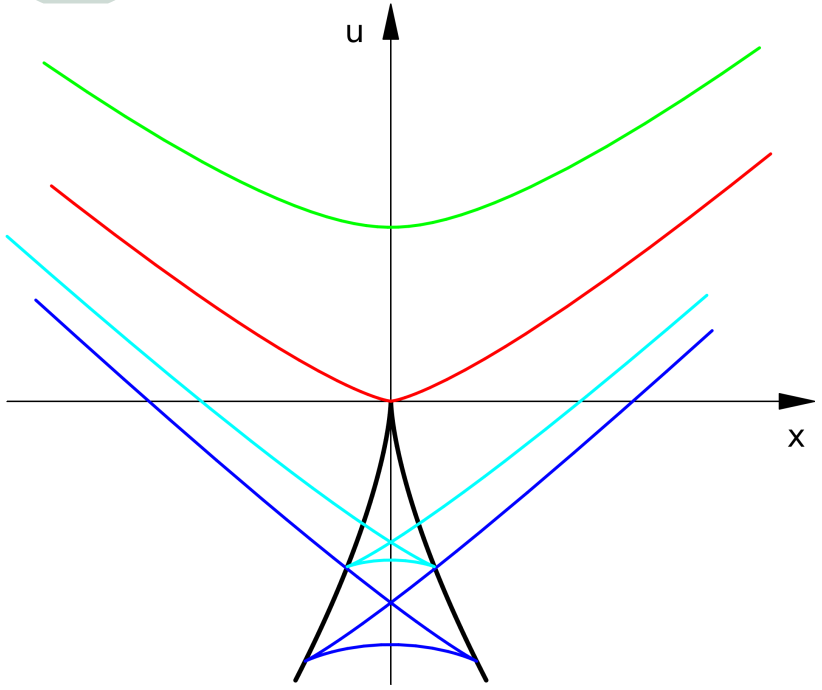
Example: $\dot{u}^3 + u\dot{u} - t = 0$ *(hyperbolic gather)*



singularity manifold (*criminant*):

$3\dot{u}^2 + u = 0$

Example: $\dot{u}^3 + u\dot{u} - t = 0$ *(hyperbolic gather)*

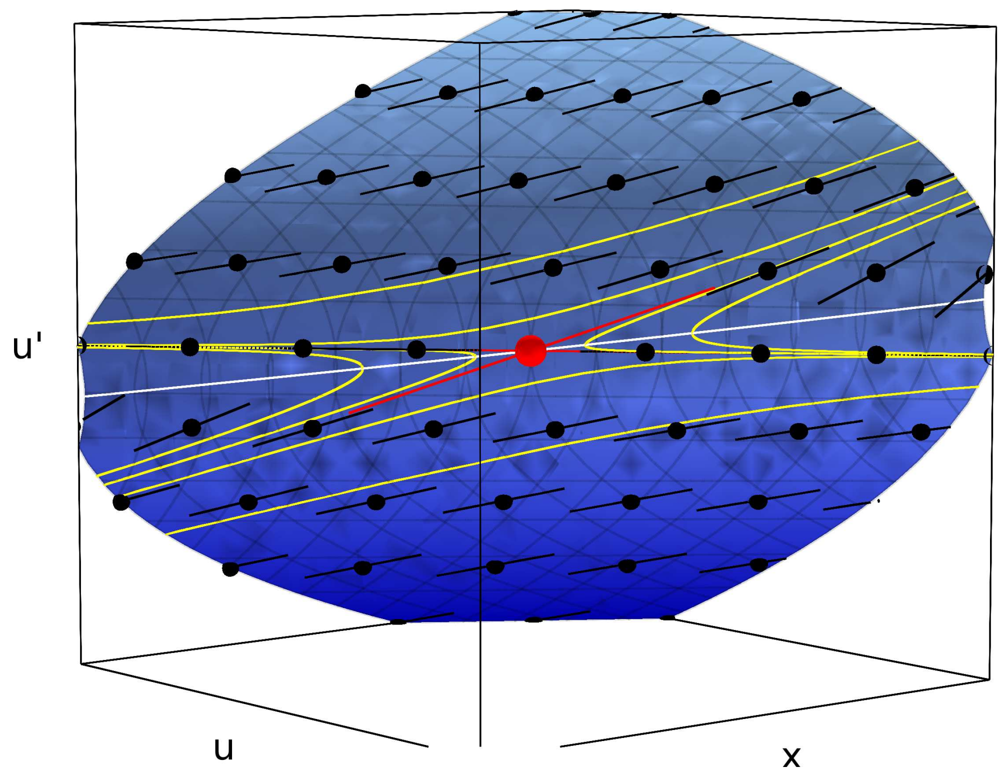


second derivative of solution touching "tip" of discriminant blows up

when crossing discriminant solutions "change direction"

Example: $\dot{u}^3 + u\dot{u} - t = 0$ *(hyperbolic gather)*

neighbourhood of an irregular singularity ρ *(folded saddle)*



generalised solutions tangential to eigenvectors of $\operatorname{Jac}_\rho X$ *intersect* in ρ

Prop: $\pi^q_{q-1} : \mathcal{J}_q\pi \to \mathcal{J}_{q-1}\pi$ affine bundle

Def: $\mathcal{R}_q \subseteq \mathcal{J}_q\pi$ *quasi-linear* equation $\rightsquigarrow$ $\mathcal{R}_q$ affine subbundle

In the sequel: square first-order equation

$$A(t,\mathbf{u})\dot{\mathbf{u}} = \mathbf{r}(t,\mathbf{u})$$

(as before: $\delta = \det A$, $C = \operatorname{adj} A$)

Lemma: Outside of irregular singularities, vector field X generating $\mathcal{V}[\mathcal{R}_1]$ *projectable* to vector field $Y \in \mathfrak{X}(\pi^1_0(\mathcal{R}_1))$ and

$$Y = (\pi^1_0)_* X = \delta\partial_t + (C\mathbf{r})^T\partial_{\mathbf{u}}$$

Note: Y *extendable* to any point $\xi \in \mathcal{E}$ where A, $\mathbf{r}$ defined

Def: point $\xi \in \mathcal{E}$ such that Y_ξ defined

- *regular* $\rightsquigarrow$ Y_ξ transversal to $\pi : \mathcal{E} \longrightarrow \mathcal{T}$
- *irregular impasse point* $\rightsquigarrow$ $Y_\xi = 0$
- *regular impasse point* $\rightsquigarrow$ otherwise

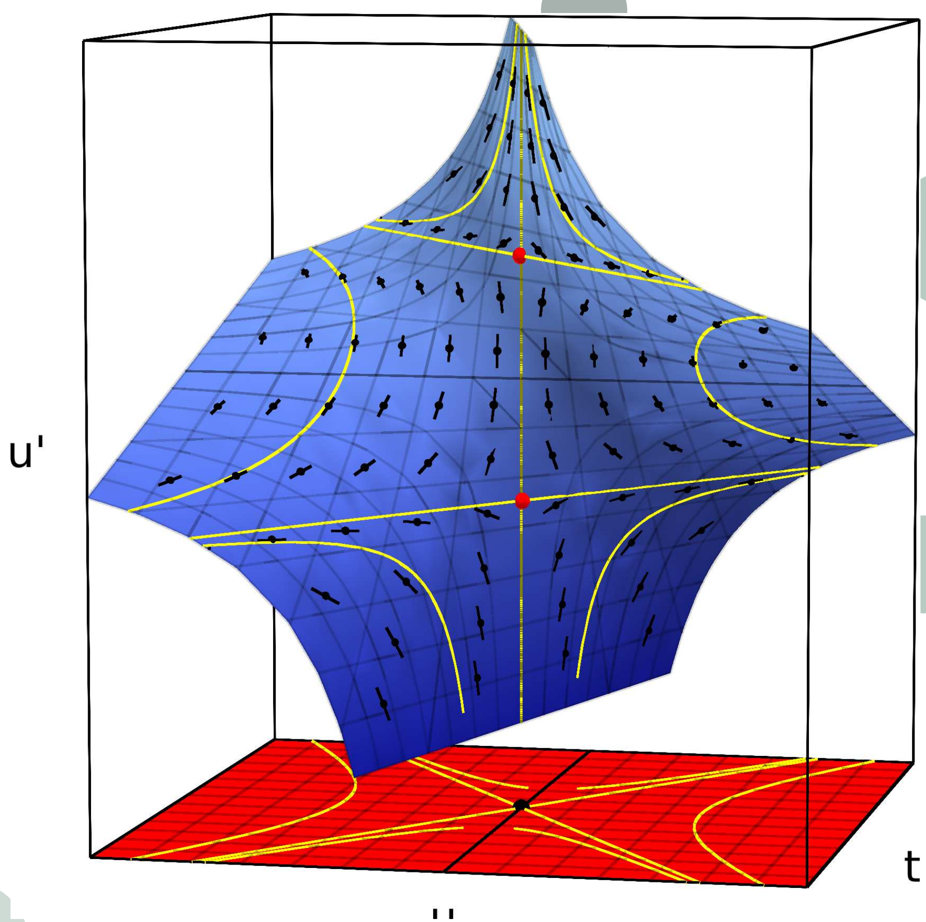
geometric solution $\rightsquigarrow$ integral curve of Y

Prop: point $\xi \in \mathcal{E}$ such that Y_ξ defined

- ξ regular $\Longleftrightarrow$ $\operatorname{rank} A(\xi) = m$
- ξ regular impasse point $\Longleftrightarrow$ $\operatorname{rank} A(\xi) = m - 1$ and $\mathbf{r}(\xi) \notin \operatorname{im} A(\xi)$

Note: $\mathbf{r}(\xi) \notin \operatorname{im} A(\xi)$ means $\xi \notin \pi_0^1(\mathcal{R}_1)$

Example: $2u\dot{u} - t = 0$



singularity manifold:

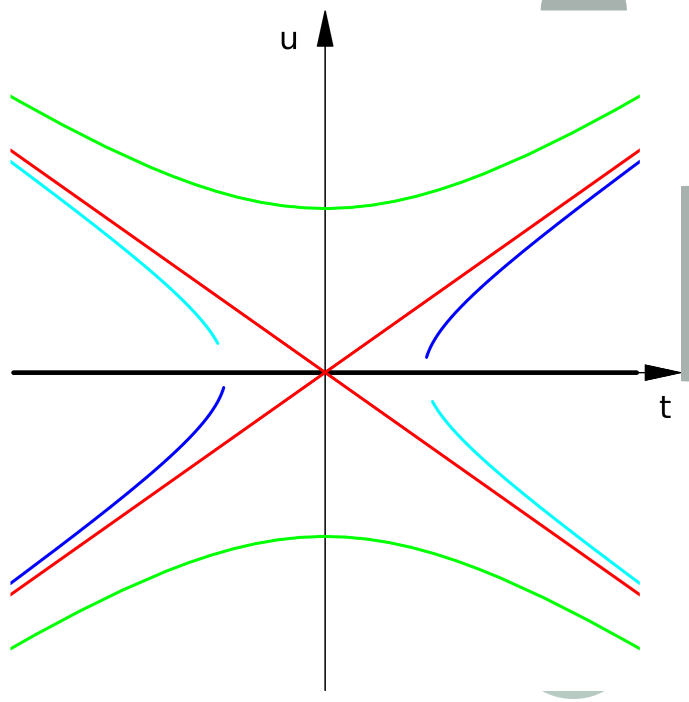
$t = u = 0$

(generalised solution!)

irregular singularities:

$\dot{u} = \pm 1/\sqrt{2}$

Example: $2u\dot{u} - t = 0$



impasse manifold $\rightsquigarrow$ t-axis
origin *irregular* impasse point

solutions through origin regular, but *intersecting*

Consider *parametrised* algebraic equation for $\mathbf{u}$: $\varphi(t,\mathbf{u}) = 0 \quad \rightsquigarrow$ how do solutions change when parameter t varies?

Def: point $\xi = (t, \mathbf{u}) \in \mathcal{E}$

- *turning point*, if

$$\varphi(\xi) = 0 \quad \dim\ker\frac{\partial\varphi}{\partial\mathbf{u}}(\xi) = 1 \quad \frac{\partial\varphi}{\partial t}(\xi) \notin \operatorname{im}\frac{\partial\varphi}{\partial\mathbf{u}}(\xi)$$

- *bifurcation point*, if

$$\varphi(\xi) = 0 \quad \dim\ker\frac{\partial\varphi}{\partial\mathbf{u}}(\xi) = 1 \quad \frac{\partial\varphi}{\partial t}(\xi) \in \operatorname{im}\frac{\partial\varphi}{\partial\mathbf{u}}(\xi)$$

(Literature: further distinction into *simple* and *higher* turning/bifurcation points — here irrelevant)

Consider *parametrised* algebraic equation for $\mathbf{u}$: $\boldsymbol{\varphi}(t,\mathbf{u}) = 0$ $\rightsquigarrow$
how do solutions change when parameter t varies?

try to construct bifurcation diagram as graph of function $\mathbf{u}(t)$ $\rightsquigarrow$
quasi-linear differential equation

$$\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{u}}(t,\mathbf{u})\dot{\mathbf{u}} + \frac{\partial \boldsymbol{\varphi}}{\partial t}(t,\mathbf{u}) = 0$$

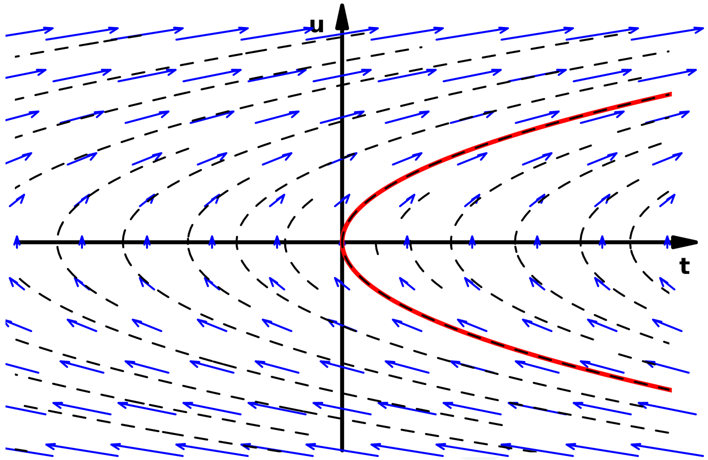
Prop:

- ξ turning point $\Longleftrightarrow$ ξ regular impasse point
- ξ bifurcation point $\Longleftrightarrow$
  ξ irregular impasse point with $\operatorname{rank} A(\xi) = m - 1$

  (each branch tangent to real eigenvector of $\operatorname{Jac} Y_\xi$)

Simple turning point

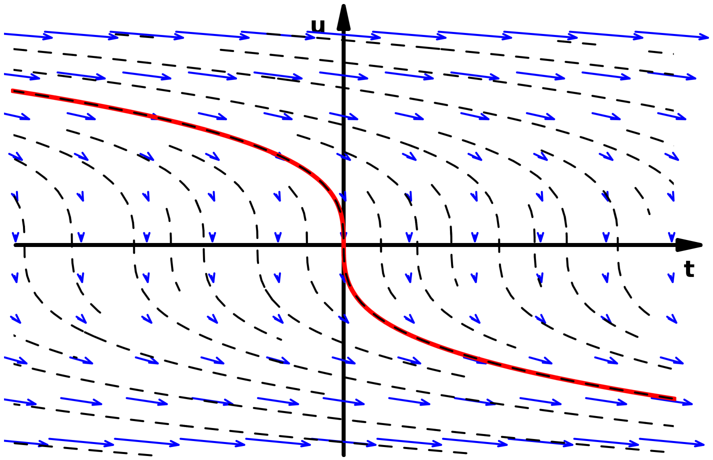
$$\varphi(t,u) = t - u^2$$



$$Y = 2u\partial_t + \partial_u$$

Hysteresis point

$$\varphi(t,u) = t - u^3$$

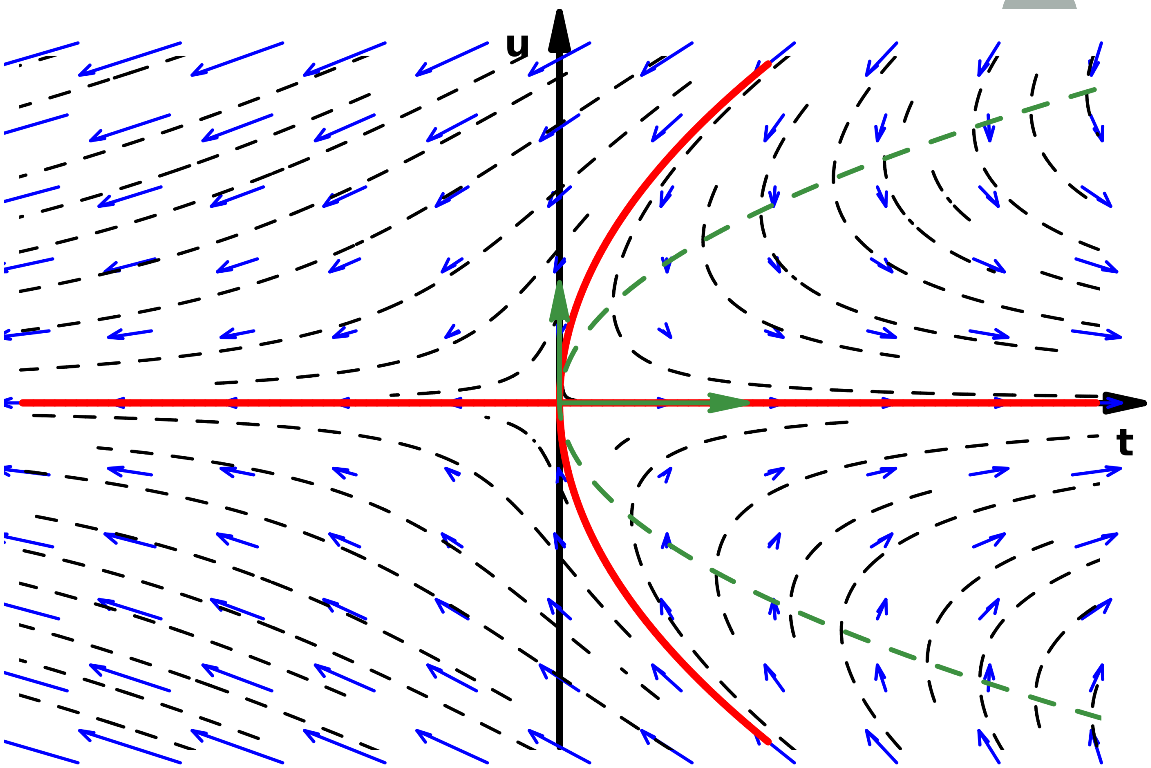


$$Y = 3u^2\partial_t + \partial_u$$

in *numerical* computations

- no difference in determining *integral curves*
- hysteresis point *multiple* zero of ∂_t-component of Y

Pitchfork bifurcation $\varphi(t,u) = tu - u^3$



$$Y = (t - 3u^2)\partial_t - u\partial_u \qquad \operatorname{Jac} Y_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

non-generic bifurcation ⇝ two simple turning points collide