On Stationary Sets of Euler's Equations on $so(3,1)$ and their Stability

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Abstract. With the use of computer algebra methods we investigate two recent found cases of integrability of Euler's equations on the Lie algebra $so(3,1)$ when the equations possess additional polynomial first integrals of degrees 3 and 6. The problems of obtaining stationary sets of the equations and investigation of their stability are considered. In addition to the sets obtained earlier [1], we have found new zero-dimensional and nonzero-dimensional stationary sets. For a number of the sets we have derived sufficient conditions of their stability and instability.

Keywords: Euler's equations, stationary sets, stability

1 Introduction

It is known that configuration space in dynamics of a rigid body is, as a rule, some natural Lie group $SO(3), E(3), SO(4)$, etc. In [2], the connection of classical Euler's equations, describing the motion of a rigid body, with Lie algebras is shown. Many problems of mechanics, mathematical physics and etc. [3] reduce to Euler's equations on Lie algebras. These equations are also a good model for the study of singularities of integrable Hamiltonian systems. Therefore an increased interest takes place to systems of a such type, in particular, finding new cases of integrability of the systems and qualitative analysis of these cases. So, the works [6]-[8] are devoted to investigation of new integrable cases of Euler's equations on Lie algebras $e(3), so(4)$ [4], [5] when the equations admit additional polynomial first integral of 4th degree. In these works, a topological analysis of the cases has been conducted.

In the present paper, two recent found cases of integrability of Euler's equations on the Lie algebra $so(3,1)$, when the equations possess additional polynomial first integrals of degrees 3 and 6 [9], [10], are investigated. We find peculiar solutions (stationary sets) of these equations and investigate their stability. For solving these problems, we apply a technique based on computer algebra methods. The latter allows us not only to obtain desired solutions in an analytical form, but and to investigate their stability, e.g., by Lyapunov's methods. In [1], some general analysis of stationary sets of the equations considered has been conducted. Nonzero-dimensional stationary sets (stationary invariant manifolds

(IMs)) have been found, their stability and bifurcations have been investigated. It was shown that the equations have zero-dimensional stationary sets in capacity of their solutions. In the given paper, we have found these sets and investigated their stability. We have also found complex IMs which possess, in our opinion, an interest property: under some additional constraints, the motion on these manifolds is described by real functions of time. For the purpose of solving computational problems, we used the computer algebra systems (CAS) *Mathematica* and *Maple* [11].

2 On Stationary Sets of Euler's Equations with an additional cubic first integral

2.1 Problem Formulation

In [10], the problem of revealing integrable equations in the family of Euler's equations with Hamiltonians of the form

$$H = c_1\,\alpha_3(M_1^2+M_2^2+M_3^2)+c_2M_3(\alpha_1M_1+\alpha_3M_3)+M_1\gamma_2 - M_2\gamma_1 \qquad (1)$$

is considered. Here M_i, γ_i $(i = 1, 2, 3)$ are the components of the two 3-dimensional vectors, α_j, c_j are some constants.

It is noted that Hamiltonians (1) have numerous applications (two-spin interactions, motion of a three dimensional rigid body in a constant-curvature space or in an ideal fluid, motion of a body with ellipsoidal cavity filled with fluid around a fixed point, etc.).

Several integrable cases of the equations discussed were found. The Hamiltonians corresponding to them are separated from family (1) by the constraints imposed on parameters c_1 and c_2: $(a)\,c_1$ is arbitrary, $c_2 = 0$; $(b)\,c_1 = 1, c_2 = -2$; $(c)\,c_1 = 1, c_2 = -1$; $(d)\,c_1 = 1, c_2 = -1/2$. In the present paper, cases (b) and (d) are studied.

According to [10], in integrable case (b) there exists an additional cubic integral of the form

$$\begin{aligned} F = \{&2[\alpha_3(M_1\gamma_2 - M_2\gamma_1) + \alpha_1(M_2\gamma_3 - M_3\gamma_2)] - k(M_1^2 + M_2^2 + M_3^2) \\ &+\gamma_1^2 + \gamma_2^2 + \gamma_3^2\}M_3 = h_1 = const. \end{aligned} \qquad (2)$$

The latter integral has been found earlier in [9].

Euler's equations corresponding to the Hamiltonian (b) write

$$\begin{aligned} &\dot M_1 = 2M_2(\alpha_1M_1+2\alpha_3M_3)-(M_3\gamma_1-M_1\gamma_3), \\ &\dot M_2 = M_2\gamma_3-M_3\gamma_2-2\big[\alpha_1(M_1^2-M_3^2)+2\alpha_3M_1M_3\big], \quad \dot M_3 = -2\alpha_1M_2M_3, \\ &\dot\gamma_1 = 2(\alpha_1M_1+\alpha_3M_3)\,\gamma_2+(2\alpha_3M_2-\gamma_1)\,\gamma_3+kM_1M_3, \\ &\dot\gamma_2 = 2\big[(\alpha_1M_3-\alpha_3M_1)\,\gamma_3-(\alpha_1M_1+\alpha_3M_3)\,\gamma_1\big]+kM_2M_3-\gamma_2\gamma_3, \\ &\dot\gamma_3 = 2\big[\alpha_3(M_1\gamma_2 - M_2\gamma_1)-\alpha_1M_3\gamma_2\big]+\gamma_1^2+\gamma_2^2-k(M_1^2+M_2^2). \end{aligned} \qquad (3)$$

The rest of the integrals of equations (3) has the form:

$$V_1 = \sum_{i=1}^{3} M_i\gamma_i = h_2 = const, \quad V_2 = \sum_{i=1}^{3}(kM_i^2 + \gamma_i^2) = h_3 = const. \tag{4}$$

We state the problem of finding stationary sets of equations (3) and investigation of their stability in the sense of Lyapunov.

2.2 Finding Stationary Sets

In order to solve the stated problem, we shall apply the Routh-Lyapunov method [12] and some its generalizations (see [13]). This method in combination with computer algebra tools allows one not only to find desired solutions, but and to investigate their stability. According to the method, stationary invariant sets of the above differential equations are called solutions of conditional extremum problem for the elements of algebra of the first integrals of these equations. To obtain these sets, some linear or nonlinear combination from the problem's first integrals (a family of the first integrals) is constructed and necessary conditions for this family to have an extremum with respect to phase variables are written. Thereby, the problem of finding stationary invariant sets for the system of differential equations with polynomial first integrals is reduced to obtaining solutions of some algebraic system. In our case, it will be a system of nonlinear equations.

Following the technique chosen, we construct the complete linear combination

$$K = \lambda_0 H - \lambda_1 V_1 - \frac{\lambda_2}{2} V_2 - \lambda_3 F \tag{5}$$

from the first integrals of the problem, and write down the necessary conditions for the integral K to have an extremum with respect to phase variables M_i, γ_i:

$$\begin{aligned}
&\partial K/\partial M_1 = \lambda_0\gamma_2 - \lambda_1\gamma_1 + (2\alpha_3\lambda_0 - k\lambda_2)M_1 - 2\,[\alpha_1\lambda_0 + \lambda_3(\alpha_3\gamma_2 - kM_1)]M_3 = 0,\\
&\partial K/\partial M_2 = -\lambda_0\gamma_1 - \lambda_1\gamma_2 + (2\alpha_3\lambda_0 - k\lambda_2)M_2 + 2\lambda_3\,(\alpha_3\gamma_1 + kM_2 - \alpha_1\gamma_3)M_3 = 0,\\
&\partial K/\partial M_3 = -\lambda_1\gamma_3 - 2\alpha_1\lambda_0 M_1 - (2\alpha_3\lambda_0 + k\lambda_2)M_3 + \lambda_3\,[k(M_1^2 + M_2^2 + 3M_3^2)\\
&\qquad +2\,(\alpha_3\gamma_1 - \alpha_1\gamma_3)M_2 + 2\,(2\alpha_1 M_3 - \alpha_3 M_1)\gamma_2 - (\gamma_1^2 + \gamma_2^2 + \gamma_3^2)] = 0, \quad (6)\\
&\partial K/\partial \gamma_1 = -\lambda_1 M_1 - \lambda_0 M_2 - \lambda_2\gamma_1 + 2\lambda_3(\alpha_3 M_2 - \gamma_1)M_3 = 0,\\
&\partial K/\partial \gamma_2 = \lambda_0 M_1 - \lambda_1 M_2 - \lambda_2\gamma_2 + 2\lambda_3\,[(\alpha_1 M_3 - \alpha_3 M_1) - \gamma_2]M_3 = 0,\\
&\partial K/\partial \gamma_3 = -\lambda_1 M_3 - \lambda_2\gamma_3 - 2\lambda_3(\alpha_1 M_2 + \gamma_3)M_3 = 0.
\end{aligned}$$

Here $\lambda_i = const$ are the family parameters of the integral K.

Equations (6) (the equations of stationarity of the integral K) represent a system of polynomial equations of 2nd degree with parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \alpha_1, \alpha_3$. It should be noted that if some part of parameters λ_i in K assumes zero values then we obtain an "incomplete" combination of the integrals. In this case, both stationary equations and solutions of these equations correspond to this "incomplete" combination of the integrals.

We applied computer algebra methods for qualitative analysis of the solution set of equations (6) and for finding solutions of these equations.

It was shown in [1], equations (6) have a finite number of solutions (6 solutions) over field $Q(\lambda_i, \alpha_j)[M_1, M_2, M_3, \gamma_1, \gamma_2, \gamma_3]$. The latter was revealed by the programs *IsZeroDimentional, NumberOfSolutions* which are included in the *Maple*-program package *PolynomialIdeals*.

The program *IsPrime* (which is also part of this package) allowed us to find out that system (6) can be decomposed into more "simple" subsystems over the above field. To this end, we applied triangular sets method [14]. The method decomposes the algebraic variety of a polynomial system into subvarieties which correspond to one or several solutions of the system. A special form, called "triangular set", is used for representing solutions. In the problem considered, application of this method has not caused any computational difficulties. We used the *Maple*-program *Triangularize*. The result of application of this program to system (6) writes

$$\begin{array}{l}\big[\,[\,\gamma_1, \gamma_2, \gamma_3, M_1, M_2, M_3],\\ [\,M_2,\, \gamma_1(2\lambda_3 M_3 + \lambda_2) + \lambda_1 M_1,\, \gamma_3(2\lambda_3 M_3 + \lambda_2) + \lambda_1 M_3,\\ \quad \gamma_2(2\lambda_3 M_3 + \lambda_2) + M_1(2\alpha_3\lambda_3 M_3 - \lambda_0) - 2\alpha_1\lambda_3 M_3^2,\\ \quad 2M_1\,(\alpha_1^2(4\lambda_3^2 M_3^2 - \lambda_2^2) - (\lambda_0 + \alpha_3\lambda_2)^2 - \lambda_1^2)\\ \quad +\alpha_1(4\alpha_3\lambda_3^2 M_3^2 + \lambda_0(2\lambda_3 M_3 + \lambda_2))M_3,\\ \quad a_0 M_3^5 + a_1 M_3^4 + a_2 M_3^3 + a_3 M_3^2 + a_4 M_3 + a_5]\,\big].\end{array} \tag{7}$$

Here $a_0, \ldots, a_5$ are some expressions of λ_i, α_j.

The program result is a list of the system solutions represented in "triangular" form. The first element of the list defines a trivial solution of system (6), the 2nd element defines five solutions of this system, because the polynomial of variable M_3 has five roots. We have found multiple roots of this polynomial. The conditions of existence of such roots can be obtained as conditions under which the resultant of two polynomials (the polynomial considered and its derivative) vanishes. The found conditions in the form of restrictions imposed on parameters λ_i, α_j and the solutions, corresponding to them, are given below.

i) $\lambda_1 = 0, \alpha_3 = 0:$

$$\gamma_1 = 0,\ \gamma_2 = -2\alpha_1\lambda_2\lambda_3^{-1},\ \gamma_3 = 0,\ M_1 = 0,\ M_2 = 0,\ M_3 = -\lambda_2\lambda_3^{-1}; \tag{8}$$

ii) $\lambda_1 = 0, \alpha_3 = \alpha_1:$

$$\gamma_1 = 0,\ \gamma_2 = 2(\alpha_1\lambda_2 + \lambda_0)\lambda_3^{-1},\ \gamma_3 = 0,\ M_1 = -2(\alpha_1\lambda_2 + \lambda_0)\,\alpha_1^{-1}\lambda_3^{-1},\ M_2 = 0,\ M_3 = (\alpha_1\lambda_2 + \lambda_0)\,\alpha_1^{-1}\lambda_3^{-1}; \tag{9}$$

iii) $\lambda_2 = 0, \alpha_1 = 0:$

$$\gamma_1 = 0,\ \gamma_2 = 0,\ \gamma_3 = -\lambda_1\,\lambda_3^{-1}/2,\ M_1 = 0,\ M_2 = 0,\ M_3 = -(2\lambda_0 \pm \sqrt{4\lambda_0^2 + 3\lambda_1^2})\,\alpha_3^{-1}\,\lambda_3^{-1}/6; \tag{10}$$

iv) $\lambda_0 = \dfrac{\lambda_1^2}{2\alpha_3\lambda_2} + \dfrac{1}{2}\alpha_3\lambda_2,\ \alpha_1 = 0:$

$$\gamma_1 = 0,\ \gamma_2 = 0,\ \gamma_3 = -(2\lambda_1 \pm \sqrt{\lambda_1^2 - 3\alpha_3^2\lambda_2^2}\,)\,\lambda_3^{-1}/2,\ M_1 = 0,$$

$$M_2 = 0,\ M_3 = -(\lambda_1^2 + 3\alpha_3^2\lambda_2^2 \mp \lambda_1\sqrt{\lambda_1^2 - 3\alpha_3^2\lambda_2^2})\,\alpha_3^{-2}\lambda_2^{-1}\lambda_3^{-1}/6. \qquad (11)$$

The above expressions are the families of the stationary solutions of differential equations (3) parameterized by λ_i. The latter is proved by direct substitution of (8)–(11) into equations (3) and (6). As a result, these equations turn into identities.

Geometrically, the elements of the families of real solutions (8)-(11) correspond to the families of points in R^6. According to [10], from a mechanical point of view, under corresponding interpretation of the parameters and the phase variables, these elements correspond to helical motions of a rigid body in fluid (generalized Kirchhoff's model), and permanent rotations of a rigid body with cavity filled with fluid (generalized Poincaré's model).

Next, we apply the triangular sets method for finding of nonzero-dimensional stationary sets. Likewise [1], we consider some part of the phase variables and some part of the parameters, e.g., $\gamma_1, \gamma_2, \gamma_3, M_1, M_2, \lambda_1$, in capacity of unknowns. The *Maple*-programs *IsZeroDimension, HilbertDimention, IsPrime* have revealed that equations (6) have infinite number of solutions with respect to the unknowns, the dimension of these solutions is 1, and the algebraic variety of the equations can be decomposed into subvarieties. Below, the result of application of the program *Triangularize* to system (6), when $\gamma_1, \gamma_2, \gamma_3, M_1, M_2, \lambda_1$ are the unknowns, is given.

$$\begin{aligned}
&[\,[\,\gamma_1(2\lambda_3 M_3 + \lambda_2) + \lambda_1 M_1 - M_2(2\alpha_3\lambda_3 M_3 - \lambda_0),\\
&\quad \gamma_2(2\lambda_3 M_3 + \lambda_2) + +M_1(2\alpha_3\lambda_3 M_3 - \lambda_0) + \lambda_1 M_2 - 2\alpha_1\lambda_3 M_3^2,\\
&\quad \gamma_3(2\lambda_3 M_3 + \lambda_2) + (2\alpha_1\lambda_3 M_2 + \lambda_1)M_3,\\
&\quad (2\alpha_1\lambda_3^2 M_1 + \lambda_0\lambda_3)M_3 + 2\alpha_3\lambda_3^2 M_3^2 + \lambda_0\lambda_2,\\
&\quad \lambda_1^2 + (\lambda_0 + \alpha_3\lambda_2)^2 + \alpha_1^2\lambda_2^2],\\
&[\,M_2,\ \gamma_1(2\lambda_3 M_3 + \lambda_2) + \lambda_1 M_1,\ \gamma_3(2\lambda_3 M_3 + \lambda_2) + \lambda_1 M_3,\\
&\quad \gamma_2(2\lambda_3 M_3 + \lambda_2) + M_1(2\alpha_3\lambda_3 M_3 - \lambda_0) - 2\alpha_1\lambda_3 M_3^2,\\
&\quad -M_1((\lambda_0 + \alpha_3\lambda_2)^2 + \lambda_1^2 + \alpha_1^2\lambda_2^2 - 4\alpha_1^2\lambda_3^2 M_3^2)\\
&\quad +2\alpha_1(\lambda_3(2\alpha_3\lambda_3 M_3 + \lambda_0)M_3 + \lambda_0\lambda_2)M_3),\\
&\quad b_0\lambda_1^4 + b_1\lambda_1^2 + b_0]\,].
\end{aligned} \qquad (12)$$

Here b_0, b_1, b_2 are some expressions of $\lambda_0, \lambda_2, \lambda_3, \alpha_j, M_3$.

The first element of list (12) defines two solutions of equations (6) which write

$$\begin{aligned}
&M_1 = -\frac{\lambda_3(2\alpha_3\lambda_3 M_3 + \lambda_0)M_3 + \lambda_0\lambda_2}{2\alpha_1\lambda_3^2 M_3},\ M_2 = -\frac{(2\lambda_3 M_3 + \lambda_2)\,\gamma_3 + \lambda_1 M_3}{2\alpha_1\lambda_3 M_3},\\
&\gamma_1 = \frac{\lambda_0(\gamma_3\lambda_3 + \lambda_1) - 2\alpha_3\lambda_3^2 M_3\gamma_3}{2\alpha_1\lambda_3^2 M_3},\\
&\gamma_2 = \frac{\lambda_3((\alpha_1^2 + \alpha_3^2)(2\lambda_3 M_3 - \lambda_2)M_3 + \lambda_1\gamma_3) - \lambda_0^2}{2\alpha_1\lambda_3^2 M_3},
\end{aligned} \qquad (13)$$

$$\lambda_1 = \pm\sqrt{-((\lambda_0 + \alpha_3\lambda_2)^2 + \alpha_1^2\lambda_2^2)}. \tag{14}$$

Expressions (13), taking into account (14), correspond to two families of stationary invariant manifolds of the initial differential equations. The latter is proved by substitution of these expressions into (6) and by IMs definition (the derivative of (13) calculated by virtue of equations (3) vanishes on these sets). The calculations are trivial and are not represented here.

Expression λ_1 (14) is the first integral of a vector field on the found IMs that is proved by first integral definition (the derivative of λ_1 calculated by virtue of equations of the vector field is identically equal to zero). In the case considered, the integral is trivial.

The 2nd element of list (12) defines four solutions:

$$M_1 = \frac{2\alpha_1(2\alpha_3\lambda_3^2M_3^2 + \lambda_0(\lambda_3M_3 + \lambda_2))M_3}{\lambda_1^2 + (\alpha_3\lambda_2 + \lambda_0)^2 + \alpha_1^2(\lambda_2^2 - 4\lambda_3^2M_3^2)},\ M_2 = 0,$$

$$\gamma_1 = -\frac{2\alpha_1\lambda_1(2\alpha_3\lambda_3^2M_3^2 + \lambda_0(\lambda_3M_3 + \lambda_2))M_3}{(2\lambda_3M_3 + \lambda_2)(\lambda_1^2 + (\alpha_3\lambda_2 + \lambda_0)^2 + \alpha_1^2(\lambda_2^2 - 4\lambda_3^2M_3^2))},$$

$$\gamma_2 = \frac{2\alpha_1\left[\lambda_0^2(2\lambda_3M_3 + \lambda_2) + \lambda_3M_3(\lambda_1^2 + (\alpha_1^2 + \alpha_3^2)(\lambda_2^2 - 4\lambda_3^2M_3^2))\right]M_3}{(2\lambda_3M_3 + \lambda_2)[\lambda_1^2 + (\alpha_3\lambda_2 + \lambda_0)^2 + \alpha_1^2(\lambda_2^2 - 4\lambda_3^2M_3^2)]},$$

$$\gamma_3 = -\frac{\lambda_1M_3}{2\lambda_3M_3 + \lambda_2}, \tag{15}$$

$$\lambda_1 = \pm\frac{1}{\sqrt{2}}\sqrt{p_0 \pm \frac{[2\alpha_3\lambda_3^2M_3^2 + \lambda_0(\lambda_2 + \lambda_3M_3)]\sqrt{z_0}}{\lambda_3M_3 + \lambda_2}}, \tag{16}$$

where $p_0 = -\lambda_0(\lambda_0 - 6\alpha_3\lambda_3M_3) + \dfrac{2\alpha_3\lambda_3^2M_3^2(\lambda_0 + 2\alpha_3\lambda_3M_3)}{\lambda_3M_3 + \lambda_2}$
$-2(\alpha_1^2 + \alpha_3^2)(\lambda_2^2 - 4\lambda_3^2M_3^2),$

$$z_0 = 4\alpha_3(3\lambda_3M_3 + 2\lambda_2)[2\alpha_3(3\lambda_3M_3 + 2\lambda_2) + \lambda_0] + 16\alpha_1^2(\lambda_2 + \lambda_3M_3)(\lambda_2 + 2\lambda_3M_3) + \lambda_0^2.$$

Expressions (15), taking into account (16), define four families of one-dimensional stationary IMs of equations (3) which deliver a stationary value to already nonlinear combination of the basic integrals. Expression λ_1 (16) is the first integral of a vector field on the found IMs. These statements are proved as above.

2.3 Motions on the Invariant Manifolds

To analyze solutions on the above IMs, we investigate the differential equations on these IMs. Computer algebra tools play an auxiliary role here.

The equations of the vector field on the elements of families IMs (13) write

$$\dot{M}_3 = \frac{(2\lambda_3M_3 + \lambda_2)\gamma_3 + \lambda_1 M_3}{\lambda_3},$$

$$\dot{\gamma}_3 = \frac{(\lambda_3 M_3 + \lambda_2)\gamma_3^2}{\lambda_3 M_3} + \frac{\lambda_1}{\lambda_3}\gamma_3 - (\alpha_1^2 + \alpha_3^2)M_3^2 + \frac{\lambda_0^2\lambda_2}{\lambda_3^3 M_3}. \qquad (17)$$

These are derived from equations (3) by eliminating variables $\gamma_1, \gamma_2, M_1, M_2$ from them with the help of (13). Here λ_1 has the form (14).

Next, we consider the case when $\lambda_1 = -\sqrt{-((\lambda_0 + \alpha_3\lambda_2)^2 + \alpha_1^2\lambda_2^2)}$.

Equations (17) admit first integrals which are obtained from initial first integrals by eliminating variables $\gamma_1, \gamma_2, M_1, M_2$ from them with the help of (13). The integrals found by this technique will be, generally speaking, dependent. Take one of them, e.g.,

$$\begin{aligned}\tilde{V} = \frac{1}{4\alpha_1^2\lambda_3^4 M_3^2}\Big(&(\alpha_3\lambda_2 + \lambda_0)\lambda_0^2\lambda_2 + \lambda_2\lambda_3^2(\lambda_0 + \alpha_3\lambda_2)\gamma_3^2 \\ &-2(\lambda_0 + \alpha_3\lambda_2)\lambda_3^2\sqrt{-((\lambda_0 + \alpha_3\lambda_2)^2 + \alpha_1^2\lambda_2^2)}M_3\gamma_3 + 2\lambda_3^3(\lambda_0 + \alpha_3\lambda_2)M_3\gamma_3^2 \\ &+\lambda_3^2(2\alpha_3\lambda_0^2 + (3\alpha_1^2 + \alpha_3^2)\lambda_0\lambda_2 - \alpha_3(\alpha_1^2 + \alpha_3^2)\lambda_2^2)M_3^2 \\ &+2\lambda_3^3(\alpha_1^2 + \alpha_3^2)(\lambda_0 + \alpha_3\lambda_2)M_3^3\Big) = \tilde{c}_1 = const. \qquad (18)\end{aligned}$$

Eliminate variable γ_3 from equations (17) by expression (18). As a result, we have the differential equations (written in corresponding maps) with respect to M_3:

$$\begin{aligned}&\dot{M}_3 = \pm\frac{\sqrt{z}}{\sqrt{\lambda_0 + \alpha_3\lambda_2}\lambda_3^2}, \\ &z = -4\lambda_3^4(\alpha_1^2 + \alpha_3^2)(\lambda_0 + \alpha_3\lambda_2)M_3^4 \\ &\quad +\lambda_3^3(8\alpha_1^2\tilde{c}_1\lambda_3^2 - 4\lambda_0(2\alpha_1^2\lambda_2 + \alpha_3(\lambda_0 + \alpha_3\lambda_2)))M_3^3 \\ &\quad +\lambda_3^2(4\alpha_1^2\tilde{c}_1\lambda_2\lambda_3^2 - \lambda_0(\lambda_0(\lambda_0 + 5\alpha_3\lambda_2) + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2))M_3^2 \\ &\quad -\lambda_0^2\lambda_2^2(\lambda_0 + \alpha_3\lambda_2) - 2\lambda_0^2\lambda_2(\lambda_0 + \alpha_3\lambda_2)\lambda_3 M_3 - \lambda_0^2\lambda_2^2(\lambda_0 + \alpha_3\lambda_2).\end{aligned}$$

The above equations are integrated in elliptic functions. We have the analogous result when we take λ_1 with positive sign. Hence, the motion on invariant submanifolds of IMs (13), the equations of which are obtained by addition of integral (18) to equations (13), is described by elliptic functions of time.

The families of IMs (13) are complex, because the equations of the IMs contain complex coefficients. When $\lambda_0 = \lambda_1 = \lambda_2 = 0$, these families have the real invariant submanifold

$$M_1 = -\frac{\alpha_3 M_3}{\alpha_1},\ M_2 = -\frac{\gamma_3}{\alpha_1},\ \gamma_1 = -\frac{\alpha_3\gamma_3}{\alpha_1},\ \gamma_2 = \frac{(\alpha_1^2 + \alpha_3^2)M_3}{\alpha_1},$$

the motion on which is described by real elementary functions. Note, the submanifold delivers a stationary value to integral F (2).

When $\lambda_0 = 0$, $\lambda_1 = -\sqrt{-(\alpha_1^2 + \alpha_3^2)\lambda_2}$, the family of IMs (13) has the subfamily of complex IMs

$$M_1 = -\frac{\alpha_3 M_3}{\alpha_1},\ M_2 = \frac{\sqrt{-\alpha_1^2 - \alpha_3^2}\,\lambda_2 M_3 - (2\lambda_3 M_3 + \lambda_2)\gamma_3}{2\alpha_1\lambda_3 M_3},\ \gamma_1 = -\frac{\alpha_3\gamma_3}{\alpha_1},$$

$$\gamma_2 = -\frac{\sqrt{-\alpha_1^2-\alpha_3^2}\,\lambda_2\gamma_3 - (\alpha_1^2+\alpha_3^2)(2\lambda_3 M_3 - \lambda_2)M_3}{2\alpha_1\lambda_3 M_3}, \tag{19}$$

which delivers a stationary value to integral $\tilde{K} = -\sqrt{-(\alpha_1^2+\alpha_3^2)}\,V_1 - \lambda_2 V_2/2 - \lambda_3 F$.

The motion on the elements of subfamily (19) is defined by the differential equations

$$\dot{M}_3 = \pm\frac{2M_3\sqrt{\alpha_1^2\lambda_2\tilde{c}_1 + 2\alpha_1^2\lambda_3\tilde{c}_1 M_3 - \alpha_3(\alpha_1^2+\alpha_3^2)\lambda_2 M_3^2}}{\sqrt{\alpha_3\lambda_2}}. \tag{20}$$

Equations (20) are integrated in the elementary functions

$$M_3(t) = \frac{2\alpha_1^2\tilde{c}_1\,\lambda_2}{e^{\mp\alpha_1\sqrt{\tilde{c}_1}(\frac{2t}{\sqrt{\alpha_3}}\pm\sqrt{\lambda_2}\tilde{c}_2)} + \alpha_1^2\tilde{c}_1\,[e^{\pm\alpha_1\sqrt{\tilde{c}_1}(\frac{2t}{\sqrt{\alpha_3}}\pm\sqrt{\lambda_2}\tilde{c}_2)}(\alpha_1^2\tilde{c}_1\lambda_3^2 - \alpha_3\lambda_2^2 k) - 2\lambda_3]}.$$

Here $k = -(\alpha_1^2+\alpha_3^2)$, $\tilde{c}_2$ is a constant of integration.

As obvious from the latter expression, it assumes real values when $\alpha_3 > 0, \lambda_2 > 0, \tilde{c}_1 > 0$ or $\alpha_3 < 0, \lambda_2 < 0, \tilde{c}_1 < 0$. Hence, under the above restrictions imposed on parameters $\alpha_3, \lambda_2, \tilde{c}_1$, the motion on the elements of the submanifold of complex IMs (19) is described by the real functions of time.

Finally, let us consider the motion on the elements of families (15).

The vector field on the elements of these families is defined by equation $\dot{M}_3 = 0$. Hence, geometrically, the elements of the families of real solutions (15) correspond to curves in R^6, each point of which is the degenerated stationary solution of the initial differential equations.

3 On Stationary Sets of Euler's Equations with an additional first integral of 6th degree

3.1 Problem Formulation

According to [10], in integrable case (d) there exists an additional first integral of 6th degree of the form

$$\begin{aligned} F = M_3^2\Big[&(M_1^2+M_3^2)[(\alpha_3 M_1+\gamma_2)^2 + \alpha_1 M_3(\alpha_1 M_3 - 2(\alpha_3 M_1+\gamma_2))] \\ &+((\alpha_1 M_2+\gamma_3)M_3 - (\alpha_3 M_2-\gamma_1)M_1)^2\Big] = h_1 = const \end{aligned}$$

Euler's equations corresponding to the Hamiltonian (d) write

$$\begin{aligned} &\dot{M}_1 = 2(M_3\gamma_1 - M_1\gamma_3) - M_2(\alpha_1 M_1 + 2\alpha_3 M_3),\\ &\dot{M}_2 = 2(M_2\gamma_3 - M_3\gamma_2) + \alpha_1(M_1^2 - M_3^2) + 2\alpha_3 M_1 M_3, \quad \dot{M}_3 = \alpha_1 M_2 M_3,\\ &\dot{\gamma}_1 = (2\alpha_3 M_3 - \alpha_1 M_1)\,\gamma_2 - 2(2\alpha_3 M_3 - \gamma_1)\,\gamma_3 - 2kM_1M_3,\\ &\dot{\gamma}_2 = \alpha_1(M_1\gamma_1 - M_3\gamma_3) + 2\alpha_3(2M_1\gamma_3 - M_3\gamma_1) - 2(kM_2M_3 + \gamma_2\gamma_3),\\ &\dot{\gamma}_3 = 4\alpha_3(M_2\gamma_1 - M_1\gamma_2) + \alpha_1 M_3\gamma_2 + 2[k(M_1^2+M_2^2) - (\gamma_1^2+\gamma_2^2)]. \end{aligned} \tag{21}$$

The rest of the integrals of these equations has the form (4).

Once again let us state the problem of finding stationary sets, now for equations (21), and investigation of their stability in the sense of Lyapunov.

3.2 Finding Stationary Sets

Following the technique chosen, we construct the complete linear combination

$$2K = 4\lambda_0 H - 2\lambda_1 V_1 - \lambda_2 V_2 - \lambda_3 F \quad (\lambda_i = const)$$

from the problem's first integrals, and write down the necessary conditions for the integral K to have an extremum with respect to phase variables M_i, γ_i:

$$\begin{aligned}
\partial K/\partial M_1 &= \lambda_0(2\eta - \alpha_1 M_3) - \lambda_1\gamma_1 - k\lambda_2 M_1 + \lambda_3\Big[\alpha_3(\alpha_1(M_1^2+M_3^2) - \gamma_2 M_3)M_3 \\
&\quad + kM_1M_3^2 + [(2\alpha_1 M_3 - \alpha_3 M_1)\sigma - (\rho^2+\sigma^2)]M_1 - \rho\chi M_3\Big]M_3^2 = 0, \\
\partial K/\partial M_2 &= 2\lambda_0 x5 - \lambda_1\gamma_2 - k\lambda_2 M_2 + \lambda_3(\alpha_3 M_1 - \alpha_1 M_3)(\rho M_1 + \chi M_3)M_3^2 = 0, \\
\partial K/\partial M_3 &= \lambda_0(2\alpha_3 M_3 - \alpha_1 M_1) - \lambda_1\gamma_3 - k\lambda_2 M_3 - \lambda_3\Big[(\rho^2+\sigma^2)M_1^2 \\
&\quad + 3\rho\chi M_1 M_3 + 2\chi^2 M_3^2 + (2\alpha_3^2 M_1^2 + \alpha_1^2(2M_1^2+3M_3^2) + 2\eta\gamma_2)M_3^2 \\
&\quad - \alpha_1(3M_1^2+5M_3^2)\,\sigma M_3\Big]M_3 = 0, \\
\partial K/\partial \gamma_1 &= -2\lambda_0 M_2 - \lambda_1 M_1 - \lambda_2\gamma_1 - \lambda_3(\rho M_1 + \chi M_3)M_1 M_3^2 = 0, \\
\partial K/\partial \gamma_2 &= 2\lambda_0 M_1 - \lambda_1 M_2 - \lambda_2\gamma_2 + \lambda_3(M_1^2 + M_3^2)(\alpha_1 M_3 - \sigma)M_3^2 = 0, \\
\partial K/\partial \gamma_3 &= -\lambda_1 M_3 - \lambda_2\gamma_3 - \lambda_3(\rho M_1 + \chi M_3)M_3^3 = 0.
\end{aligned} \tag{22}$$

Here the following denotations $\eta = 2\alpha_3 M_1 + \gamma_2, \rho = \gamma_1 - \alpha_3 M_2, \sigma = \alpha_3 M_1 + \gamma_2, \chi = \alpha_1 M_2 + \gamma_3$ are used.

The above equations (the equations of stationarity of the integral K) represent a system of polynomial algebraic equations of 5th degree with parameters λ_i, α_j.

Likewise the previous problem, some qualitative analysis of the solution set of equations (22) by the *Maple*-programs *IsZeroDimentional, NumberOfSolutions, HilbertDimendion* was conducted [1]. It was revealed that system (22) has infinite number of solutions with respect to the phase variables, and the dimension of these solutions is 1.

The above system was decomposed into two subsystems. To this end, Gröbner basis and polynomial factorization methods were used. The 1st subsystem consists of 20 polynomial equations of degrees from 2 to 9. The 2nd subsystem consists of 16 polynomial equations of degrees from 1 to 9. The coefficients of equations belong to field $Q(\lambda_i, \alpha_j)$. The solutions, the dimension of which is 1, form the solution set of the 1st subsystem. The 2nd subsystem has a finite number of solutions which is 33. Each of solutions of the subsystem corresponds to a single point.

The solutions of the 1st subsystem obtained with respect to the phase variables, and the solutions of the 2nd subsystem obtained with respect to some part of the phase variables and some part of the parameters can be found in [1]. These represent IMs of equations (21).

In the given work, we stated the problem of finding solutions for the 2nd subsystem with respect to the phase variables. We tried to obtain solutions corresponding to the complete linear combination of the problem's first integrals (the integral K).

It is known, lexicographic basis is the most suitable for computing the roots of a polynomial algebraic system. We have not managed to construct a lexicographic basis for the subsystem considered without imposing any additional restrictions on parameters λ_i, α_j neither with the use of Gröbner basis method (the programs *GroebnerBasis* and *gbasis* of CAS *Mathematica* and *Maple* were applied), neither with the use of the *Maple*-program *FGLM*. The computations were executed for over 12 hours on Intel CPU 3.6 GHz, 32 GB RAM running under Windows 7 Professional.

To find the desired solutions we have constructed the Gröbner basis with respect to elimination monomial order. It writes:

$$M_2 = 0,\ f_1(\gamma_1, M_1, M_3, \lambda_i, \alpha_i) = 0,\ f_2(\gamma_2, M_1, M_3, \lambda_i, \alpha_i) = 0,$$
$$f_3(\gamma_3, M_1, M_3, \lambda_i, \alpha_i) = 0, \tag{23}$$

$$g_r(M_1, M_3, \lambda_i, \alpha_i) = 0\ (r = 1, \ldots, 7). \tag{24}$$

Here f_s are the 5th degree polynomials of variables $M_1, M_3, \gamma_1, \gamma_2, \gamma_3$, g_r are the 6-9 degree polynomials of variables M_1, M_3. The coefficients of the polynomials f_s, g_r represent some expressions of λ_i, α_j.

A number of solutions of equations (24) was obtained by computing resultants for polynomials g_r. Indeed, having computed the resultant of any two polynomials g_l, g_m (24) with respect to, e.g., variable M_1, we obtain some polynomial of variable M_3. Each root M_3 of this polynomial corresponds to common roots M_1 of polynomials g_l, g_m. Since system (24) is compatible, some of the common roots of polynomials g_l, g_m are the roots of all the rest of the polynomials g_r.

Following the technique chosen, compute the resultant for two polynomials of system (24) (e.g., for polynomials having the least degree with respect to variable M_1). It writes:

$$Res = \alpha_1((2\lambda_0 + \alpha_3\lambda_2^2)^2 + \alpha_1^2\lambda_2^2)\, D\, M_3\, (a_0 M_3^{12} + a_1 M_3^8 + a_2 M_3^4 + a_4)$$
$$\times(b_0 M_3^{20} + b_1 M_3^{16} + b_2 M_3^{12} + b_3 M_3^8 + b_4 M_3^4 + b_5) \tag{25}$$

Here D, a_σ, b_ρ are some expressions of λ_i, α_j.

As obvious from (25), the resultant expression is factorized and, consequently, computing the roots of the polynomials chosen is reduced to finding the roots of 12th and 20th degree bipolynomials. Under some restrictions imposed on parameters λ_i, α_j, we have found several solutions of these equations. Substituting the obtained values of M_3 into the polynomials chosen, we find common roots of these polynomials and equations (24).

Next, we substitute the found values of M_1, M_3 into equations (23) and find the values of $\gamma_1, \gamma_2, \gamma_3$ corresponding to them. Some of the solutions of equations (23), (24) obtained by the technique described are given below.

i) $\lambda_1 = 0, \lambda_2 = \lambda_3^5, \alpha_3 = 0:$

$$\gamma_1 = 0, \gamma_2 = \pm\frac{1}{2}\alpha_1\lambda_3, \gamma_3 = 0, M_1 = 0, M_2 = 0, M_3 = \pm\lambda_3;$$

ii) $\lambda_1 = 0, \lambda_2 = \lambda_5^3, \alpha_3 = 0:$

$$\gamma_1 = 0, \gamma_2 = \pm\frac{1}{2}\alpha_1\sqrt{-\lambda_3}, \gamma_3 = 0, M_1 = 0, M_2 = 0, M_3 = \pm\sqrt{-\lambda_3};$$

iii) $\lambda_1 = 0, \lambda_2 = -\lambda_3$:

$$\gamma_1 = \frac{\alpha_1\sqrt{[(2\lambda_0 - \alpha_3\lambda_3)^2(4\alpha_3\lambda_0 - (3\alpha_1^2 + 2\alpha_3^2)\lambda_3) - \alpha_1^4\lambda_3^3]\,\lambda_3}}{2(\alpha_3\lambda_3 - 2\lambda_0)^{3/2}[(2\lambda_0 - \alpha_3\lambda_3)^2 + \alpha_1^2\lambda_3^2]^{1/4}},$$

$$\gamma_2 = \frac{\alpha_1((\alpha_1^2 + \alpha_3^2)\lambda_3^2) - 4\lambda_0^2}{2(\alpha_3\lambda_3 - 2\lambda_0)^{3/2}[(2\lambda_0 - \alpha_3\lambda_3)^2 + \alpha_1^2\lambda_3^2]^{1/4}},$$

$$\gamma_3 = \frac{\sqrt{(2\lambda_0 - \alpha_3\lambda_3)^2(4\alpha_3\lambda_0 - (3\alpha_1^2 + 2\alpha_3^2)\lambda_3) - \alpha_1^4\lambda_3^3}}{2\sqrt{(\alpha_3\lambda_3 - 2\lambda_0)\,\lambda_3}\,[(2\lambda_0 - \alpha_3\lambda_3)^2 + \alpha_1^2\lambda_3^2]^{1/4}},$$

$$M_1 = -\frac{\alpha_1\lambda_3}{\sqrt{(\alpha_3\lambda_3 - 2\lambda_0)\,\lambda_3}\,[(2\lambda_0 - \alpha_3\lambda_3)^2 + \alpha_1^2\lambda_3^2]^{1/4}},$$

$$M_2 = 0,\ M_3 = -\frac{\sqrt{\alpha_3\lambda_3 - 2\lambda_0}}{[(2\lambda_0 - \alpha_3\lambda_3)^2 + \alpha_1^2\lambda_3^2]^{1/4}}. \tag{26}$$

These represent the families of stationary solutions of equations (22) parameterized by λ_i. Likewise above, the latter is proved by direct substitution of expressions (26) into the stationary equations and the initial differential ones (i.e., equations (21) and (22)).

From a mechanical point of view and geometrically, the above solutions are interpreted similar to the stationary solutions of integrable case (b).

4 On Stability of the Stationary Sets

Now, we shall consider the problem of stability for the above stationary solutions. We shall investigate stability of the solutions on the base of Lyapunov's stability theorems: the 2nd method [15], in particular, the Routh-Lyapunov method [12] which is its modification, theorems for linear approximation [15] and theorems for stability with respect to part of variables [16].

First, let us investigate a trivial solution by the Routh-Lyapunov method. This method allows one to obtain sufficient conditions of stability.

The family of Hamiltonians (1) corresponds to a family of Euler's equations on Lie algebras $e(3), so(4)$ $so(3,1)$. We shall investigate stability of a trivial solution of equations of this family. This solution is peculiar stationary one, because it delivers a stationary value to all the above first integrals, and it is the solution for all equations of the family. Hence, investigation of its stability has a special interest.

Application of the Routh-Lyapunov method, in the case considered, reduces the stability problem to analysing the sign definiteness of variation of integral $\tilde{K} = \lambda_0 H - \lambda_1 V_1 - \lambda_2 V_2$ obtained in the neighbourhood of the given solution. It writes

$$\Delta\tilde{K} = -\lambda_2(z_1^2 + z_2^2 + z_3^2) - \lambda_1(z_1z_4 + z_2z_5 + z_3z_6) + \lambda_0(z_2z_4 - z_1z_5 + \alpha_1c_2z_4z_6) + (\alpha_3c_1\lambda_0 + (\alpha_1^2 + \alpha_3^2)\lambda_2)z_4^2 + (\alpha_3c_1\lambda_0 + (\alpha_1^2 + \alpha_3^2)\lambda_2)z_5^2 + (\alpha_3(c_1 + c_2)\lambda_0 + (\alpha_1^2 + \alpha_3^2)\lambda_2)z_6^2.$$

Here z_i are deviations of the perturbed solution from the unperturbed one.

According to Sylvester's criterion, the quadratic form $\Delta\tilde{K}$ is sign definite when the following conditions

$$\Delta_1 = -\lambda_2 > 0,\ \Delta_2 = -(\lambda_0^2 + \lambda_1^2 + 4\alpha_3 c_1\lambda_0\lambda_2 + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2) > 0,$$
$$\Delta_3 = \Delta_2(4\alpha_3(c_1 + c_2)\lambda_0^3\lambda_2 + 4\alpha_3(2c_1 + c_2)\lambda_0\lambda_2(\lambda_1^2 + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2)$$
$$+(\lambda_1^2 + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2)^2 + \lambda_0^2(\lambda_1^2 + 4(\alpha_3^2(4c_1(c_1 + c_2) + 1) - \alpha_1^2(c_2^2 - 1))\lambda_2^2)).$$

hold.

The above inequalities are compatible under the following restrictions:

$$\Big(\alpha_3 > 0 \wedge \alpha_1 \neq 0 \wedge \lambda_2 < 0 \wedge \Big((\lambda_0 < 0 \wedge c_1 > A_1 \wedge B_2 < c_2 < B_1)$$
$$\vee(\lambda_0 > 0 \wedge c_1 < A_1 \wedge B_2 < c_2 < B_1)\Big)\Big) \vee \Big(\alpha_3 < 0 \wedge \alpha_1 \neq 0 \wedge \lambda_2 < 0$$
$$\wedge\Big((\lambda_0 < 0 \wedge c_1 < A_1 \wedge B_2 < c_2 < B_1) \vee (\lambda_0 > 0 \wedge c_1 > A_1 \wedge B_2 < c_2 < B_1)\Big)\Big)$$
$$\vee\Big(\alpha_3 > 0 \wedge \alpha_1 = 0 \wedge \lambda_2 < 0 \wedge \Big((\lambda_0 < 0 \wedge c_1 > A_2 \wedge c_2 > B_3)$$
$$\vee(\lambda_0 > 0 \wedge c_1 < A_2 \wedge c_2 < B_3)\Big)\Big) \vee \Big(\alpha_3 < 0 \wedge \alpha_1 = 0 \wedge \lambda_2 < 0$$
$$\wedge\Big((\lambda_0 < 0 \wedge c_1 < A_2 \wedge c_2 < B_3) \vee (\lambda_0 > 0 \wedge c_1 > A_2 \wedge c_2 > B_3)\Big)\Big). \qquad (27)$$

These have been obtained with the *Mathematica*-program *Reduce*. Here $A_1 = -(\lambda_0^2 + \lambda_1^2 + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2)/(4\alpha_3\lambda_0\lambda_2)$, $A_2 = -(\lambda_0^2 + \lambda_1^2 + 4\alpha_3^2\lambda_2^2)/(4\alpha_3\lambda_0\lambda_2)$, $B_{1,2} = 1/(2\alpha_1^2\lambda_0\lambda_2)\,[\alpha_3(\lambda_0^2 + \lambda_1^2 + 4\alpha_3 c_1\lambda_0\lambda_2 + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2) \pm \sqrt{D}]$, $D = (\lambda_0^2 + \lambda_1^2 + 4\alpha_3 c_1\lambda_0\lambda_2 + 4(\alpha_1^2 + \alpha_3^2)\lambda_2^2)(\alpha_1^2\lambda_1^2 + \alpha_3^2(\lambda_0^2 + \lambda_1^2) + 4\alpha_3(\alpha_1^2 + \alpha_3^2)c_1\lambda_0\lambda_2 + 4(\alpha_1^2 + \alpha_3^2)^2\lambda_2^2)$, $B_3 = -(\lambda_1^2 + 4\alpha_3 c_1\lambda_0\lambda_2 + 4\alpha_3^2\lambda_2^2)/(4\alpha_3\lambda_0\lambda_2)$.

It should be noted that correctness of the above conditions can be verified, e.g., by other CAS programm (similar to *Reduce*) or by special numeric tests. In the given work, the latter way was used.

Conditions (27) saparate from the family of Euler's equations with Hamiltonians (1) some subfamily of systems of the equations, a trivial solution of which is stable.

To verify whether or not integrable cases $(a) - (d)$ (section 2.1) enter into this subfamily it is sufficient, e.g., to test the compatibility of the conditions of integrability with conditions (27) by applying the program *Reduce*. As a result, we have that one integrable case (when c_1 is arbitrary, $c_2 = 0$) enters into the above subfamily only.

We have managed to obtain conditions of stability for the trivial solution of system (21) (the integrable case (d)) with respect to some part of the phase variables [16]. To this end, we analyzed the variation of integral $\bar{K} = -\lambda_2(2\alpha_3 H + V_2)$ (where $\alpha_1 = 0$) written in the neighbourhood of this solution. It writes:

$$2\Delta\tilde{K} = -\lambda_2\,[z_3^2 + (z_2 + \alpha_3 z_4)^2 + (z_1 - \alpha_3 z_5)^2].$$

Introduce variables $y_1 = z_2 + \alpha_3 z_4$, $y_2 = z_1 - \alpha_3 z_5$ and write down $\Delta\tilde{K}$ in terms of y_1, y_2, z_3:

$$2\Delta\tilde{K} = -\lambda_2(y_1^2 + y_2^2 + z_3^2). \qquad (28)$$

Quadratic form (28) is sign definite with respect to y_1, y_2, z_3 when $\lambda_2 \neq 0$. Hence, the conditions $\alpha_1 = 0$, $\lambda_2 \neq 0$ are sufficient for the stability of the trivial solution with respect to variables y_1, y_2, z_3.

Next, we investigate one of the families of stationary solutions (26), e.g.,

$$\gamma_1 = 0,\ \gamma_2 = \frac{1}{2}\alpha_1\lambda_3,\ \gamma_3 = 0,\ M_1 = 0,\ \ M_2 = 0,\ M_3 = \lambda_3. \tag{29}$$

We show that the elements of family (29) are instability when $\alpha_1 \neq 0$ and $\lambda_3 \neq 0$. To this end, we consider this solution in capacity of the unperturbed one and write down the equations of first approximation:

$$\dot{z}_1 = \frac{3}{2}\alpha_1^2\lambda_3 z_4,\ \ \dot{z}_2 = 2\alpha_1^2\lambda_3 z_5,\ \ \dot{z}_3 = \frac{1}{2}\alpha_1\lambda_3(-2z_2 + \alpha_1 z_6),$$
$$\dot{z}_4 = 2\lambda_3 z_1,\ \ \dot{z}_5 = \lambda_3(2z_2 - \alpha_1 z_6),\ \ \dot{z}_6 = \alpha_1\lambda_3 z_5.$$

The characteristic equation of the above linear system writes $\mu^2(\mu^2 – 3\alpha_1^2\lambda_3^2)^2 = 0$ and has four nonzero roots $\mu_1 = -\sqrt{3}\,\alpha_1\lambda_3$, $\mu_2 = -\sqrt{3}\,\alpha_1\lambda_3$, $\mu_3 = \sqrt{3}\,\alpha_1\lambda_3$, $\mu_4 = \sqrt{3}\,\alpha_1\lambda_3$, among of which are real positive when $\alpha_1 \neq 0$, $\lambda_3 \neq 0$. The latter, according to Lyapunov's stability theorem for linear approximation [15], means instability of the elements of family (29).

The rest of the families of stationary solutions (26) have been investigated by this technique. We have obtained results analogous above.

5 Conclusion

Practically, the completed analysis for stationary sets of Euler's equations on the Lie algebra $so(3,1)$ when the equations possess additional polynomial first integrals of degrees 3 and 6 has been performed. We considered the case when the sets correspond to the complete linear combination of the problem's first integrals. The obtained results can be a base for further qualitative analysis of the considered integrable cases. The approach applied in this work to investigation of integrable systems may be of interest for the study of new integrable cases of equations of a such type when the problem's algebraic first integrals have degree higher than 2. This approach may also be of interest for the problems of parametric analysis where properties of solutions in relation to continuous change of parameters of these solutions are investigated.

The work was supported by the Presidium of the Russian Academy of Sciences, basic research program no. 17.1.

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