# Algebraic Attacks Using IP-Solvers

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# Outline

# Finding $\mathbb{F}_2$-rational Solutions

Consider the field $\mathbb{F}_2$ with two elements. Let $f_1, f_2 \in \mathbb{F}_2[x_1, x_2]$, where $f_1 = x_1x_2 + x_2$, and $f_2 = x_1x_2 + x_1 + 1$. Find a solution of the system

$$f_1 = 0, \ \ f_2 = 0$$

in $\mathbb{F}_2^2$. The four possible solution candidates are:

$$(0,0), \ (0,1), \ (1,0), \ (1,1).$$

The only candidate that qualifies for a solution is $(1,0)$.

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### Gröbner Bases Approach

Compute a Gröbner basis of the ideal

$$\langle f_1, f_2, x_1^2 + x_1, x_2^2 + x_2 \rangle$$

which is

$$\{x_1 + x_2 + 1, \;\; x_2\}$$

# Finding $\mathbb{F}_q$-rational Solutions

Let $p$ be a prime and $q = p^e$ with $e > 0$. Let $K = \mathbb{F}_q$ be the finite field and let $F = \{f_1, \ldots, f_\ell\} \subseteq P = K[x_1, \ldots, x_n]$ be a set of polynomials. Find the $K$-rational solutions of the following system of equations.

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_\ell(x_1, \ldots, x_n) &= 0 \end{aligned}$$

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Special Properties of the System:

- The so-called field polynomials $x_1^q - x_1, \ldots, x_n^q - x_n$ play an essential role. For instance, the ideal
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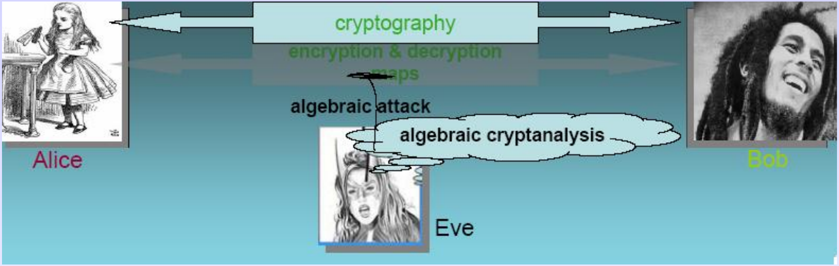
  is a 0-dimensional radical ideal.
- The system has a unique (or a few) $K$-rational solution(s). The polynomials $f_1, \ldots, f_\ell$ are quadratic and $p = 2$.




## Cryptosystem

A cryptosystem consists of the following components:

- a set $\mathcal{P}$ called plaintext space,
- a set $\mathcal{C}$ called ciphertext space,
- a set $\mathcal{K}$ called key space,
- for every $k \in \mathcal{K}$ an encryption map, $\varepsilon_k : \mathcal{P} \longrightarrow \mathcal{C}$ and a decryption map, $\delta_k : \mathcal{C} \longrightarrow \mathcal{P}$ such that $\delta_k \circ \varepsilon_k = \mathrm{id}_{\mathcal{P}}$.

# Algebraic Attacks

## Idea

Reduce the task of breaking a cryptosystem to the task of solving a polynomial system!

How: Let the plaintext space and the ciphertext space be of the form $\mathcal{P} = K^n$ and $\mathcal{C} = K^m$ with a finite field $K$ (usually $K = \mathbb{F}_2$). Then every map $\varphi : K^n \longrightarrow K^m$ is given by polynomials, i.e. there exist polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ such that

$$\varphi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)),$$

for all $(x_1, \ldots, x_n) \in K^n$.

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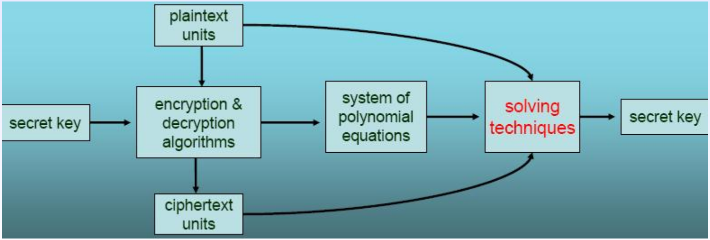

# Standard Cryptographic Polynomial Systems

Courtois Toy Cipher (CTC)

| CTC(S-Boxes,Rounds) | CTC(3,3) | CTC(4,4) | CTC(5,5) | CTC(6,6) | CTC(7,7) | CTC(8,8) |
|---|---|---|---|---|---|---|
| equations | 216 | 380 | 605 | 864 | 1169 | 1496 |
| variables | 117 | 204 | 330 | 468 | 630 | 795 |
| non-linear terms | 162 | 288 | 450 | 648 | 882 | 1152 |

Small Scale Advanced Encryption Standard (AES)

| AES(n,r,c,w) | AES(9,1,1,4) | AES(10,1,1,4) | AES(4,2,1,4) | AES(2,2,2,4) | AES(3,2,2,4) | AES(1,1,1,8) |
|---|---|---|---|---|---|---|
| equations | 1184 | 1312 | 1088 | 1024 | 1472 | 640 |
| variables | 592 | 656 | 544 | 512 | 736 | 320 |
| non-lin. terms | 1584 | 1760 | 1408 | 1056 | 1584 | 2416 |

Systems used are available at `http://apcocoa.org/polynomialsystems/`

## Our Approach for Solving over Finite Fields

system of equations over $\mathbb{F}_q$

system of equations over $\mathbb{F}_q$

Traditional techniques using Gröbner basis:

Improvements: Buchberg's algorithm using strategies such as normal selection, sugar cube, etc.

Variants: $\mathrm{F}_4$ and $\mathrm{F}_5$ algorithms, XL-algorithm and its mutant variants.

Border basis algorithm and its improvements.

SAT-Solvers and characteristic set methods.

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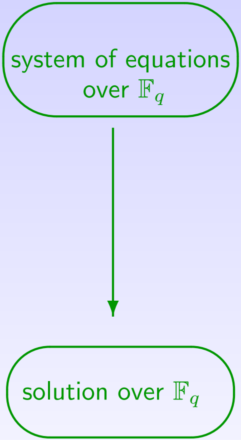


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### Border basis algorithm and its improvements.

### SAT-Solvers and characteristic set methods.

system of equations over $\mathbb{F}_q$

system of equations over $\mathbb{R}$ (resp. $\mathbb{Z}$)

$\downarrow$

solution over $\mathbb{F}_q$

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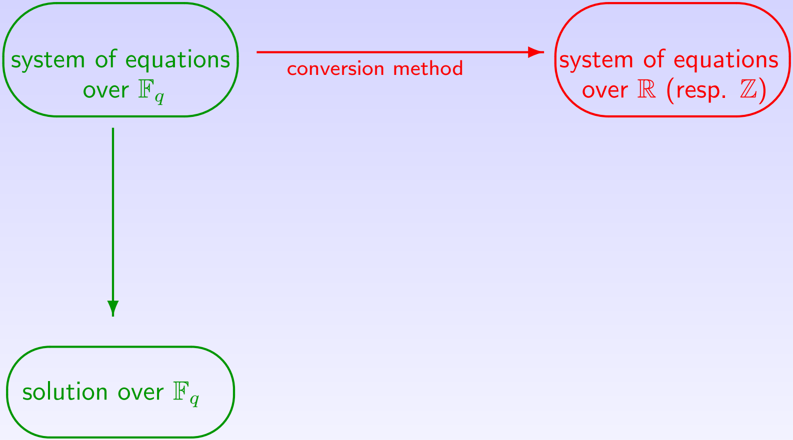


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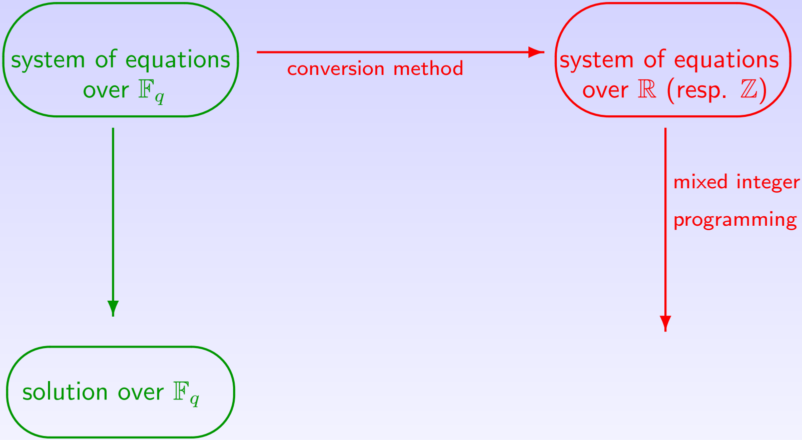


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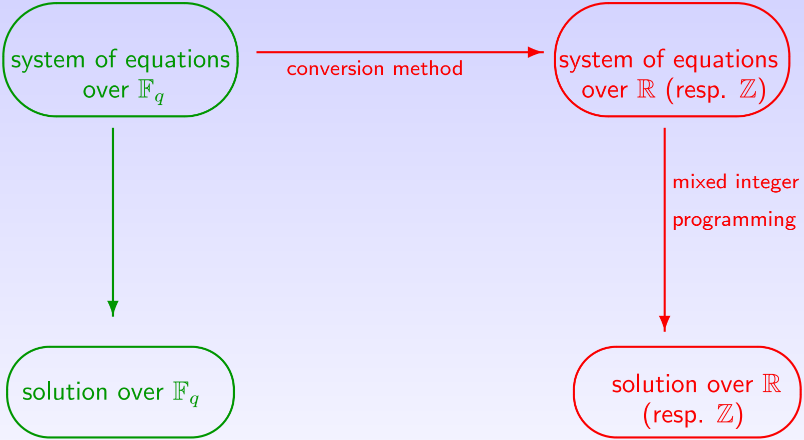


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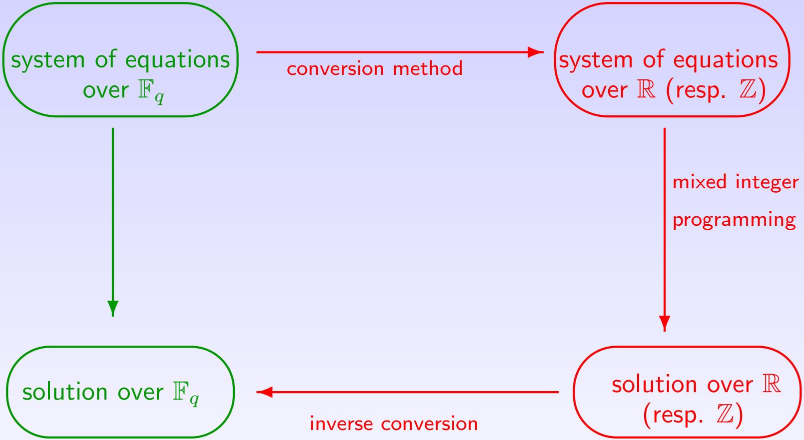


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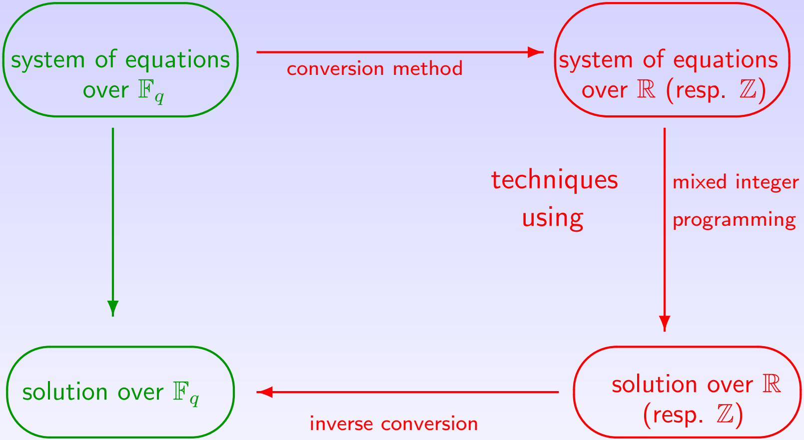


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# Objectives

- Look for new techniques and strategies
- Study the impact of various newly developed strategies
- Get advantage of parallel computing, state-of-the-art solvers
- Provide tools for algebraic cryptanalysis through ApCoCoA

system of equations over $\mathbb{F}_2$

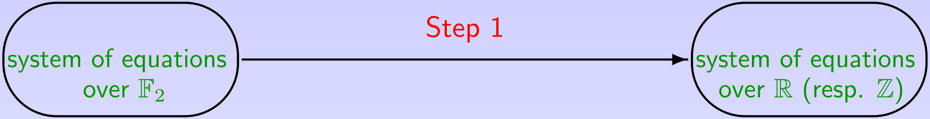
$$\text{system of equations over } \mathbb{F}_2 \xrightarrow{\text{Step 1}} \text{system of equations over } \mathbb{R} \text{ (resp. } \mathbb{Z}\text{)}$$

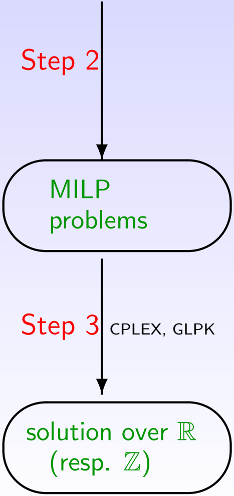
Step 1: transformation to $\mathbb{R}$ or $\mathbb{Z}$

system of equations over $\mathbb{F}_2$ $\xrightarrow{\text{Step 1}}$ system of equations over $\mathbb{R}$ (resp. $\mathbb{Z}$) $\xrightarrow{\text{Step 2}}$ MILP problems

Step 1: transformation to $\mathbb{R}$ or $\mathbb{Z}$

Step 2: modeling MILP problem,
linearization (using strategies) $+$ modeling




Step 1: transformation to $\mathbb{R}$ or $\mathbb{Z}$

Step 2: modeling MILP problem,
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Step 3: using an IP solver

system of equations over $\mathbb{F}_2$ $\xrightarrow{\text{Step 1}}$ system of equations over $\mathbb{R}$ (resp. $\mathbb{Z}$)

Step 2

MILP problems

Step 3 CPLEX, GLPK

solution over $\mathbb{R}$ (resp. $\mathbb{Z}$)

Step 4

solution over $\mathbb{F}_2$

Step 1: transformation to $\mathbb{R}$ or $\mathbb{Z}$

Step 2: modeling MILP problem, linearization (using strategies) + modeling

Step 3: using an IP solver

Step 4: inverse transformation

Find a $\mathbb{F}_2$-rational solution of

$$f_1(x_1, \ldots, x_n) = 0, \ldots, f_m(x_1, \ldots, x_n) = 0,$$

where $f_1, \ldots, f_m \in \mathbb{F}_2[x_1, \ldots, x_n]$.

An overview given by R. Beigel (1993).

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We are looking for a tuple $(a_1, \dots, a_n) \in \{0, 1\}^n$ such that

$$\begin{aligned} F_1(a_1, \dots, a_n) &\equiv 0 \pmod 2 \\ &\vdots \\ F_m(a_1, \dots, a_n) &\equiv 0 \pmod 2 \end{aligned}$$

where $F_i \in \mathbb{Z}[X_1, \dots, X_n]$ ($\mathbb{R}[X_1, \dots, X_n]$) are liftings of the polynomials $f_i$.

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- standard representation:

$$\begin{array}{ccc}\overline{0} & \rightarrow & 0\\ \overline{1} & \rightarrow & 1\end{array}$$

iteratively replace each sum $X_1+X_2$ by $X_1+X_2-2X_1X_2$

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Let $f = x_1x_2 + x_3x_4 + x_5 + x_6 + 1 \in \mathbb{F}_2[x_1, \ldots, x_6]$.

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## Example

*The standard representation of $f$ is*

$$\begin{aligned} F = {} & 8X_1X_2X_3X_4X_5X_6 - 4X_1X_2X_3X_4X_5 - 4X_1X_2X_3X_4X_6 \\ & +2X_1X_2X_3X_4 - 4X_1X_2X_5X_6 - 4X_3X_4X_5X_6 + 2X_1X_2X_5 \\ & +2X_3X_4X_5 + 2X_1X_2X_6 + 2X_3X_4X_6 - X_1X_2 - X_3X_4 \\ & +2X_5X_6 - X_5 - X_6 + 1 \in \mathbb{R}[X_1, \ldots, X_6] \end{aligned}$$

*The polynomial $F$ has 16 terms in its support and degree 6.*

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## Splitting

- $y_1 + x_1x_2 = x_3x_4 + x_5$, $y_1 = x_6 + 1$.
- $y_1 + y_2 = y_3 + x_5$, $y_1 = x_6 + 1$, $y_2 = x_1x_2$, $y_3 = x_3x_4$.
- $Y_1 + Y_2 - 2Y_1Y_2 = Y_3 + X_5 - 2Y_3X_5$, $Y_1 = 1 - X_6$, $Y_2 - X_1X_2 = 0$, $Y_2 - X_3X_4 = 0$.

We require the solution of a polynomial system of equations

$$f_1(x_1, \ldots, x_n) = 0, \ldots, f_m(x_1, \ldots, x_n) = 0.$$

with polynomials $f_1, \ldots, f_m \in \mathbb{F}_2[x_1, \ldots, x_n]$.

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In other words, find a tuple $(a_1,\dots,a_n)\in\{0,1\}^n$ such that

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Idea: Formulate these congruences as a system of linear equalities or inequalities over $\mathbb{Z}$ and solve it using an IP-solver.

# Integer Polynomial Conversion (IPC)

Assume we are given a congruence

$$F(a_1, \dots, a_n) \equiv 0 \pmod 2$$

with $F \in \mathbb{Z}[X_1, \dots, X_n]$ and we are looking for solutions with $0 \leq a_i \leq 1$. For simplicity, assume $\deg(F) = 2$ and $F$ is squarefree.

This technique is known due to M. Kreuzer, Algebraic Attacks (2009).

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2. For each term $X_i X_j$ in the support of $F$ introduce a new indeterminate $Y_{ij}$. Let $L$ be the linear part of $F$. Form the equation

$$\sum_{i,j} Y_{ij} + L - 2K = 0.$$

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$$\sum_{i,j} Y_{ij} + L - 2K = 0.$$
3. Form the inequalities
$$X_i \leq 1, Y_{ij} \leq X_i, Y_{ij} \leq X_j \text{ and } Y_{ij} \geq X_i + X_j - 1.$$

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# Real Polynomial Conversion (RPC)

Consider the polynomial equation $x_1 + x_2 = x_3 + x_4x_5$ over $\mathbb{F}_2$.

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1. $x_1 + x_2 = x_3 + x_6$, where $x_6 = x_4 x_5$
2. Lift over $\mathbb{R}$ using standard representation:

$$X_1 + X_2 - 2X_1X_2 - X_3 - X_6 + 2X_3X_6 = 0, X_6 - X_4X_5 = 0$$

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4. $X_i \leq 1,\ Z_k - X_i \leq 0,\ Z_k - X_j \leq 0,\ -Z_k + X_i + X_j - 1 \leq 0$

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# Converting Boolean Polynomials to CNF Clauses

Let $f \in \mathbb{F}_2[x_1, \dots, x_n]$ be a (squarefree) polynomial. Let $X = \{X_1, \dots, X_n\}$ be a set of boolean variables (atomic formulas), and let $\widehat{X}$ be the set of all (propositional) logical formulas that can be constructed (using $\neg$, $\wedge$, and $\vee$ operations) from them.

## Definition

*A logical representation of $f$ is a logical formula $F \in \widehat{X}$ such that $\varphi_a(F) = f(a_1, \dots, a_n) + 1$ for every $a = (a_1, \dots, a_n) \in \mathbb{F}_2^n$, where $\varphi_a$ denotes the boolean value of $F$ at the tuple of boolean values $a$ with $1 =$ `true` and $0 =$ `false`.*

G. Bard (2007), B. Chen (2008), P. Jovanovic and M. Kreuzer (2010).

# Converting Boolean Polynomials to CNF Clauses

Let $f \in \mathbb{F}_2[x_1, \dots, x_n]$ be a (squarefree) polynomial. Let $X = \{X_1, \dots, X_n\}$ be a set of boolean variables (atomic formulas), and let $\widehat{X}$ be the set of all (propositional) logical formulas that can be constructed (using $\neg$, $\wedge$, and $\vee$ operations) from them.

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## Conversion Procedure

- Linearize: introduce a new indeterminate for each nonlinear term
- Cutting: cut the resulting linear polynomial after certain no. of terms
- Logical Equivalent: find logical equivalents using a XOR-CNF conversion

G. Bard (2007), B. Chen (2008), P. Jovanovic and M. Kreuzer (2010).

- Standard Strategy (SS): substitute a new variable $y$ for $t$ in $f$ and form the clauses corresponding to $t + y$.
- Linear Partner Strategy (LPS): replace $x_i x_j + x_i$ in $f$ by $y$ and form the clauses corresponding to $x_i(x_j + 1) + y$.
- Double Partner Strategy (DPS): replace $x_i x_j + x_i + x_j + 1$ in $f$ by $y$ and form the clauses corresponding to $(x_i + 1)(x_j + 1) + y$.
- Quadratic Partner Substitution: replaces combinations of the form $x_i x_j + x_i x_k$.
- Cubic Partner Substitution: replaces combinations of the form $x_i x_j x_k + x_i x_j x_l$.

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Added by P. Jovanovic and M. Kreuzer (2010).

# The Logic of 0-1 Inequalities

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Any clause in propositional logic

$$X_1 \vee \cdots \vee X_r \vee \neg Y_1 \vee \cdots \vee \neg Y_s$$

can be translated into a clausal inequality

$$X_1 + \cdots + X_r + (1 - Y_1) + \cdots + (1 - Y_s) \geq 1$$

$$X_1 + \cdots + X_r - Y_1 + \cdots + -Y_s \geq 1 - s$$

A clause set is satisfiable if and only if the corresponding system of clausal inequalities together with the bounds $0 \leq X_i, Y_j \leq 1$ has an integer solution.

The LPC (using the LP strategy) of the polynomial $f = x_1 + x_2 + x_3 + x_3x_4$ is:

1. Let $x_1 + x_2 + y_1 = 0$ and form the clauses

$$\neg Y_1 \vee X_3, \neg Y_1 \vee \neg X_4, Y_1 \vee \neg X_3 \vee X_4$$

corresponding to $y_1 = x_3 + x_3x_4$.

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corresponding to $x_1 + x_2 + y_1 = 0$.
3. The clausal inequalities are
$$-X_3 + Y_1 \leq 0, X_4 + Y_1 - 1 \leq 0, X_3 - X_4 - Y_1 \leq 0,$$
$$X_1 - X_2 - Y_1 \leq 0, -X_1 + X_2 - Y_1 \leq 0,$$
$$-X_1 - X_2 + Y_1 \leq 0, X_1 + X_2 + Y_1 - 2 \leq 0.$$

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2. Hybrid RPC

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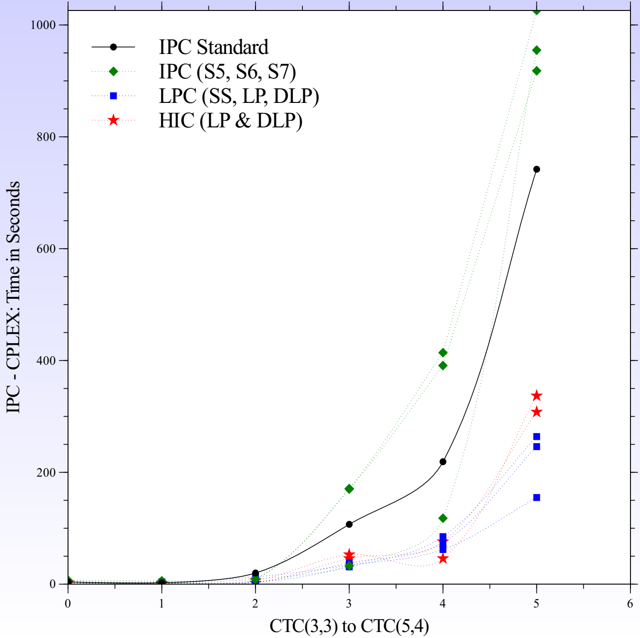
4. $X_i \leq 1$, $-X_4 + Y \leq 0$, $X_5 + Y - 1 \leq 0$, $-X_5 - Y + X_4 \leq 0$.

## The Hybrid IPC and RPC Conversions



# Comparison With IPC and RPC

| system | m | n | LPC(QPS) | HRPC(SS) | HRPC(QPS) | HIPC(QPS) | RPC | IPC |
|---|---|---|---|---|---|---|---|---|
| AES(8,1,1,4) | 1056 | 528 | 3908 | 21921 | 298 | **226** | 8351 | 1986 |
| AES(9,1,1,4) | 1184 | 592 | 26406 | 2493 | 814 | **236** | 2493 | 417 |
| AES(10,1,1,4) | 1312 | 656 | 6994 | 9521 | 13211 | **1982** | 9521 | 2655 |
| AES(4,2,1,4) | 1088 | 544 | 1377 | 6391 | 62338 | 3147 | 6391 | **789** |
| AES(2,2,2,4) | 1024 | 512 | 19970 | 19243 | 74982 | 81014 | 19243 | **7830** |
| AES(3,2,2,4) | 1472 | 736 | 523240 | 339069 | 279126 | **61020** | 532100 | 525226 |
| AES(1,1,1,8) | 640 | 220 | 42354 | **2043** | 207180 | 9323 | 10370 | 4684 |

| system | m | n | LPC(LP) | LPC(DLP) | HIPC(LP) | HIPC(DLP) | HRPC(SS) | RPC | IPC |
|---|---|---|---|---|---|---|---|---|---|
| CTC(5,5) | 605 | 330 | 691 | 679 | 1798 | 552 | **480** | 1356 | 2708 |
| CTC(5,6) | 705 | 375 | **270** | 1875 | 9332 | 2421 | 1041 | 1227 | 3088 |
| CTC(6,5) | 708 | 378 | 15540 | 16707 | 14661 | 11621 | 10723 | **7743** | 15656 |
| CTC(6,6) | 864 | 458 | 16941 | 12264 | **10716** | 16757 | 11572 | 25978 | 45272 |
| CTC(6,7) | 984 | 522 | 30868 | 18660 | **2285** | 11031 | 7716 | 9224 | 26209 |
| CTC(7,6) | 987 | 525 | 91358 | 97985 | 68146 | **9436** | 73090 | 11904 | 22198 |

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CPLEX running on a laptop with a 2.13 GHz Intel Pentium P6200 Dual Core processor and 4GB RAM.

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- We study the IPC and RPC conversions
- A new conversion technique called LPC
- Several new strategies to use with IPC and RPC

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*for your attention.*

*Questions ?*

*Remarks ?*