The Three-Body Problem

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1 Abstract

This article is devoted to the Three-Body Problem of interactions. We are interested in the numerical simulation of this problem. We formulate the equations of motion and see some common periodic solutions. In order to get the numerical solution of this problem we use the Runge-Kutta method.

2 Context

- Motivation and History
- Periodic solutions to the three-body problem
- The restricted three-body problem
- Runge-Kutta method
- Numerical simulation

3 Motivations and History

Few-body problem of interactions is called the N-body problem. The two-body problem was analyzed by Johannes Kepler in 1609 and solved by Isaac Newton in 1687. The three-body problem was a central topic in mathematical physics from the mid-1700s until the early 1900s. Various exact results were obtained - notably the existence of stable equilateral triangle configurations corresponding to the so-called Lagrange points. Many approximate practical calculations, particularly on the Earth-Moon-Sun system, were done using series expansions involving thousands of algebraic terms. (It is now possible to get most results just by direct numerical computation using for example NDSolve.) From its basic setup the three-body system conserves standard mechanical quantities like energy and angular momentum. But it was thought it might also conserve other quantities (or so-called integrals of the motion). In 1887, however, Heinrich Bruns showed that there could be no such quantities expressible as algebraic functions of the positions and velocities of the bodies (in standard Cartesian coordinates). In the mid-1890s Henri Poincar then showed that there could also be no such quantities analytic in positions, velocities and mass ratios. From these results, the conclusion was drawn that the three-body problem could not be solved in terms of algebraic formulas

and integrals. In 1912 Karl Sundman did however find an infinite series that could in principle be summed to give the solution, but these series converges exceptionally slowly.

In Poincaré's study of the collection of possible trajectories for three-body systems he identified sensitive dependence on initial conditions, noted the general complexity of what could happen (particularly in connection with so-called homoclinic tangles), and developed topology to provide a simpler overall description. With appropriate initial conditions one can get various forms of simple behavior.

We begin with the statement of the N-body problem and some of its solutions. Newton told us that two masses attract each other, the force of attraction being directed along the line joining them, proportional to the product of the masses, and inversely proportional to the square of the distance between them. If we have N masses, then the force on any one is the sum of the forces exerted on it by all the others. This gives us the nonlinear system of second-order differential equations

$$m_i \frac{d^2x}{dt^2} = -\sum_{i \neq j} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3} \tag{1}$$

$$i = 1, ..., N$$

m_i being the numerical value of the i^{th} mass, $x_i(t) \in \mathrm{R}^d$ its position vector, and r_{ij} is the distance between masses i and j. A solution to the N-body problem, we are interested in the planar case d = 2, is then a solution $x(t) = (x_1(t),\ldots,x_N(t))$ to these equations.

4 Periodic Solutions

Newton solved the two-body problem. The difference vector $x = x_1 - x_2$ satisfies Kepler's problem:

$$\frac{d^2x}{dt^2} = \frac{-kx}{|x|^3} \tag{2}$$

all solutions of which are conics with one focus at the origin. The Kepler constant k is $m_1 + m_2$. Correspondingly, if we fix the center of mass of our two bodies to be the origin, then they move along similar conic sections with one focus at this origin. The periodic two-body motions are ellipses. We refer to them as Keplerian ellipses. They include degenerate ellipses, sometimes called elliptic collision-ejection orbits, which are line segments with one endpoint at the origin. They represent collision solutions to the two-body problem.

It is impossible to describe all the solutions to the three-body problem. Following

Poincaré, we focus on the periodic solutions $x_i(t) = x_i(t+T)$. Here T is called the period. The simplest periodic solutions for the three-body problem were discovered by Euler [1765] and by Lagrange [1772]. Built out of Keplerian ellipses, they are the only explicit solutions.

To form the Lagrange solution, start by placing the three masses at the vertices x_{01}, x_{02} and x_{03} of an equilateral triangle whose center of mass $m_1x_{01} + m_2x_{02} + m_3x_{03}$ is the origin. Identify the plane of the triangle with the complex plane C, so that $x_{0i} \in$C. Take any solution $\lambda(t) \in$C to the planar Kepler problem (2) where the Kepler constant k is a certain rational expression in the three masses m_i . The Lagrange solutions are $x_i(t) = \lambda(t)x_{0i}$. Each mass moves in an ellipse in such a way that the triangle formed by the three masses evolves by a composition of instantaneous dilations and rotations and hence is equilateral for all time. The Lagrange orbits are only stable when one of the three masses is much greater than the other two.

For the Euler solutions start by placing the three masses on the same line with their positions x_{0i} such that the ratios r_{ij}/r_{ik} of their distances are the roots of a certain polynomial whose coefficients depend on the masses. Again, take any solution $\lambda(t)\in$C to Kepler's (2) equation where the Kepler constant is a certain other rational expression in the masses m_i. The Euler solutions are $x_i(t) = \lambda(t)x_{0i}$. At every instant the masses are collinear, and the ratios of their distances remain constant. There are three different families of Euler solutions, according to which mass remains between the other two. Together, the Euler and Lagrange solutions form the only solutions for which the similarity class of the triangle remains constant throughout the motion. Their beginning configurations x_{0i} are called central configurations. The Euler solutions are never stable.

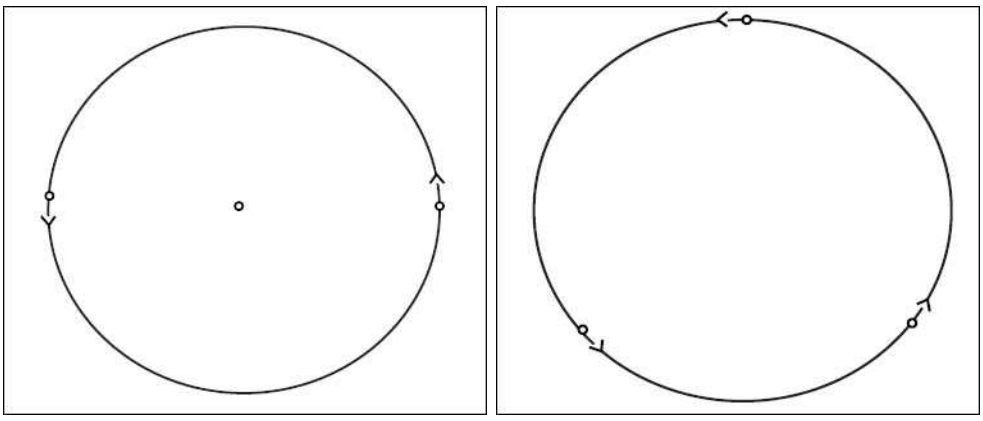

Figure 1: Euler's and Lagrange's solution (respectively) in the equal mass case.

Most important to astronomy are Hill's periodic solutions, also called tight binaries. These model the earth-moon-sun system. Two masses are close to each other while the

third remains far away. The two move in nearly circular orbits about their common center of mass. This center of mass and the third body in turn move in nearly circular orbits about the total center of mass. Like the Euler and Lagrange solutions, these Hill's solutions exist for all ratios of masses. And also they are always stable.

A new solution to the Three-Body problem is called the Figure Eight. Unlike the earlier orbits, it is particular to the case when all three masses are equal. The three equal masses chase each other around the same figure-eight-shaped curve in the plane. The eight was discovered numerically by Chris Moore [1993]. Alain Chenciner and Richard Montgomery [2001] rediscovered it and proved its existence. Carles Simó showed numerically that the figure eight is stable.

The eight is a periodic solution $x = (x_1(t), x_2(t), x_3(t))$ to the equal-mass three-body problem. If T is the period, then $x_2(t) = x_1(t - T/3)$ and $x_3(t) = x_1(t - 2T/3)$. This says that the three bodies travel the same planar curve, phase shifted from each other by one-third of a period. This curve has the form of a figure eight. There is an eight orbit of any period T, according to a scaling symmetry of the equations (1) to be described below. Modulo this scaling symmetry and the other obvious symmetries of (1), the eight is unique according to all numerical investigations. Its unicity has not been proved.

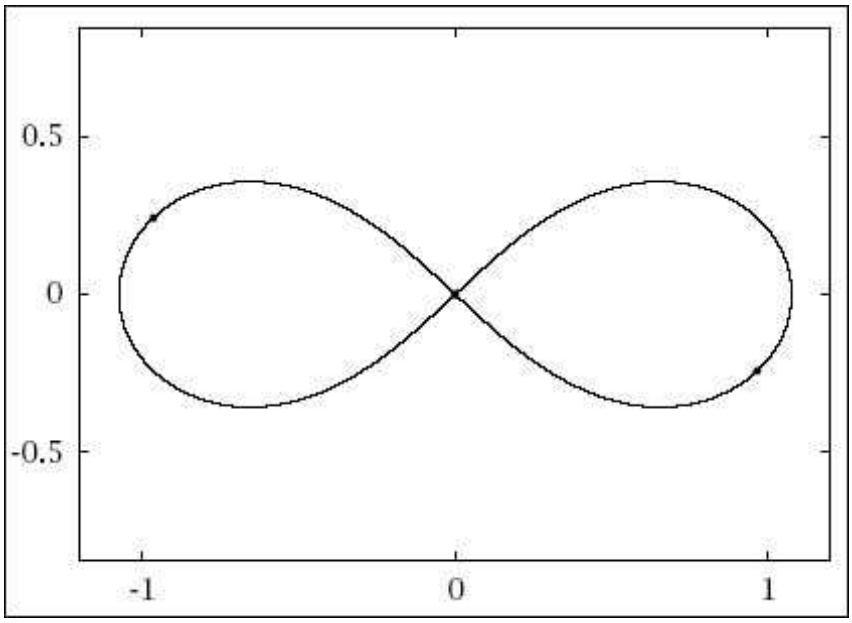

Figure 2: The figure eight solution.

5 The restricted three-body problem.

The restricted problem is said to be a limit of the three-body problem as one of the masses tends to zero.

To solve this problem we use Hamilton's equations. Let's formulate these equations in general case. We have to choose convenient coordinate system $q_k(t)$, k =1, .., n, to describe the position of the point masses as functions of the time t. The unknown functions in Hamilton's equations are the position coordinates $q_k(t)$ and the momentum coordinates $p_k(t)$ which are canonically conjugate to them. To determine the momentum coordinates one forms Lagrange function:

$$L = T(q_k, \frac{dq_k}{dt}) - V(q_k, \frac{dq_k}{dt}, t)$$

Were T is the kinetic and V the potential energy. The potential V may depend explicitly on the time. The canonical momentum coordinates p_k are defined as the partial derivatives:

$$p_k = \frac{\partial L}{\partial(dq_k/dt)}$$

One now expresses the Hamiltonian function H as a function of the coordinates p_k, q_k and dq_k/dp_k:

$$H(p_k, q_k) = \sum p_k \frac{dq_k}{dt} - L$$

Then Hamilton's equations:

$$\frac{dq_k}{dt} = \frac{dH}{dp_k}$$

$$\frac{dp_k}{dt} = -\frac{dH}{dq_k}$$

To solve Restricted Three-Body problem we choose rotating coordinate system. In order to obtain Hamilton's equations for small mass μ we first construct the Lagrange function:

$$T = \frac{\mu}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt} + \omega\right)^2\right)$$

The expression for the potential energy V of the mass μ depends only on the distances s_i=$|r_i$- $r|$ and is not influenced by the rotation of the coordinate system.

We obtain for Lagrange function and for the canonical momenta p_k:

$$L = T - V = \frac{\mu}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt} + \omega\right)^2\right) + \frac{\gamma m_1 \mu}{s_1} + \frac{\gamma m_2 \mu}{s_2}$$

$$p_r = \mu\frac{dr}{dt}$$

$$p_\varphi = \mu r^2\left(\frac{d\varphi}{dt} + \omega\right)$$

We can determine the Hamilton function:

$$H = \frac{p_r^2}{2\mu} + \frac{p_\varphi^2}{2\mu r^2} - p_\varphi\omega - \frac{\gamma m_1 \mu}{s_1} - \frac{\gamma m_2 \mu}{s_2}$$

And then Hamilton's equations:

$$\frac{dr}{dt} = \frac{p_r}{\mu}$$

$$\frac{d\varphi}{dt} = \frac{p_\varphi}{\mu r^2} - \omega$$

$$\frac{dp_r}{dt} = \frac{p_\varphi^2}{\mu r^3} - \frac{\gamma m_1 \mu}{s_1^3}(r - r_1\cos(\varphi - \varphi_1)) - \frac{\gamma m_2 \mu}{s_2^3}(r - r_2\cos(\varphi - \varphi_2))$$

$$\frac{dp_\varphi}{dt} = -\frac{\gamma m_1 \mu}{s_1^3} r r_1 \sin\varphi - \varphi_1) - \frac{\gamma m_2 \mu}{s_2^3} r r_2 \sin\varphi - \varphi_2)$$

The solution to this system of equations gives us position and velocity of the small mass as functions of time. To solve these equations we use 4^{th} order Runge-Kutta method.

6 The Runge-Kutta method.

The Runge-Kutta method was first developed by the German mathematicians C.D.T. Runge and M.W. Kutta in the latter half of the nineteenth century. It is based on difference schemes.

First we obtain Runge-Kutta scheme for the Cauchy problem:

$$\begin{cases} \frac{du}{dx} = f(x,u) \\ u(a) = u_0 \end{cases} \tag{3}$$

Let u be the solution, and let's expand it in the Taylor series:

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + ... + \frac{h^n}{n!}u^{(n)}(x) + O(h^{n+1}) \tag{4}$$

With (3) we have:

$$u''(x_i) = \frac{d}{dx}f(x,u)\bigg|_{x_i} = f'_x(x_i,u_i) + f(x_i,u_i)f'_u(x_i,u_i)$$

If we put it into (4) and substitute derivatives for the difference derivatives then:

$$y_{j+1} = y_j + h[(1-\beta)f(x_i,y_i) + \beta \cdot f(x_i + \frac{h}{2\beta}, y_j + \frac{h}{2\beta}f(x_j,y_j))]$$

Where y_{j+1} is the approximation of the solution u, $0<\beta<1$ is the parameter of difference derivative. Usually one considers β is equal to $1/2$, and in this case we obtain the Runge-Kutta scheme of second order. In the same way (if we take account of the 4^{th} order of smallness Taylor series term) we can obtain the scheme of 4^{th} order. It takes on form:

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{array}{l} k_1 = f(x_i, y_i) \\ k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) \\ k_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2) \\ k_4 = f(x_i + h, y_i + hk_3) \end{array}$$

Now let's consider a system of differential equations:

$$\begin{cases} u'' = f(x,u,u') \\ u(x_0) = u_0 \\ u'(x_0) = u'_0 \end{cases}$$

Let's denote $u' = v$, $\vec{\mathbf{u}} = \begin{pmatrix} u \\ v \end{pmatrix}$. Then system takes on form:

$$\begin{cases} \vec{\mathbf{u}} = f(x, \vec{\mathbf{u}}) \\ \vec{\mathbf{u}}(x_0) = \vec{\mathbf{u}}_0 \end{cases}$$

If $\vec{\mathbf{y}}_j = \begin{pmatrix} y_j \\ z_j \end{pmatrix}$ is a vector of approximations of the solution $\vec{\mathbf{u}}_j$, at point x_j, and $\vec{\mathbf{k}}_m = \begin{pmatrix} k_m \\ q_m \end{pmatrix}$ are vectors of design factors, then:

$$\vec{\mathbf{y}}_{j+1} = \begin{pmatrix} y_{j+1} \\ z_{j+1} \end{pmatrix} = \begin{pmatrix} y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ z_j + \frac{h}{6}(q_1 + 2q_2 + 2q_3 + q_4) \end{pmatrix}$$

$$\begin{array}{ll} k_1 = z_j & k_2 = z_j + \frac{h}{2}q_1 \\ k_3 = z_j + \frac{h}{2}q_2 & k_4 = z_j + hq_3 \\ q_1 = f(x_j, y_j, z_j) & q_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1, z_j + \frac{h}{2}q_1) \\ q_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_2, z_j + \frac{h}{2}q_2) & q_4 = f(x_j + h, y_j + hk_3, z_j + hq_3) \end{array}$$

In order to control the error that occurs on every calculation step, we have to estimate this error.

Theorem (error approximation in Runge-Kutta method):
$\varepsilon_h(t_1) = |y_h(t_1) - y(t_1)| \approx \frac{16}{15}\left|y_h(t_1) - y_{h/2}(t_1)\right|$, where ε_h is the calculation error for one step at the point t_1 with step size h.

7 Numerical simulation

Numerical simulation is based on:

- 4^{th} order Runge-Kutta method
- Adaptive stepsize control for Runge-Kutta

Program is developed in Delphi. Here are some obtained orbits:

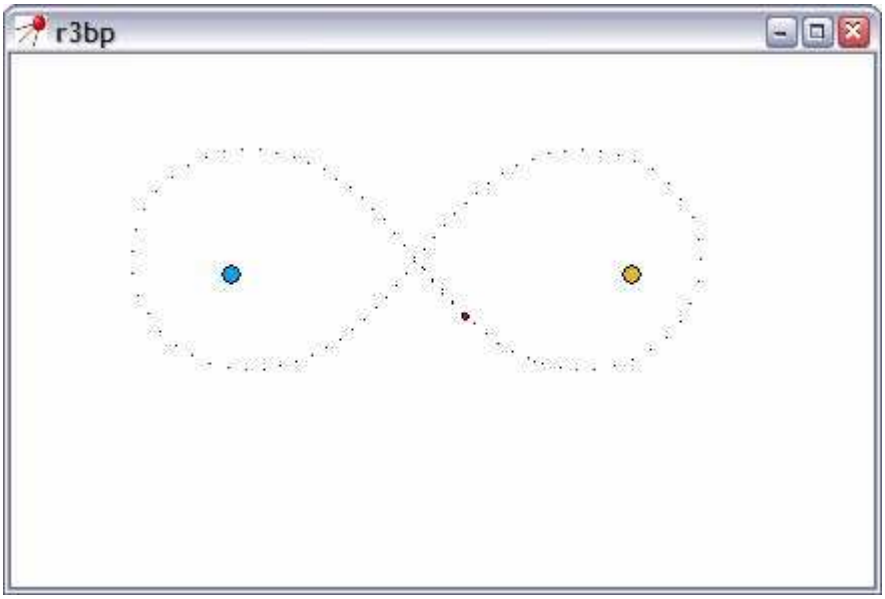

Figure 3: Eight.

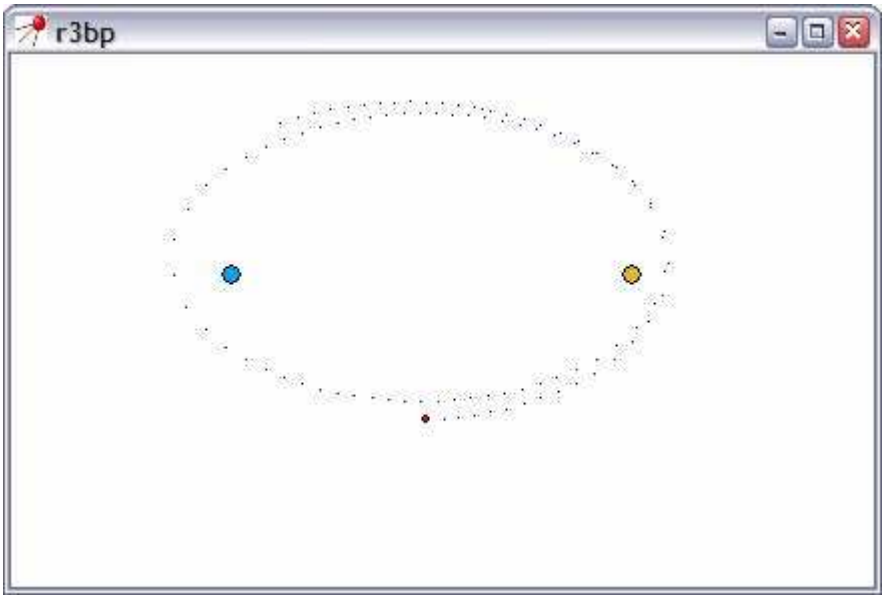

Figure 4: Ellipse.

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