Course "Proofs and Computers“, JASS’06

Introducing IP, AM, MA

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March 30, 2006
We can view NP as a proof system. For each language $L \in \text{NP}$ there exists a polynomial-time recognizable relation $R_L$ such that:

$$L = \{x | \exists y : s.t. (x, y) \in R_L\}$$

and $(x, y) \in R_L$ only if $|y| \leq \text{poly}(|x|)$. 
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### NP as a proof System

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2. Correctness requirements:
   - **Completeness**: For a true assertion, there is a convincing proof strategy (in the case of NP, if \( x \in L \) the witness \( y \) exists).
   - **Soundness**: For a false assertion, no convincing proof strategy exists (in the case of NP, if \( x \notin L \) then no witness \( y \) exists).
Interactive Proof Systems

Now we generalize the requirements from a proof system, adding interaction and randomness.
An interactive proof is a sequence of questions and answers between the prover and the verifier.

At the end of the interaction, the verifier decides based on the knowledge he acquired in the process whether the claim is true or false.
Definition 1

(*interactive proof systems.*) An *interactive proof system* for a language $L$ is a two-party game between a verifier and a prover that interact on a common input in a way satisfying the following properties:

1. The verifier strategy is a probabilistic polynomial-time procedure (where time is measured in terms of the length of the common input).
2. Correctness requirements:
   - **Completeness**: There exists a prover strategy $P$, such that for every $x \in L$, when interacting on the common input $x$, the prover $P$ convinces the verifier with probability at least $\frac{2}{3}$.
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The IP Hierarchy

**Definition 2**

(\textit{The IP Hierarchy}:) The complexity class \textbf{IP} consists of all languages having an interactive proof system. We call the number of message exchanges (a question and an answer) between the two parties, the number of \textit{rounds} in the system. After a certain number of rounds the verifier decides whether to accept or reject.

For every integer function \(r(.)\), the complexity class \(\text{IP}(r(.))\) consists of all the languages that have an interactive proof system in which, on common input \(x\), at most \(r(|x|)\) rounds are used. If we denote by \textbf{poly} the set of all integer polynomial functions, then \(\text{IP} = \text{IP}(\text{poly})\).
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Claim 3

Any language that has an interactive proof system, has one that achieves error probability of at most $2^{-p(\cdot)}$ for any polynomial $p(\cdot)$. 

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Proof.
Using Chernoff’s Bound:

\[ Pr[z < (1 - \delta)E(z)] < e^{-\frac{\delta^2 E(z)}{2}} \]

We choose \( k = O(p(.)) \) and \( \delta = \frac{1}{4} \) and note that \( E(z) = \frac{2}{3}k \) (so that \( \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2} \)) to get:

\[ Pr[z < (1 - \frac{1}{2}k] < 2^{-p(.)} \]
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- By adding interaction only, the interactive proof systems collapse to NP-proof systems.
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- By adding interaction only, the interactive proof systems collapse to NP-proof systems.
- By adding randomness only, we get a proof system in which the prover sends a witness and the verifier can run a BPP algorithm for checking its validity. We obtain the class Merlin-Arthur game - MA.
Example 4 (Graph Non-Isomorphism (GNI))

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic (denoted $G_1 \cong G_2$) if there exists a 1-1 and onto mapping $\pi : V_1 \rightarrow V_2$ such that $(u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2$. The mapping $\pi$, if existing, is called an isomorphism between the graphs. If no such mapping exists then the graphs are non-isomorphic (denoted $G_1 \not\cong G_2$).

GNI is the language containing all pairs of non-isomorphic graphs. Formally:

$$GNI = \{(G_1, G_2) : G_1 \not\cong G_2\}$$
An interactive proof system for GNI:

- $G_1$ and $G_2$ are given as input to the verifier and the prover. Assume without loss of generality that $V_1 = V_2 = \{1, 2, \ldots, n\}$. 


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- The verifier chooses $i \in_R \{1, 2\}$ and $\pi \in_R S_n$ ($S_n$ is the group of all permutations on $\{1, 2, \ldots, n\}$). He applies the mapping $\pi$ on the graph $G_i$ to obtain a graph $H$

$$H = (\{1, 2, \ldots, n\}, E_H) \text{ where } E_H = \{(\pi(u), \pi(v)) : (u, v) \in E_i\}$$

and sends the graph $H$ to the prover.
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- The prover sends $j \in \{1, 2\}$ to the verifier.
Graph Non-Isomorphism (GNI)

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   and sends the graph $H$ to the prover.

3. The prover sends $j \in \{1, 2\}$ to the verifier.

4. The verifier accepts iff $j = i$. 

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Remark: We state that the secrecy of the outcome of the coin tosses is essential to this protocol.
Definition 5

(public-coin interactive proofs - AM:) Public coin proof systems (known also as Arthur-Merlin games) are a special case of interactive proof systems, in which, at each round, the verifier can only toss coins, and send their outcome to the prover. After a certain number of rounds the verifier decides **deterministically** whether to accept or reject.
For every integer function $r(.)$, the complexity class $AM(r(.))$ consists of all the languages that have an Arthur-Merlin proof system in which, on common input $x$, at most $r(|x|)$ rounds are used.
Denote $AM = AM(1)$. 
Surprisingly it was shown Arthur-Merlin games and the general interactive proof systems are essentially equivalent:

**Theorem 6**

(Relating $IP(.)$ to $AM(.)$):

$$\forall \ r(.) : IP(r(.)) \subseteq AM(r(.)) + 1$$
The following theorem shows that power of $AM(r(.))$ is invariant under a linear change in the number of rounds:

**Theorem 7**

*(Linear Speed-up Theorem)*:

$$\forall r(.) \geq 2 : AM(2r(.)) = AM(r(.))$$
Combing the two last theorems we get:

**Corollary 8**

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(*Collapse of constant-round IP to one-round AM*):

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**Corollary 10**

*(Relating MA to AM)*

\[ MA \subseteq AM \]
Theorem 11

(Relating MA to PP):

\[ MA \subseteq PP \]

Proof. Let \( L \in MA \). Thus there are a polynomial \( p \) and a polynomial-time Turing machine \( Q \) such that:

\[
x \in L \implies \exists s \in \{0, 1\}^{p(|x|)} : Pr[Q(x, r, x)] > \frac{2}{3}
\]

\[
x \notin L \implies \forall s \in \{0, 1\}^{p(|x|)} : Pr[Q(x, r, x)] < \frac{1}{3}
\]

where probability is taken over uniform distribution in \( \{0, 1\}^{p(|x|)} \).
Using standard amplification we can construct a new polynomial $p_1$ and a new polynomial-time machine $Q_1$ such that

$$x \in L \Rightarrow \exists s \in \{0, 1\}^{p(|x|)} : Pr[Q_1(x, r, s)] > 1 - 4^{-p(|x|)}$$

$$x \notin L \Rightarrow \forall s \in \{0, 1\}^{p(|x|)} : Pr[Q_1(x, r, s)] < 4^{-p(|x|)}$$

where probability is taken over uniform distribution in $\{0, 1\}^{p_1(|x|)}$. Consider now the uniform distribution on pairs $< r, s > \in \{0, 1\}^{p(|x|) + p_1(|x|)}$. We have

$$x \in L \Rightarrow \exists Pr[Q_1(x, r, s)] > 2^{-p(|x|)}(1 - 4^{-p(|x|)}) > 4^{-p(|x|)}$$

$$x \notin L \Rightarrow Pr[Q_1(x, r, s)] < 4^{-p(|x|)}$$

This is equivalent to $L \in PP$. □
What if we require Perfect Completeness, i.e., convincing the verifier with probability 1?
**Theorem 12**

*If a language has an interactive proof system then it has one with perfect completeness.*

We will show that given a public coin proof system we can construct a perfect completeness public coin proof system. We define:

\[ AM^0(r(.)) = \{ L \mid L \text{ has a perfect completeness } r(.) \text{ round public coin proof system} \} \]
We will show:

**Lemma 13**

*If L has a public coin proof system then it has one with perfect completeness*

\[ AM(r(.)) \subseteq AM^0(r(.)) + 1 \]
Proof.

- Assume that the Arthur-Merlin proof system consists of $t$ rounds.
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- Also assume that the completeness and soundness error probabilities of the proof system are at most $\frac{1}{3tm}$. This is obtained using standard amplification.
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- We denote the messages sent by Arthur (the verifier) \( r_1, \ldots, r_t \) and the messages sent by Merlin (the prover) \( \alpha_1, \ldots, \alpha_t \).
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- We denote the messages sent by Arthur (the verifier) $r_1, \ldots, r_t$ and the messages sent by Merlin (the prover) $\alpha_1, \ldots, \alpha_t$.
- Denote by $< P, V >_x (r_1, \ldots, r_t)$ the outcome of the game on common input $x$ between the optimal prover $P$ and the verifier $V$. 
We construct a new proof system with perfect completeness, in which Arthur and Merlin play $tm$ games simultaneously.
Perfect Completeness

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Formally, we add an additional round at the beginning in which Merlin sends the \(tm\) shifts \(S^1, \ldots, S^m\) where 
\[
S^i = (S^i_1, \ldots, S^i_t), \quad S^i_j \in \{0, 1\}^m
\]
to Arthur.
For game $i$ and round $j$, Merlin considers the random coins to be $r_j \oplus S^i_j$ and sends as a message $\alpha^i_j$ where $\alpha^i_j$ is computed according to $(r_1 \oplus S^i_1, \ldots, r_t \oplus S^i_t)$. 
Perfect Completeness

- For game $i$ and round $j$, Merlin considers the random coins to be $r_j \oplus S_j^i$ and sends as a message $\alpha_j^i$ where $\alpha_j^i$ is computed according to $(r_1 \oplus S_1^i, \ldots, r_t \oplus S_t^i)$.
- The entire message in round $j$ is $\alpha_j^1, \ldots, \alpha_j^{tm}$. 
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The entire message in round $j$ is $\alpha_j^1, \ldots, \alpha_j^{tm}$.

At the end of the protocol Arthur accepts if at least one out of the $tm$ games is accepting.
In order to show perfect completeness we will show that for every \( x \in L \) there exists \( S^1, \ldots, S^{tm} \) such that for all \( r_1, \ldots, r_t \) at least one of the games is accepting.
**Perfect Completeness**

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- We use a probabilistic argument to show that the complementary event occurs with probability strictly smaller than 1.
Perfect Completeness

\[ Pr_{S_1, \ldots, S_{tm}}[\exists r_1, \ldots, r_t \bigwedge_{i=1}^{tm} (\langle P, V \rangle_x (r_1 \oplus S_1^i, \ldots, r_t \oplus S_t^i) = 0)] \]

\[ \leq (1) \sum_{r_1, \ldots, r_t \in \{0,1\}^m} Pr_{S_1, \ldots, S_{tm}}[\bigwedge_{i=1}^{tm} (\langle P, V \rangle_x (r_1 \oplus S_1^i, \ldots, r_t \oplus S_t^i) = 0)] \]

\[ \leq (2) 2^{tm} \cdot \left( \frac{1}{3tm} \right)^{tm} < 1 \]

Inequality (1) is obtained using the union bound. Inequality (2) is due to the fact that the \( r_j \oplus S_j^i \) are independent random variables so the results of the games are independent, and that the optimal prover fails to convince the verifier on a true assertion with probability at most \( \frac{1}{3tm} \).
We still have to show that the proof system suggested satisfies the soundness requirement. We show that for every $x \notin L$ and for any prover strategy $P^\star$ and choices of shifts $S^1, \ldots, S^{tm}$ the probability that one or more of the $tm$ games is accepting is at most $\frac{1}{3}$. 
Perfect Completeness

\[
Pr_{r_1, \ldots, r_t} \left[ \bigvee_{i=1}^{tm} (\langle P, V \rangle_x (r_1 \oplus S_1^i, \ldots, r_t \oplus S_t^i) = 1) \right] \\
\leq (1) \sum_{i=1}^{tm} Pr_{r_1, \ldots, r_t} \left[ \langle P^\star, V \rangle_x (r_1 \oplus S_1^i, \ldots, r_t \oplus S_t^i) = 1 \right] \\
\leq (2) \sum_{i=1}^{tm} \frac{1}{3tm} = \frac{1}{3}
\]

Inequality (1) is obtained using the union bound. Inequality (2) is due to the fact that any prover has probability of at most \(\frac{1}{3tm}\) of success for a single game. \(\square\)
Unlike the last theorem, requiring \textit{perfect soundness} reduces the model to an \textit{NP}-proof system.

\textbf{Proposition 14}

\textit{If a language \( L \) has an interactive proof system with perfect soundness then \( L \in \text{NP} \).}

\textbf{Remark:} (This is an alternative argument for interactive proof systems collapsing to \textit{NP} without randomness. This is due to the fact that perfect soundness is equivalent to a deterministic verifier.)
We shall conclude this paper with a very interesting protocol that uses interactive proofs and cryptography. Suppose that Alice is a girl with superintellectual abilities capable to solve NP-problems.
And suppose that Bob - an ordinary guy, but a good friend - is only able to compute problems in P.

Alice knows a 3-coloring of a large graph $G = (V, E)$ and wants to convince Bob that she has a coloring of $G$ without telling him the coloring.

What is required here is a **zero knowledge proof**, that is, an interactive protocol at the end of which Bob is convinced that with very high probability Alice has a legal 3-coloring of $G$, but has no clue about the actual 3-coloring.
Alice’s coloring is $\chi : V \mapsto \{00, 11, 01\}$. The protocol proceeds in rounds. At each round, Alice carries out the following steps:

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A zero knowledge proof
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- She generates $|V|$ RSA public-private key pairs, $(p_i, q_i, d_i, e_i)$, one for each node $i \in V$.
- For each node $i$ she computes the probabilistic encoding $(y_i, y'_i)$, according the $j$ the RSA system, of the color $\pi(\chi(i))$. 
Suppose that $b_i, b'_i$ are the two bits of $\pi(\chi(i))$; then $y_i = (2x_i + b_i)^e \mod p_i q_i$ and $y'_i = (2x'_i + b'_i)^e \mod p_i q_i$, where $x_i$ and $x'_i$ are random integers no greater than $pq/2$. 
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All these computations are private to Alice. Alice reveals to Bob the integers $(e_i, p_i q_i, y_i, y'_i)$ for each node $i \in V$. That is, the public part of the RSA systems, and the encrypted colors.
Bob’s turn:

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- He checks that indeed \(b_i b'_i \neq b_j b'_j\).
Alice and Bob repeat \( k|E| \) times, where \( k \) is a parameter representing the desired reliability of the protocol.

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- At each round Bob has a probability of at least $\frac{1}{|E|}$ if discovering that edge.
- After $k|E|$ rounds, the probability of Bob finding out that Alice has no legal coloring is at least $1 - e^{-k}$. 
What is remarkable about this protocol is that Bob has learned nothing about Alice’s coloring of $G$ in the process.
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As a final note the zero knowledge protocol just described works for 3-COLORING, an NP-complete problem. Using reductions, it is possible to conclude all problems in NP have zero-knowledge proofs.