Numerical Simulation
Sparse Grids

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Course 2: Numerical Simulation – From Models to Visualizations
1 Introduction

2 Hierarchical Basis
   • In 1 dimension
   • In 2 or more dimensions
   • Sparse grids

3 Conclusion
Some Example Problems

- PDE: $\triangle u = f$ in $\Omega$ and $u|_{\partial\Omega} = 0$

find $u \in V$ with $u|_{\partial\Omega} = 0$
Some Example Problems

- PDE: $\Delta u = f$ in $\Omega$ and $u|_{\partial\Omega} = 0$

find $u \in V$ with $u|_{\partial\Omega} = 0$

- numerical quadrature

compute $\int_{\Omega} f(x) \, dx$
What is the main problem we have to solve?
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- given a multivariate function \( f : \Omega \to \mathbb{R} \)
- want to construct a function \( u : \Omega \to \mathbb{R} \) with special properties
- only an approximation \( u_S \) to \( u \) is possible
- quality of \( u_S \) depends on the number of evaluations of \( f \)
Motivation

What is the main problem we have to solve?

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- quality of $u_S$ depends on the number of evaluations of $f$

So do what we have to do?

- evaluate $f(x)$ for many different states $x \in \Omega$
- but: evaluation of $f$ is expensive
Example

With a naive approach ($n$ sample points in each dimension):

- 1-dim: $n$ evaluations of $f$
- $d$-dim: $n^d$ evaluations of $f$
Introduction

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Example

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- in high dimensional problems $d$ is very large
- $n^d f$ evaluations is too expensive
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Challenge

- reduce the number of \(f\) evaluations
Introduction

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**Example**

With a naive approach \((n\text{ sample points in each dimension})\):

- \(1\text{-dim}: n\) evaluations of \(f\)
- \(d\text{-dim}: n^d\) evaluations of \(f\)

- in high dimensional problems \(d\) is very large
- \(n^d\) evaluations is too expensive

**Challenge**

- reduce the number of \(f\) evaluations
- keep quality of \(u_S\) still as high as possible
Introduction

Main Goal

a little be more in detail:
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What to do

- take a finite $n$-dimensional subspace $S \subset V$ with

$$S = \text{span} \{ \phi_i : 1 \leq i \leq n \}$$
Main Goal

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- want to find a function $u \in V$ with special properties
- $V$ is often a Sobolev Space, particularly $\text{dim}(V) = \infty$

What to do

- take a finite $n$-dimensional subspace $S \subset V$ with
  
  $$ S = \text{span}\{\phi_i : 1 \leq i \leq n\} $$

- $\Rightarrow$ get an approximative $u_S$ as linear combination of basis functions:

  $$ u_S = \sum_{i=1}^{n} \alpha_i \cdot \phi_i $$
Outline

1. Introduction

2. Hierarchical Basis
   - In 1 dimension
   - In 2 or more dimensions
   - Sparse grids

3. Conclusion
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- search for 'good' basis functions \( \phi_i \) to approximate any given \( u : \Omega \rightarrow \mathbb{R} \)
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How to get Basis Functions

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- the \( \phi_i \) should be inexpensive to evaluate
- \( \text{dim}(S) \) should be small for an optimal approximation
- w.l.o.g. we set \( \Omega := [0, 1] \), \( u(0) = u(1) = 0 \)
Hierarchical Basis

In 1 dimension

Piecewise Linear Approach

- consider \( n = 2^\ell - 1 \) equidistant (inner) knots

\[
x_{\ell,i} = i \cdot h_\ell \quad \text{with} \quad h_\ell = 2^{-\ell} \quad \text{and} \quad 1 \leq i \leq 2^\ell - 1
\]
Hierarchical Basis

In 1 dimension

Piecewise Linear Approach

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- piecewise linear approach with ordinary \textbf{hat function}
  \[
  \phi(x) = \max\{1 - |x|, 0\}
  \]
Piecewise Linear Approach

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  \[ \phi(x) = \max\{1 - |x|, 0\} \]

- for every point \( x_{\ell,i} \) we construct a function \( \phi_{\ell,i}(x) : \Omega \to \mathbb{R} \)
  \[ \phi_{\ell,i}(x) = \phi\left(\frac{x - x_{\ell,i}}{h_\ell}\right), \quad T_{\ell,i} := \text{supp}(\phi_{\ell,i}) = [x_{\ell,i-1}, x_{\ell,i+1}] \]
Hierarchical Basis

In 1 dimension

### Piecewise Linear Approach

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x_{\ell,i} = i \cdot h_\ell \quad \text{with} \quad h_\ell = 2^{-\ell} \quad \text{and} \quad 1 \leq i \leq 2^\ell - 1
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- piecewise linear approach with ordinary hat function

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Nodal Point Basis

- **the nodal point basis**

\[ \Phi_\ell := \{ \phi_{\ell,i}, i = 1, \ldots, 2^\ell - 1 \} \]
Nodal Point Basis

- the nodal point basis

\[ \Phi_\ell := \{ \phi_{\ell,i}, i = 1, \ldots, 2^\ell - 1 \} \]

- the resulting subspace of \( V \):

\[ S_\ell := \text{span}(\Phi_\ell), \quad \text{dim}(S_\ell) = 2^\ell - 1 \]
Nodal Point Basis

- \( u_\ell \in S_\ell \) can be written as:

\[
    u_\ell(x) = \sum_{i=1}^{2^\ell - 1} \alpha_i \cdot \phi_{\ell,i}(x)
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Nodal Point Basis

- \( u_\ell \in S_\ell \) can be written as:

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u_\ell(x) = \sum_{i=1}^{2^\ell - 1} \alpha_i \cdot \phi_{\ell,i}(x)\]

- the coefficients can be computed very fast

\[
\alpha_i = u_\ell(x_{\ell,i}) = u(x_{\ell,i})
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- The coefficients can be computed very fast

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- Example for an arbitrary function $u_\ell \in S_\ell$
Nodal Point Basis

Example

For the function $u(x) = 4x(1 - x)$ we have the following coefficients:

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\alpha_i = u_\ell(x_{\ell,i})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{3}{4}$ $1$ $\frac{3}{4}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{7}{16}$ $\frac{3}{4}$ $\frac{15}{16}$ $1$ $\frac{15}{16}$ $\frac{3}{4}$ $\frac{7}{16}$</td>
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![Graph of $u(x) = 4x(1 - x)$](image-url)
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<tr>
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<td>$3/4$</td>
<td>1</td>
</tr>
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The plot shows the function $u(x) = 4x(1 - x)$ with nodal points at $x = 0, 0.5, 1$. The coefficients are plotted at these points, with $\alpha_1 = 1$, $\alpha_2 = 3/4$, and $\alpha_3 = 7/16$. The graph illustrates how the function behaves across the interval $[0, 1]$. The vertical line at $x = 0.5$ represents the midpoint where the function reaches its maximum value of $1$.
### Example

For the function \( u(x) = 4x(1-x) \) we have the following coefficients:

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Nodal Point Basis

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\[
\alpha_1 = u_1(x_{1,1}) = 1, \quad \alpha_2 = u_2(x_{2,1}) = 3/4, \quad \alpha_3 = u_3(x_{3,1}) = 7/16.
\]
What do we have so far?

\[ V = \bigcup_{\ell=1}^{\infty} S_{\ell} = \text{span}(\Phi) \quad \text{with} \quad \Phi := \bigcup_{\ell=1}^{\infty} \Phi_{\ell} \]
Alternative Basis

What do we have so far?

\[ V = \bigcup_{\ell=1}^{\infty} S_\ell = \text{span}(\Phi) \quad \text{with} \quad \Phi := \bigcup_{\ell=1}^{\infty} \Phi_\ell \]

But still big problems remain:
- \( \Phi \) is not a basis
- the coefficients remain large for increasing \( \ell \)
Hierarchical Basis

In 1 dimension

Alternative Basis

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But still big problems remain:

- \( \Phi \) is not a basis
- the coefficients remain large for increasing \( \ell \)

Search for an alternative basis for \( S_\ell \)!
Hierarchical Increments

we define the **hierarchical increments**

\[ W_\ell := \text{span}\{\phi_{\ell,i} : i \in I_\ell\} \quad I_\ell := \{i : 1 \leq i \leq 2^\ell - 1, i \text{ odd}\} \]
Hierarchical Basis

In 1 dimension, we define the hierarchical increments

\[ W_\ell := \text{span}\{\phi_{\ell,i} : i \in I_\ell\} \quad I_\ell := \{i : 1 \leq i \leq 2^\ell - 1, i \text{ odd}\} \]

the basis functions for the hierarchical increments \( W_1, W_2, W_3 \):
Hierarchical Basis and Hierarchical Surpluses

- we get a new view of \( S_\ell \) and \( V \)

\[
S_\ell = \bigoplus_{k=1}^{\ell} W_k \quad \text{and} \quad V = \bigoplus_{k=1}^{\infty} W_k
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Hierarchical Basis and Hierarchical Surpluses

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$$S_\ell = \bigoplus_{k=1}^{\ell} W_k \quad \text{and} \quad V = \bigoplus_{k=1}^{\infty} W_k$$

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- with the hierarchical surpluses $w_\ell$

$$u(x) = \sum_{\ell=1}^{\infty} w_\ell(x), \quad w_\ell = \sum_{i \in I_\ell} v_{\ell,i} \cdot \phi_{\ell,i} \in W_\ell$$

$$v_{\ell,i} = u(x_{\ell,i}) - \frac{u(x_{\ell,i-1}) + u(x_{\ell,i+1})}{2}$$
Hierarchical Basis

Example

Again given the function \( u(x) = 4x(1 - x) \) but different coefficients:

\[
\begin{array}{ccc|c|c|c|c}
\ell = 1 & v_i \\
\ell = 2 & 1/4 & 1 \\
\ell = 3 & 1/16 & 1/4 & 1/16 & 1/16 & 1/4 & 1/16
\end{array}
\]

![Graph of the function](image)
Hierarchical Basis

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Convergence

We have

\[ u(x) = \sum_{\ell=1}^{\infty} w_{\ell}(x) \quad \text{and} \quad w_{\ell} = \sum_{i \in I_{\ell}} v_{\ell,i} \cdot \phi_{\ell,i} \]
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- It seems that the \( v_{\ell,i} \) decrease very fast with increasing \( \ell \).
Convergence

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\[ u(x) = \sum_{\ell=1}^{\infty} w_\ell(x) \quad \text{and} \quad w_\ell = \sum_{i \in I_\ell} v_{\ell,i} \cdot \phi_{\ell,i} \]

- it seems that the \( v_{\ell,i} \) decrease very fast with increasing \( \ell \)
- are the \( w_\ell \) really less important for large \( \ell \)?
We have

\[ u(x) = \sum_{\ell=1}^{\infty} w_\ell(x) \quad \text{and} \quad w_\ell = \sum_{i \in I_\ell} v_{\ell,i} \cdot \phi_{\ell,i} \]

- it seems that the \( v_{\ell,i} \) decrease very fast with increasing \( \ell \)
- are the \( w_\ell \) really less important for large \( \ell \)?
- to decide about the importance of each \( w_\ell \): define some norms
There are different possibilities to define norms in $V$

Important norms in our applications:

**maximum norm:**

$$\| \phi_{\ell,i} \|_{\infty} := \max_{x \in \Omega} \{ \phi_{\ell,i}(x) \} = 1$$
Norms

There are different possibilities to define norms in $V$

Important norms in our applications:

maximum norm:

$$\|\phi_{\ell,i}\|_{\infty} := \max_{x \in \Omega} \{\phi_{\ell,i}(x)\} = 1$$

$L_2$ norm:

$$\|\phi_{\ell,i}\|_2 := \sqrt{\int_{\Omega} \phi_{\ell,i}^2(x) \, dx} = \sqrt{\frac{2h_\ell}{3}}$$
There are different possibilities to define norms in $V$

Important norms in our applications:

**maximum norm:**

$$\| \phi_{\ell,i} \|_\infty := \max_{x \in \Omega} \{ \phi_{\ell,i}(x) \} = 1$$

**$L_2$ norm:**

$$\| \phi_{\ell,i} \|_2 := \sqrt{\int_\Omega \phi_{\ell,i}^2(x) \, dx} = \sqrt{\frac{2h_{\ell}}{3}}$$

**energy norm:**

$$\| \phi_{\ell,i} \|_E := \sqrt{\int_\Omega (\phi'_{\ell,i}(x))^2 \, dx} = \sqrt{\frac{2}{h_{\ell}}}$$
Norms

We have norms now, but we don't know $u$. 
Norms

We have norms now, but we don’t know $u$

- we need another representation of $v_{\ell,i}$:

$$v_{\ell,i} = \int_0^1 -\frac{h_\ell}{2} \cdot \phi_{\ell,i} \cdot u''(x) \, dx$$
Norms

We have norms now, but we don’t know \( u \)

- we need another representation of \( v_{\ell,i} \):

\[
v_{\ell,i} = \int_{0}^{1} \left( \frac{h_{\ell}}{2} \right) \cdot \phi_{\ell,i} \cdot u''(x) \, dx
\]

- now we can find some upper bounds for the \( v_{\ell,i} \)

\[
|v_{\ell,i}| \leq \frac{h_{\ell}^2}{2} \cdot ||u''||_{T_{\ell,i}} \infty
\]

\[
|v_{\ell,i}| \leq \frac{h_{\ell}^3}{6} \cdot ||u''||_{T_{\ell,i}} ^2
\]
And what we originally wanted to quantify:

\[
\begin{align*}
\|w_\ell\|_\infty & \leq \frac{h_\ell^2}{2} \cdot \|u''\|_\infty \\
\|w_\ell\|_2 & \leq \frac{h_\ell^2}{3} \cdot \|u''\|_2 \\
\|w_\ell\|_W & \leq \frac{h_\ell}{2} \cdot \|u''\|_\infty
\end{align*}
\]
With these results it is possible to quantify the approximation error:

\[ \| u - u_n \|_\infty \leq \frac{\| u'' \|_\infty}{6}. \]

\[ h_n^2 = O(h_n^2) \]

Often an estimation of the error is possible:

\[ e.g. \text{finite elements} \]

\[ u'' = f \Rightarrow \| u'' \|_{\ast} = \| f \|_{\ast} \]
Approximation Error

With these results it is possible to quantify the approximation error:

\[
\| u - u_n \|_{\infty} \leq \frac{\| u'' \|_{\infty}}{6} \cdot h_n^2 = O(h_n^2)
\]

\[
\| u - u_n \|_2 \leq \frac{\| u'' \|_2}{9} \cdot h_n^2 = O(h_n^2)
\]
Hierarchical Basis

In 1 dimension

Approximation Error

With these results it is possible to quantify the approximation error:

\[
\begin{align*}
\|u - u_n\|_\infty & \leq \frac{\|u''\|_\infty}{6} \cdot h_n^2 = O(h_n^2) \\
\|u - u_n\|_2 & \leq \frac{\|u''\|_2}{9} \cdot h_n^2 = O(h_n^2) \\
\|u - u_n\|_E & \leq \frac{\|u''\|_\infty}{2} \cdot h_n = O(h_n) \\
\end{align*}
\]

(with \(d = 1: n = \ell\))
Approximation Error

With these results it is possible to quantify the approximation error:

\[
\| u - u_n \|_\infty \leq \frac{\| u'' \|_\infty}{6} \cdot h_n^2 = O(h_n^2)
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\| u - u_n \|_E \leq \frac{\| u'' \|_\infty}{2} \cdot h_n = O(h_n)
\]

(with \( d = 1: n = \ell \))

Often an estimation of the error is possible:

- e.g. finite elements \( u'' = f \)

\[
\Rightarrow \| u'' \|_\ast = \| f \|_\ast
\]
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2. Hierarchical Basis
   - In 1 dimension
   - In 2 or more dimensions
   - Sparse grids

3. Conclusion
Some Definitions

- w.l.o.g. $\Omega = [0, 1]^d$, $u|_{\partial \Omega} = 0$ and $u(x) = u(x_1, \ldots, x_d)$
- now $\ell$ is a grid vector:

$$\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$$
Some Definitions

- w.l.o.g. $\Omega = [0, 1]^d$, $u|_{\partial \Omega} = 0$ and $u(x) = u(x_1, \ldots, x_d)$
- now $\ell$ is a grid vector:
  $$\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$$

- 2 norms used to shorten the syntax:
  $$|\ell|_1 := \ell_1 + \cdots + \ell_d \quad |\ell|_\infty := \max\{\ell_1, \ldots, \ell_d\}$$
Some Definitions

- w.l.o.g. $\Omega = [0, 1]^d$, $u|_{\partial\Omega} = 0$ and $u(x) = u(x_1, \ldots, x_d)$
- now $\ell$ is a grid vector:

$\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$

- 2 norms used to shorten the syntax:

$|\ell|_1 := \ell_1 + \cdots + \ell_d \quad |\ell|_\infty := \max\{\ell_1, \ldots, \ell_d\}$

- as the grid is not necessarily quadratic:

$h_\ell = (h_{\ell_1}, \ldots, h_{\ell_d}) = (2^{-\ell_1}, \ldots, 2^{-\ell_d})$
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- as the grid is not necessarily quadratic:
  \[ h_\ell = (h_{\ell_1}, \ldots, h_{\ell_d}) = (2^{-\ell_1}, \ldots, 2^{-\ell_d}) \]

- the grid points:
  \[ x_{\ell,i} := (i_1 \cdot h_{\ell_1}, \ldots, i_d \cdot h_{\ell_d}) \text{ with } 1 \leq i < 2^\ell \]
  (the $\leq$ is componentwise, i.e. $\forall j: 1 \leq i_j \leq 2^{\ell_j} - 1$)
Some Definitions

Here are the first grid points for $d = 2$ ($l$ up to $(3, 3)$):

$$x_{l,i} := (i_1 \cdot h_{l_1}, \ldots, i_d \cdot h_{l_d}) \text{ with } 1 \leq i < 2^l$$
d-linear Functions

We have to construct some basis functions:
We have to construct some basis functions:

- multiply one dimensional hats for each coordinate:

\[
\phi_{\ell,j,i_j}(x_j) = \phi \left( \frac{x_j - x_{\ell,j,i_j}}{h_{\ell,j}} \right)
\]

\[
\phi_{\ell,i}(x) = \prod_{j=1}^{d} \phi_{\ell,j,i_j}(x_j)
\]
We have to construct some basis functions:

- multiply one dimensional hats for each coordinate:

\[ \phi_{\ell_j,i_j}(x_j) = \phi \left( \frac{x_j - x_{\ell_j,i_j}}{h_{\ell_j}} \right) \]

\[ \phi_{\ell,i}(x) = \prod_{j=1}^{d} \phi_{\ell_j,i_j}(x_j) \]

- get d-linear basis functions
  \( \text{(i.e. for fixed } (d-1) \text{ coordinates, function is linear in the remaining variable)} \)
Example

The functions $\phi(1,1), (1,1)$ and $\phi(2,3), (3,5)$
Now we have to make some definitions, analog to the 1 dimensional case for a fixed $\ell \in \mathbb{N}^d$ we define:
Basis and Subspaces

Now we have to make some definitions, analog to the 1 dimensional case. For a fixed $\ell \in \mathbb{N}^d$ we define:

- We get the subspace

$$S_\ell := \text{span}\{\Phi_\ell\} = \text{span}\{\psi_\ell\} = \bigoplus_{k \leq \ell} W_k$$

- The dimension of $S_\ell$

$$\dim(S_\ell) = (2^{\ell_1} - 1) \cdot \ldots \cdot (2^{\ell_d} - 1) = \mathcal{O}(2^{||\ell||_1})$$

- The whole space can be represented by the $W_k$

$$V = S_\infty = \bigoplus_{k \in \mathbb{N}^d} W_k$$
Basis and Subspaces

Now we have to make some definitions, analog to the 1 dimensional case for a fixed \( \ell \in \mathbb{N}^d \) we define:

- the hierarchical surpluses of \( u \in V \)
  \[
  u(x) = \sum_{\ell \in \mathbb{N}^d} w_\ell(x), \quad w_\ell = \sum_{i \in I_\ell} v_{\ell,i} \cdot \phi_{\ell,i} \in W_\ell
  \]

- and the \( v_{\ell,i} \), same as in 1 dimension (\textit{maybe a little more complicated ;})
  \[
  v_{\ell,i} = \left( \prod_{j=1}^{d} \left[ -\frac{1}{2} \ 1 \ -\frac{1}{2} \right] x_{\ell,j,i,j,\ell_j} \right) u
  \]
Hierarchical Surpluses

**Example**

For 2 dimensions, \( v_{\ell,(i_1,i_2)} \) is as follows:

\[
v_{\ell,(i_1,i_2)} = \frac{u(x_{\ell,(i_1-1,i_2-1)}) - 2u(x_{\ell,(i_1,i_2-1)}) + u(x_{\ell,(i_1+1,i_2-1)})}{4} - 2u(x_{\ell,(i_1-1,i_2)}) + 4u(x_{\ell,(i_1,i_2)}) - 2u(x_{\ell,(i_1+1,i_2)}) + \frac{u(x_{\ell,(i_1-1,i_2+1)}) - 2u(x_{\ell,(i_1,i_2+1)}) + u(x_{\ell,(i_1+1,i_2+1)})}{4} + 4u(x_{\ell,(i_1,i_2+1)}) - 2u(x_{\ell,(i_1+1,i_2+1)}) + u(x_{\ell,(i_1+1,i_2+1)})\]
There is an integral representation for $v_{\ell,i}$, analogous to 1-dim
There is an integral representation for \( v_{\ell,i} \), analogous to 1-dim

- first of all, a different definition for \( u'' \):

\[
}\frac{\partial^{2d} u}{\partial x_1^2 \cdots \partial x_d^2}
\]
Hierarchical Basis

In 2 or more dimensions

\( v_{\ell,i} \) - Another Representation

There is an integral representation for \( v_{\ell,i} \), analogous to 1-dim

- first of all, a different definition for \( u'' \):

\[
   u'' := \frac{\partial^2 d u}{\partial x_1^2 \cdots \partial x_d^2}
\]

- the new representation:

\[
   v_{\ell,i} = \int_{\Omega} \left( \prod_{j=1}^{d} \left( -\frac{h_{\ell,j}}{2} \cdot \phi_{\ell,j,i_j}(x_j) \right) \right) u''(x) \, dx
\]
Hierarchical Basis

In 2 or more dimensions

Norms – $d$ Dimensions

for the basis functions we have:

$$\| \phi_{\ell,i} \|_\infty = 1$$

$$\| \phi_{\ell,i} \|_2 = \left( \frac{2}{3} \right)^{\frac{d}{2}} \cdot 2^{-|\ell|/2}$$

$$\| \phi_{\ell,i} \|_E = \sqrt{2} \cdot \left( \frac{2}{3} \right)^{\frac{d-1}{2}} \cdot 2^{-|\ell|/2} \left( \sum_{j=1}^{d} 2^{2\ell_j} \right)^{\frac{1}{2}}$$
Norms – $d$ Dimensions

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and for the coefficients:

$$|v_{\ell,i}| \leq 2^{-d} \cdot 2^{-2|\ell|} \cdot \|u''\|_{\infty}$$
$$|v_{\ell,i}| \leq 2^{-d} \cdot \left( \frac{2}{3} \right)^{\frac{d}{2}} 2^{-3/2|\ell|} \cdot \|u''\|_{T_{\ell,i}}^2$$
Hierarchical Basis

In 2 or more dimensions

Norms – $d$ Dimensions

we are interested in the surpluses:

\[ \| w_\ell \|_\infty \leq 2^{-d} \cdot 2^{-2|\ell|_1} \cdot \| u'' \|_\infty = O(h_1^2 \cdots h_d^2) \]

\[ \| w_\ell \|_2 \leq 3^{-d} \cdot 2^{-2|\ell|_1} \cdot \| u'' \|_2 = O(h_1^2 \cdots h_d^2) \]
we are interested in the surpluses:

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\|w_\ell\|_2 \leq 3^{-d} \cdot 2^{-2|\ell|_1} \cdot \|u''\|_2 = O(h_1^2 \cdots h_d^2)
\]

\[
\|w_\ell\|_\infty \leq \frac{1}{2 \cdot 12^{(d-1)/2}} \cdot 2^{-2|\ell|_1} \left( \sum_{j=1}^{d} 2^{2\ell_j} \right)^{1/2} \cdot \|u''\|_\infty
\]

\[
= O \left( h_1^2 \cdots h_d^2 \cdot \sqrt{\sum_{j=1}^{d} \frac{1}{h_j^2}} \right)
\]
the approximation error depends on the approximation but:
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approximation in 2 or more dimensions is not clear
Hierarchical Basis

In 2 or more dimensions

Approximation Error – $d$ Dimensions

the approximation error depends on the approximation but:

approximation in 2 or more dimensions is not clear

---

**approximation error**

- take a set $L \subset \mathbb{N}^d$
- approximation error with this $L$:

$$
\|u - u_L\| \leq \sum_{\ell \notin L} \|w_\ell\|
$$
Hierarchical Basis

In 2 or more dimensions

Approximation Error – $d$ Dimensions

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approximation error

- take a set $L \subset \mathbb{N}^d$
- approximation error with this $L$:

$$\|u - u_L\| \leq \sum_{\ell \notin L} \|w_\ell\|$$

which $L$ is the best to take?
Outline

1 Introduction

2 Hierarchical Basis
   - In 1 dimension
   - In 2 or more dimensions
   - Sparse grids

3 Conclusion
First Idea

We have to search for an optimal $L \subset \mathbb{N}^d$
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We have to search for an optimal \( L \subset \mathbb{N}^d \)

- first idea: multidimensional equidistant grid

\[
L_n^\infty := \{ \ell \in \mathbb{N}^d : |\ell|_\infty \leq n \}, \quad S_n^\infty := \bigoplus_{\ell \in L_n^\infty} W_{\ell}
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First Idea

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First Idea

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We want to decide, how good a given $L$ is
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But how to evaluate the quality of $L$?
Evaluation – Costs

We want to decide, how good a given $L$ is

But how to evaluate the quality of $L$ ?

- Look at the costs of each $\ell$:

\[
c(\ell) := |I_\ell| = |\{1 \leq i \leq 2^\ell - 1, \text{ all } i_j \text{ odd}\}| = 2^{|\ell|1-d}
\]

\textit{the more points} $\supseteq$ \textit{functions in} $W_\ell$, \textit{the higher the costs}
We want to decide, how good a given $L$ is

But how to evaluate the quality of $L$?

- Look at the costs of each $\ell$:
  \[ c(\ell) := |I_\ell| = \{|1 \leq i \leq 2^\ell - 1, \text{ all } i_j \text{ odd}| = 2|\ell|^{1-d} \]

  _the more points $\hat{=} functions in W_\ell$, the higher the costs_

- Costs of the full grid $S_\infty^n$: In each coordinate $2^n - 1$ possibilities
  \[ \Rightarrow C(S_\infty^n) = O(2^{nd}) \]
and look at the benefits of each $\ell$:

$$\text{maxfail}(L \cup \{\ell\}) - \text{maxfail}(L) = \sum_{k \notin L \cup \{\ell\}} \|w_k\|_* - \sum_{k \notin L} \|w_k\|_* = \|w_\ell\|_*$$

*benefit of $\ell$ is a better approximation of $u_L$ to $u$ if $\ell \in L$*
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$$\text{maxfail}(L \cup \{\ell\}) - \text{maxfail}(L) = \sum_{k \notin L \cup \{\ell\}} \|w_k\|_* - \sum_{k \notin L} \|w_k\|_* = \|w_\ell\|_*$$

*benefit of $\ell$ is a better approximation of $u_L$ to $u$ if $\ell \in L$*

$$b_\infty(\ell) = b_2(\ell) := 2^{-2|\ell|_1}$$

$$b_E(\ell) := 2^{-2|\ell|_1} \cdot \left( \sum_{j=1}^{d} 2^{2\ell_j} \right)^{\frac{1}{2}}$$

*benefit $b(\ell)$ is the on $\ell$ dependend part of the bound of $w_\ell$*
Now we can evaluate the quality of each $\ell$: the ratio of benefit and cost.
Now we can evaluate the quality of each $\ell$: the ratio of benefit and cost for $L_2$- and $\infty$-norm ("\("*\)"):

\[
\text{cbr}_2(\ell) = \text{cbr}_\infty(\ell) = \frac{b_*(\ell)}{c(\ell)} = \frac{2^{-2|\ell|_1}}{2|\ell|_1 - d} = 2^{-3|\ell|_1 + d}
\]
Cost-Benefit Ratio

Now we can evaluate the quality of each $\ell$: the ratio of benefit and cost

- for $L_2$- and $\infty$-norm (="\*"):

$$cbr_2(\ell) = cbr_\infty(\ell) = \frac{b_*(\ell)}{c(\ell)} = \frac{2^{-2|\ell|_1}}{2|\ell|_1-d} = 2^{-3|\ell|_1+d}$$

- the ratio is best for small $|\ell|_1$

- so we define

$$L^1_n := \{ \ell \in \mathbb{N}^d : |\ell|_1 \leq n + d - 1 \}$$

$$S^1_n := \bigoplus_{\ell \in L^1_n} W_\ell$$
Sparse Grids

So, what is a **SPARSE GRID**?
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- $L_1^n$ specifies a diagonal cut
  
  \[(alongside\ every\ diagonal\ line,\ cmr(\ell)\ is\ constant)\]
**Sparse Grids**

So, what is a **SPARSE GRID**?

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  *(alongside every diagonal line, $cbr(\ell)$ is constant)*

- resulting grid is called "sparse grid"
So, what is a **SPARSE GRID**?

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  - (alongside every diagonal line, $cbr(\ell)$ is constant)
- resulting grid is called “sparse grid”

**Example**

example with $d = 2, n = 5$

points with same $cbr / |\ell|_1$ have same color)
Sparse Grids

So, what is a SPARSE GRID?

- $L_1^n$ specifies a diagonal cut
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Example

example with $d = 2, n = 5$

<table>
<thead>
<tr>
<th>\ell_1</th>
<th>\ell_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

points with same $cbr / |\ell|_1$ have same color
How good is a sparse grid?

Analysis of the approximation error – \( \infty \)-norm:

\[
\| u - u_n^\infty \|_\infty \leq \frac{d}{6^d} \cdot 2^{-2n} \cdot \| u'' \|_\infty \\
\| u - u_n^1 \|_\infty \leq \frac{2}{8^d} \cdot 2^{-2n} \cdot \| u'' \|_\infty \cdot \left( \frac{n^{d-1}}{(d-1)!} + \mathcal{O}(n^{d-2}) \right)
\]

\[
\| u - u_n^\infty \|_\infty = \mathcal{O}(h_n^2), \quad \| u - u_n^1 \|_\infty = \mathcal{O}(h_n^2 \cdot n^{d-1})
\]
Comparison: Full – Sparse Grid

**How good is a sparse grid?**

Analysis of the approximation error – \(L_2\)-norm:

\[
\|u - u_n^\infty\|_2 \leq \frac{d}{9d} \cdot 2^{-2n} \cdot \|u''\|_2
\]

\[
\|u - u_n^1\|_2 \leq \frac{2}{12d} \cdot 2^{-2n} \cdot \|u''\|_2 \cdot \left(\frac{n^{d-1}}{(d-1)!} + \mathcal{O}(n^{d-2})\right)
\]

\[
\|u - u_n^\infty\|_2 = \mathcal{O}(h_n^2), \quad \|u - u_n^1\|_2 = \mathcal{O}(h_n^2 \cdot n^{d-1})
\]
Comparison: Full – Sparse Grid

How good is a sparse grid?

Analysis of the approximation error – $E$-norm:

\[
\| u - u_n^\infty \|_E \leq \frac{d^{3/2}}{2 \cdot 3^{(d-1)/2} \cdot 6^{d-1}} \cdot 2^{-n} \cdot \| u'' \|_\infty
\]

\[
\| u - u_n^1 \|_E \leq \frac{d}{2 \cdot 3^{(d-1)/2} \cdot 4^{d-1}} \cdot 2^{-n} \cdot \| u'' \|_\infty
\]

\[
\| u - u_n^\infty \|_E = \mathcal{O}(h_n), \quad \| u - u_n^1 \|_E = \mathcal{O}(h_n)
\]
**Comparison: Full – Sparse Grid**

**How good is a sparse grid?**

Indeed – it is very good, especially for high dimensional problems.

**approximation errors**

\[
\| u - u_n^\infty \|_\infty = O(h_n^2), \quad \| u - u_n^1 \|_\infty = O(h_n^2 \cdot n^{d-1})
\]

\[
\| u - u_n^\infty \|_2 = O(h_n^2), \quad \| u - u_n^1 \|_2 = O(h_n^2 \cdot n^{d-1})
\]

\[
\| u - u_n^\infty \|_E = O(h_n), \quad \| u - u_n^1 \|_E = O(h_n)
\]

**dimensions:**

\[
\text{dim}(S_n^\infty) = O(2^{nd}), \quad \text{dim}(S_n^1) = O(2^n \cdot n^{d-1})
\]
Sizes of Dimensions

Already with small $d$, the effect is quite drastic

<table>
<thead>
<tr>
<th>Dimension Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 2$</td>
</tr>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$\dim(S_{n}^{\infty})$</td>
</tr>
<tr>
<td>$\dim(S_{n}^{1})$</td>
</tr>
<tr>
<td>$d = 3$</td>
</tr>
<tr>
<td>$n$</td>
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A short remark on the $E$-norm sparse grid:
A short remark on the $E$-norm sparse grid:

- a grid based on $cbr_E$ is not the same as one based on $cbr_2/cbr_\infty$
A short remark on the $E$-norm sparse grid:

- A grid based on $\text{cbr}_E$ is not the same as one based on $\text{cbr}_{2/\infty}$
- We have

$$b_E(\ell) = 2^{-2|\ell|_1} \left( \sum_{j=1}^{d} 2^{2\ell_j} \right)^{\frac{1}{2}}$$
E-Norm Sparse Grid

A short remark on the $E$-norm sparse grid:

- a grid based on $cbr_E$ is not the same as one based on $cbr_2/cbr_\infty$

- we have

$$b_E(\ell) = 2^{-2|\ell|_1} \cdot \left( \sum_{j=1}^{d} 2^{2\ell_j} \right)^{1/2}$$

- one can show

$$\dim(S^E_n) \leq 2^n \cdot \frac{d}{2} \cdot e^d = \mathcal{O}(2^n) \quad , \quad S^E_n \subset S^1_n$$

$$\|u - u^E_n\|_E = \mathcal{O}(h_n) = \|u - u^\infty_n / 1\|_E$$
Outline

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We always assumed $\Omega = [0, 1]^d$. What’s about $\Omega \neq [0, 1]^d$?
Different $\Omega$

We always assumed $\Omega = [0, 1]^d$. What’s about $\Omega \neq [0, 1]^d$?

- for $\Omega$ cuboid: linear transformation of coordinates of $x_{i,\ell}$:

$$x_{i,\ell} = \left( a_j + i_j \cdot \frac{b_j - a_j}{2\ell} \right)_{j=1,...,d}$$
Different $\Omega$

We always assumed $\Omega = [0, 1]^d$. What’s about $\Omega \neq [0, 1]^d$?

- for $\Omega$ cuboid: linear transformation of coordinates of $x_{i,\ell}$:

$$x_{i,\ell} = \left( a_j + i_j \cdot \frac{b_j - a_j}{2\ell} \right)_{j=1,...,d}$$

- for arbitrary $\Omega$:
  - approximate $\Omega$ with cuboids $C$
    
    *(additional approximation error, take care of special properties of $u$!)*
  - transform cuboid into a fitting shape e.g. circle or sphere
Adaptive Refinement

- a sparse grid is not yet an adaptive refinement
Adaptive Refinement

a sparse grid is not yet an adaptive refinement for adaptivity:

- partition $\Omega$ (e.g. halves, quarters)
- evaluate error for each part $T$

$$\| (u - u_n)|_T \|_* = \| u - u_n \|_* \cdot \frac{\| u''|_T \|_*}{\| u'' \|_*} \cdot \frac{\text{area}(T)}{\text{area}(\Omega)}, \quad (* = 2/E)$$

- new sparse grid on $T$ where the error is maximal
Adaptive Refinement

- a sparse grid is not yet an adaptive refinement for adaptivity:
  - partition $\Omega$ (e.g. halfs, quarters)
  - evaluate error for each part $T$

$$
\|(u - u_n)|_T\|_* = \|u - u_n\|_* \cdot \frac{\|u''|_T\|_*}{\|u''\|_*} \cdot \frac{\text{area}(T)}{\text{area}(\Omega)}, \quad (\ast = 2/E)
$$

- new sparse grid on $T$ where the error is maximal

Caution: we usually don’t know $u'' = \frac{\partial^{2d}u}{\partial x^2_1 \cdots \partial x^2_d}$ for $d > 2$

if at all, we only know $\triangle u := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x^2_i}$ (e.g. PDE: $\triangle u = -f$)
Other Approaches and Applications

- not only piecewise linear approaches are possible:
Other Approaches and Applications

- not only piecewise linear approaches are possible:
  - quadratic functions
  - polynomial functions in general
  - wavelets
  - etc.
- many applications of sparse grids:
not only piecewise linear approaches are possible:
- quadratic functions
- polynomial functions in general
- wavelets
- etc.

many applications of sparse grids:
- numerical quadrature
- solving PDEs
- data-mining
- etc.
The End

Thanks for listening!

For further reading:

- H.-J. Bungartz, M. Griebel
  Sparse grids
  *Acta Numerica*, pp. 147-269, 2004

- M. Bader, S. Zimmer
  lecture’s slides “Algorithmen des Wissenschaftlichen Rechnens”
  [link](http://www5.in.tum.de/lehre/vorlesungen/algowiss/ss05/material.html)
  *TU München*, summer term 2005
4 Sparse Grids on Finite Elements
Given a PDE: $\Delta u = f$ in $\Omega$ and $u|_{\partial \Omega} = 0$
The PDE and it’s weak form

- Given a PDE: \( \nabla u = f \) in \( \Omega \) and \( u|_{\partial\Omega} = 0 \)
- Find \( u \in V \) with \( u|_{\partial\Omega} = 0 \) and

\[
\int_{\Omega} u' \cdot v' \, dx = \int_{\Omega} f \cdot v \, dx, \forall v \in V
\]

\( \iff \)

\[
\int_{\Omega} \nabla u^T \cdot \nabla v \, dx = \int_{\Omega} f \cdot v \, dx, \forall v \in V
\]
Galerkin Projection

- Take finite $n$-dimensional subspace $S \subset V$ with

$$S = \text{span}\{\phi_i : 1 \leq i \leq n\}$$
Galerkin Projection

- Take finite $n$-dimensional subspace $S \subset V$ with

$$S = \text{span}\{\phi_i : 1 \leq i \leq n\}$$

- Receive an approximative $u_S$ as linear combination of basis functions:

$$u_S = \sum_{i=1}^{n} \alpha_i \cdot \phi_i$$
We get a linear equation system for $z = (\alpha_1, \ldots, \alpha_n)$

$$Az = b$$
We get a *linear* equation system for $z = (\alpha_1, \ldots, \alpha_n)$

$$A z = b$$

$$A = \left( \int_{\Omega} \phi_i' \cdot \phi_j' \, dx \right)_{i,j=1,\ldots,n}$$

$$b = \left( \int_{\Omega} f \cdot \phi_i \, dx \right)_{i=1,\ldots,n}$$
Matrix Conditions

- matrix is sparse, if \( \phi \) has small support - e.g. using the nodal point basis, but then:

\[
\text{cond}(A) = \mathcal{O}(h_n^{-2}) \quad \text{and} \quad \text{dim}(A) = \mathcal{O}(2^{dn}) \times \mathcal{O}(2^{dn})
\]
Matrix Conditions

- matrix is sparse, if $\phi$ has small support - e.g. using the nodal point basis, but then:

  $$\text{cond}(A) = \mathcal{O}(h_n^{-2}) \quad \text{and} \quad \text{dim}(A) = \mathcal{O}(2^{dn}) \times \mathcal{O}(2^{dn})$$

- sparse grid functions: bigger support $\Rightarrow A$ is (nearly) fully covered but:

  $$\text{cond}(A) = \mathcal{O}(h_n^{-1}) \quad \text{and} \quad \text{dim}(A) = \mathcal{O}(2^n \cdot n^{d-1}) \times \mathcal{O}(2^n \cdot n^{d-1})$$
Matrix Conditions

- matrix is sparse, if \( \phi \) has small support - e.g. using the nodal point basis, but then:

\[
\text{cond}(A) = \mathcal{O}(h^{-2}_n) \quad \text{and} \quad \text{dim}(A) = \mathcal{O}(2^{dn}) \times \mathcal{O}(2^{dn})
\]

- sparse grid functions: bigger support \( \Rightarrow \) \( A \) is (nearly) fully covered but:

\[
\text{cond}(A) = \mathcal{O}(h^{-1}_n) \quad \text{and} \quad \text{dim}(A) = \mathcal{O}(2^n \cdot n^{d-1}) \times \mathcal{O}(2^n \cdot n^{d-1})
\]

- using iterative linear equation solvers (e.g. CG method):
  \( \Rightarrow \) don’t need \( A \) explicitly but only \( Av \)
Matrix Conditions

- matrix is sparse, if $\phi$ has small support - e.g. using the nodal point basis, but then:

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- sparse grid functions: bigger support $\Rightarrow$ $A$ is (nearly) fully covered but:

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- using iterative linear equation solvers (e.g. CG method):
  $\Rightarrow$ don’t need $A$ explicitly but only $Av$

- there are algorithms for evaluation of $Av$ in $O(N)$ time