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Fundamental Algorithms

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Fall Semester 2007
1. (a,b)-Trees

As we saw in the previous section, the efficiency of standard operations on binary search trees depends on the maximum tree height. Using **height balancing**, we ensure that trees cannot degenerate linearly but instead have logarithmic height. Let us extend this approach to more general trees.

**Motivation**: assume tree nodes are stored in secondary storage (hard disk). Comparisons of keys of binary trees would be too time expensive due to mechanical positioning of the read-write head of the hard drive. Reading blocks of data (sectors, pages, etc.) is relatively fast provided the read-write head is positioned.

**Idea**: store blocks of data in nodes of trees!

**Advantages**: faster access to the data and decreasing height of trees!

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*(a,b)-Trees* have been invented by Rudolf Bayer and Edward M. McCreight (1976).
**Definition 1**
Consider a node $v$ of a search tree and let $deg(v)$ be the number of sons of $v$. *(a,b)-Tree* is a tree with following properties:

- All keys are located on the same level
- For every vertex $v$ internal $b \geq deg(v) \geq a$
  
- $a \geq 2$ and $b \geq 2a - 1$
- For the root $b \geq deg(v) \geq 2$
- For every vertex $v$ all keys stored in the $i$th subtree are less than keys stored in the $(i + 1)$th subtree
- For every internal node $v$, let $m_v = deg(v)$. Then
  - $v$ has $(m_v - 1)$ key values
  - $k_1 < k_2 < \cdots < k_{m_v - 1}$
  - For $1 \leq i \leq m_v$ the following is satisfied: $k_{i-1} < \text{keys in the } i\text{th subtree} \leq k_i$
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Example 2
The (2,3)-tree:
Example 3
The (2,3)-tree:
2. Operations for (a,b)-Trees

2.1 is_element

This operation has to be implemented like for general binary search trees. The only difference is that the higher branching factor has to be treated appropriately.

Algorithm:

data is_element(key k) {
  v := root of the tree
  while (v is not a leaf) do
    i := \min\{j | 1 \leq j \leq \deg(v) \land k \leq k_j\}
    v := ith child of v
  od
  location := v
  if (v.key = k) then location := v; return v.data
  else return NULL;
  fi
}

2.2 insert

- is_element finds the position for the element to be inserted (stored in $location$)
- attach new leaf to the leaf in $location$
- If the branching factor of the leaf in $location$ $\geq b + 1$ – do rebalancing
2.3 insert: Rebalancing

- Split the node \( w \), \( \text{deg}(w) \geq b + 1 \) into \( v_1 \) and \( v_2 \).
- Assign first \( a \) sons of \( w \) to \( v_1 \), and the remaining \( b + 1 - a \) sons to \( v_2 \).
  - Since \( b \geq 2a - 1 \) (see Definition 1), we obtain that \( \text{deg}(v_1) \geq a, \text{deg}(v_2) \geq a \).
- This may increase the degree of the ancestor of \( w \) – repeat splitting for the ancestor of \( w \).
- In necessary – proceed up to the root.
- The root may also be divided into two nodes, then create a new root – the height of the tree increases.
  - Since according to Definition 1 \( b \geq \text{deg}(\text{root}) \geq 2 \), splitting of the root into two nodes is valid.
Example 4

Insert 6:
Example 5

Rebalancing:
Example 6
Rebalancing:
Example 7

Rebalancing:
2.4 delete

- is_element finds the position for the element to be removed (stored in location)
- remove the element stored in location
- If the branching factor of the ancestor of the node in location $< a$ – do rebalancing
2.5 delete: Rebalancing

- Let $\text{deg}(v_1) < a$ – merge $v_1$ and its brother $v_2$ into the new node $w$.
- If $\text{deg}(w) > b$, split $w$ into two new nodes $v_1, v_2$ and assign first $a$ sons to the first node.
  - Since $b \geq 2a - 1$ (see Definition 1), we obtain that $\text{deg}(v_1) \geq a, \text{deg}(v_2) \geq a$.
  - The number of sons of the ancestor of $w$ is not changed in this case.
- If $\text{deg}(w) \leq b$ the merging may decrease the degree of the ancestor of $w$ – repeat merging for the ancestor of $w$.
- In necessary – proceed up to the root.
Theorem 8

For the \((a,b)\)-Tree with \(n\) nodes and height \(h\) the following is satisfied:

\[2a^{h-1} \leq n \leq b^h\]

\[\log_b(n) \leq h \leq \log_a(n/2) + 1\]

Proof.

Easy. Homework.
Theorem 8
For the \((a,b)\)-Tree with \(n\) nodes and height \(h\) the following is satisfied:

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- \(\log_b(n) \leq h \leq \log_a(n/2) + 1\)

Proof.
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