11 Augmenting Path Algorithms

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an $s$-$t$ path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible
The Residual Graph

From the graph $G = (V, E, c)$ and the current flow $f$ we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

- Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between $u$ and $v$.

- $G_f$ has edge $e_1'$ with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and $e_2'$ with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

![Graph](image-url)
Augmenting Path Algorithm

Definition 1
An **augmenting path** with respect to flow \( f \), is a path from \( s \) to \( t \) in the auxiliary graph \( G_f \) that contains only edges with non-zero capacity.

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Algorithm 1 FordFulkerson(G = (V, E, c))
1: Initialize \( f(e) \) ← 0 for all edges.
2: while \( \exists \) augmenting path \( p \) in \( G_f \) do
3: augment as much flow along \( p \) as possible.
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Animation for augmenting path algorithms is only available in the lecture version of the slides.
Augmenting Path Algorithm

Theorem 2
A flow $f$ is a maximum flow iff there are no augmenting paths.

Theorem 3
The value of a maximum flow is equal to the value of a minimum cut.

Proof.
Let $f$ be a flow. The following are equivalent:

1. There exists a cut $A, B$ such that $\text{val}(f) = \text{cap}(A, B)$.
2. Flow $f$ is a maximum flow.
3. There is no augmenting path w.r.t. $f$.

\qed
Augmenting Path Algorithm

1. \( \Rightarrow \) 2.
This we already showed.

2. \( \Rightarrow \) 3.
If there were an augmenting path, we could improve the flow. Contradiction.

3. \( \Rightarrow \) 1.
   - Let \( f \) be a flow with no augmenting paths.
   - Let \( A \) be the set of vertices reachable from \( s \) in the residual graph along non-zero capacity edges.
   - Since there is no augmenting path we have \( s \in A \) and \( t \notin A \).
Augmenting Path Algorithm

\[ \text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \]

\[ = \sum_{e \in \text{out}(A)} c(e) \]

\[ = \text{cap}(A, V \setminus A) \]

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving \( A \).
Assumption:
All capacities are integers between 1 and $C$.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.
Lemma 4
The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where $f^*$ denotes the maximum flow. Each iteration can be implemented in time $O(m)$. This gives a total running time of $O(nmC)$.

Theorem 5
If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.
A Bad Input

Problem: The running time may not be polynomial.

Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?
A Bad Input

Problem: The running time may not be polynomial.

Question: Can we tweak the algorithm so that the running time is polynomial in the input length?

See the lecture-version of the slides for the animation.
A Pathological Input

Let \( r = \frac{1}{2}(\sqrt{5} - 1) \). Then \( r^{n+2} = r^n - r^{n+1} \).

Running time may be infinite!!!

See the lecture-version of the slides for the animation.
How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.
Overview: Shortest Augmenting Paths

Lemma 6
The length of the shortest augmenting path never decreases.

Lemma 7
After at most $\Theta(m)$ augmentations, the length of the shortest augmenting path strictly increases.
These two lemmas give the following theorem:

**Theorem 8**

*The shortest augmenting path algorithm performs at most $O(mn)$ augmentations. This gives a running time of $O(m^2n)$.***

**Proof.**

- We can find the shortest augmenting paths in time $O(m)$ via BFS.
- $O(m)$ augmentations for paths of exactly $k < n$ edges.
Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest $s-v$ path in $G_f$.

Let $L_G$ denote the subgraph of the residual graph $G_f$ that contains only those edges $(u, v)$ with $\ell(v) = \ell(u) + 1$.

A path $P$ is a shortest $s-u$ path in $G_f$ if it is a shortest $s-u$ path in $L_G$. 
In the following we assume that the residual graph $G_f$ does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.
Shortest Augmenting Path

First Lemma:
The length of the shortest augmenting path never decreases.

After an augmentation $G_f$ changes as follows:
- Bottleneck edges on the chosen path are deleted.
- Back edges are added to all edges that don’t have back edges so far.

These changes cannot decrease the distance between $s$ and $t$. 
**Shortest Augmenting Path**

**Second Lemma:** After at most $m$ augmentations the length of the shortest augmenting path strictly increases.

Let $E_L$ denote the set of edges in graph $L_G$ at the beginning of a round when the distance between $s$ and $t$ is $k$.

An $s$-$t$ path in $G_f$ that uses edges not in $E_L$ has length larger than $k$, even when considering edges added to $G_f$ during the round.

In each augmentation one edge is deleted from $E_L$. 

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![Graph Diagram](image)
Shortest Augmenting Paths

Theorem 9
The shortest augmenting path algorithm performs at most $O(mn)$ augmentations. Each augmentation can be performed in time $O(m)$.

Theorem 10 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $O(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:
There always exists a set of $m$ augmentations that gives a maximum flow (why?).
When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $O(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $O(m)$ per augmentation for this).
Shortest Augmenting Paths

We maintain a subset $E_L$ of the edges of $G_f$ with the guarantee that a shortest $s$-$t$ path using only edges from $E_L$ is a shortest augmenting path.

With each augmentation some edges are deleted from $E_L$.

When $E_L$ does not contain an $s$-$t$ path anymore the distance between $s$ and $t$ strictly increases.

Note that $E_L$ is not the set of edges of the level graph but a subset of level-graph edges.
Suppose that the initial distance between $s$ and $t$ in $G_f$ is $k$.

$E_L$ is initialized as the level graph $L_G$.

Perform a **DFS search** to find a path from $s$ to $t$ using edges from $E_L$.

Either you find $t$ after at most $n$ steps, or you end at a node $v$ that does not have any outgoing edges.

You can delete incoming edges of $v$ from $E_L$. 

11.2 Shortest Augmenting Paths
Let a phase of the algorithm be defined by the time between two augmentations during which the distance between \( s \) and \( t \) strictly increases.

Initializing \( E_L \) for the phase takes time \( O(m) \).

The total cost for searching for augmenting paths during a phase is at most \( O(mn) \), since every search (successful (i.e., reaching \( t \)) or unsuccessful) decreases the number of edges in \( E_L \) and takes time \( O(n) \).

The total cost for performing an augmentation during a phase is only \( O(n) \). For every edge in the augmenting path one has to update the residual graph \( G_f \) and has to check whether the edge is still in \( E_L \) for the next search.

There are at most \( n \) phases. Hence, total cost is \( O(mn^2) \).
How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.
Capacity Scaling

Intuition:

▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
▶ Don’t worry about finding the exact bottleneck.
▶ Maintain scaling parameter $\Delta$.
▶ $G_f(\Delta)$ is a sub-graph of the residual graph $G_f$ that contains only edges with capacity at least $\Delta$. 

$G_f$

$G_f(99)$
Algorithm 2 maxflow($G, s, t, c$)

1: foreach $e \in E$ do $f_e \leftarrow 0$;
2: $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
3: while $\Delta \geq 1$ do
4: $G_f(\Delta) \leftarrow \Delta$-residual graph
5: while there is augmenting path $P$ in $G_f(\Delta)$ do
6: $f \leftarrow$ augment($f, c, P$)
7: update($G_f(\Delta)$)
8: $\Delta \leftarrow \Delta/2$
9: return $f$
Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:
- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.
Capacity Scaling

**Lemma 11**
*There are \( \lceil \log C \rceil \) iterations over \( \Delta \).*

**Proof:** obvious.

**Lemma 12**
*Let \( f \) be the flow at the end of a \( \Delta \)-phase. Then the maximum flow is smaller than \( \text{val}(f) + m\Delta \).*

**Proof:** less obvious, but simple:

- There must exist an \( s-t \) cut in \( G_f(\Delta) \) of zero capacity.
- In \( G_f \) this cut can have capacity at most \( m\Delta \).
- This gives me an upper bound on the flow that I can still add.
Lemma 13
There are at most $2m$ augmentations per scaling-phase.

Proof:
- Let $f$ be the flow at the end of the previous phase.
- $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- Each augmentation increases flow by $\Delta$.

Theorem 14
We need $\Theta(m \log C)$ augmentations. The algorithm can be implemented in time $\Theta(m^2 \log C)$. 