11 Augmenting Path Algorithms

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an $s-t$ path with $f(e) < c(e)$ on every edge
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The Residual Graph

From the graph $G = (V, E, c)$ and the current flow $f$ we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):
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- $G_f$ has edge $e'_1$ with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and $e'_2$ with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$. 
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Augmenting Path Algorithm

Definition 1
An augmenting path with respect to flow $f$, is a path from $s$ to $t$ in the auxiliary graph $G_f$ that contains only edges with non-zero capacity.

Algorithm 1

```
FordFulkerson(G = (V, E, c))
1: Initialize $f(e) \leftarrow 0$ for all edges.
2: while $\exists$ augmenting path $p$ in $G_f$ do
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Augmenting Path Algorithm

11.1 The Generic Augmenting Path Algorithm

Ernst Mayr, Harald Räcke
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$G$

Flow value = 10

$G_f$

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$G$

$G_f$

Flow value = 16

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\[ G \]

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\[ G \]

\[ G_f \]

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\[ G \]

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Augmenting Path Algorithm

$G$

$G_{f}$

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\[ G \]

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Augmenting Path Algorithm

Theorem 2
A flow $f$ is a maximum flow iff there are no augmenting paths.

Theorem 3
The value of a maximum flow is equal to the value of a minimum cut.

Proof.
Let $f$ be a flow. The following are equivalent:
1. There exists a cut $A,B$ such that $\text{val}(f) = \text{cap}(A,B)$.
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This we already showed.

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If there were an augmenting path, we could improve the flow. Contradiction.

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Let $f$ be a flow with no augmenting paths.
Let $A$ be the set of vertices reachable from $s$ in the residual graph along non-zero capacity edges.
Since there is no augmenting path we have $s \in A$ and $t \not\in A$. 

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11.1 The Generic Augmenting Path Algorithm
Augmenting Path Algorithm

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Augmenting Path Algorithm

\[ \text{val}(f) \]

This finishes the proof. Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving \( A \).

11.1 The Generic Augmenting Path Algorithm

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Augmenting Path Algorithm

\[
\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) = \sum_{e \in \text{out}(A)} c(e) = \text{cap}(A, V - A)
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Assumption:
All capacities are integers between 1 and $C$.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.
Analysis

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Lemma 4
The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where $f^*$ denotes the maximum flow. Each iteration can be implemented in time $\Theta(m)$. This gives a total running time of $\Theta(nmC)$.

Theorem 5
If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.
Lemma 4
The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where $f^*$ denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Theorem 5
If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.
A Bad Input

Problem: The running time may not be polynomial.

Question: Can we tweak the algorithm so that the running time is polynomial in the input length?
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Graphics: A directed graph with nodes s, 1, 2, t and edges labeled with 1998 and 1999.

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Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.
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How to choose augmenting paths?

▶ We need to find paths efficiently.
▶ We want to guarantee a small number of iterations.

Several possibilities:
▶ Choose path with maximum bottleneck capacity.
▶ Choose path with sufficiently large bottleneck capacity.
▶ Choose the shortest augmenting path.
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Overview: Shortest Augmenting Paths

Lemma 6
The length of the shortest augmenting path never decreases.

Lemma 7
After at most $O(m)$ augmentations, the length of the shortest augmenting path strictly increases.
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These two lemmas give the following theorem:

**Theorem 8**
The shortest augmenting path algorithm performs at most $O(mn)$ augmentations. This gives a running time of $O(m^2n)$.

**Proof.**

We can find the shortest augmenting paths in time $O(m)$ via BFS.

$O(m)$ augmentations for paths of exactly $k$ new edges.
Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

**Theorem 8**

*The shortest augmenting path algorithm performs at most \(O(mn)\) augmentations. This gives a running time of \(O(m^2n)\).*

**Proof.**

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**Theorem 8**

The shortest augmenting path algorithm performs at most \( \Theta(mn) \) augmentations. This gives a running time of \( \Theta(m^2n) \).

**Proof.**

- We can find the shortest augmenting paths in time \( \Theta(m) \) via BFS.
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Overview: Shortest Augmenting Paths

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Define the level $\ell(v)$ of a node as the length of the shortest $s-v$ path in $G_f$. 
Shortest Augmenting Paths

Define the level \( \ell(v) \) of a node as the length of the shortest \( s-v \) path in \( G_f \).

Let \( L_G \) denote the subgraph of the residual graph \( G_f \) that contains only those edges \( (u, v) \) with \( \ell(v) = \ell(u) + 1 \).
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A path $P$ is a shortest $s-u$ path in $G_f$ if it is a an $s-u$ path in $L_G$. 
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In the following we assume that the residual graph $G_f$ does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.
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First Lemma:
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![Graph Diagram]
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An $s$-$t$ path in $G_f$ that uses edges not in $E_L$ has length larger than $k$, even when considering edges added to $G_f$ during the round.
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![Graph diagram](image-url)
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Theorem 10 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $O(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:
There always exists a set of $m$ augmentations that gives a maximum flow (why?).
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\( E_L \) is initialized as the level graph \( L_G \).

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between $s$ and $t$ strictly increases.

Initializing $E_L$ for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching $t$) or unsuccessful) decreases the number of edges in $E_L$ and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph $G_f$ and has to check whether the edge is still in $E_L$ for the next search.

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How to choose augmenting paths?

- We need to find paths efficiently.

Several possibilities:

- Choose path with maximum bottleneck capacity.
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Capacity Scaling

Intuition:
▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
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▶ Maintain scaling parameter $\Delta$.
$G_f(\Delta)$ is a sub-graph of the residual graph $G_f$ that contains only edges with capacity at least $\Delta$. 

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Algorithm 2 maxflow\((G, s, t, c)\)

1: \textbf{foreach} \(e \in E\) \textbf{do} \(f_e \leftarrow 0\);
2: \(\Delta \leftarrow 2^{\lceil \log_2 C \rceil}\)
3: \textbf{while} \(\Delta \geq 1\) \textbf{do}
   4: \(G_f(\Delta) \leftarrow \Delta\)-residual graph
   5: \textbf{while} there is augmenting path \(P\) in \(G_f(\Delta)\) \textbf{do}
      6: \(f \leftarrow \text{augment}(f, c, P)\)
      7: \(\text{update}(G_f(\Delta))\)
   8: \(\Delta \leftarrow \Delta/2\)
9: \textbf{return} \(f\)
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Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

▶ because of integrality we have $G_{f(1)} = G_{f}$

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There are $\lceil \log C \rceil$ iterations over $\Delta$.
Proof: obvious.

Lemma 12
Let $f$ be the flow at the end of a $\Delta$-phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.
Proof: less obvious, but simple:
1. There must exist an $s$-t cut in $G_f(\Delta)$ of zero capacity.
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We need $O(m \log C)$ augmentations. The algorithm can be implemented in time $O(m^2 \log C)$.\[11.3\text{ Capacity Scaling}\]
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