16 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set $F$ and capacities $w$ that fulfills the following properties.

1. **Equivalent Flow Tree**: For any pair of vertices $s, t \in V$, $f(s, t)$ in $G$ is equal to $f_T(s, t)$.

2. **Cut Property**: A minimum $s$-$t$ cut in $T$ is also a minimum cut in $G$.

Here, $f(s, t)$ is the value of a maximum $s$-$t$ flow in $G$, and $f_T(s, t)$ is the corresponding value in $T$.

Overview of the Algorithm

The algorithm maintains a partition of $V$, (sets $S_1, \ldots, S_t$), and a spanning tree $T$ on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

- In each such split-operation it chooses a set $S_i$ with $|S_i| \geq 2$ and splits this set into two non-empty parts $X$ and $Y$.
- $S_i$ is then removed from $T$ and replaced by $X$ and $Y$.
- $X$ and $Y$ are connected by an edge, and the edges that before the split were incident to $S_i$ are attached to either $X$ or $Y$.

In the end this gives a tree on the vertex set $V$.

Details of the Split-operation

- Select $S_i$ that contains at least two nodes $a$ and $b$.
- Compute the connected components of the forest obtained from the current tree $T$ after deleting $S_i$. Each of these components corresponds to a set of vertices from $V$.
- Consider the graph $H$ obtained from $G$ by contracting these connected components into single nodes.
- Compute a minimum $a$-$b$ cut in $H$. Let $A$ and $B$ denote the two sides of this cut.
- Split $S_i$ in $T$ into two sets/nodes $S_i^a = S_i \cap A$ and $S_i^b = S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a, b)$.
- Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction
Analysis

Lemma 1
For nodes $s, t, x \in V$ we have $f(s, t) \geq \min\{f(s, x), f(x, t)\}$

Lemma 2
For nodes $s, t, x_1, \ldots, x_k \in V$ we have
$$f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k), f(x_k, t)\}$$

Lemma 3
Let $S$ be some minimum $r$-$s$ cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum $v$-$w$-cut $T$ with $T \subset S$.

Proof: Let $X$ be a minimum $v$-$w$ cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$. Note that $S \setminus X$ and $S \cap X$ are $v$-$w$ cuts inside $S$. We may assume w.l.o.g. $s \in X$.

First case $r \in X$.
$\Rightarrow \quad \text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.
$\Rightarrow \quad \text{cap}(X \setminus S) \geq \text{cap}(S)$ because $X \setminus S$ is an $r$-$s$ cut.
$\Rightarrow \quad \text{This gives cap}(S \setminus X) \leq \text{cap}(X)$.

Second case $r \notin X$.
$\Rightarrow \quad \text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$.
$\Rightarrow \quad \text{cap}(X \cup S) \geq \text{cap}(S)$ because $X \cup S$ is an $r$-$s$ cut.
$\Rightarrow \quad \text{This gives cap}(S \cap X) \leq \text{cap}(X)$.
Lemma 3 tells us that if we have a graph $G = (V,E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s,t \not\in X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum $s$-$t$ mincut in graph $H$.

Invariant [existence of representatives]:
For any edge $\{S_i,S_j\}$ in $T$, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i,S_j) = f(a,b)$ and the cut defined by edge $\{S_i,S_j\}$ is a minimum $a$-$b$ cut in $G$.

We first show that the invariant implies that at the end of the algorithm $T$ is indeed a cut-tree.

$\triangleright$ Let $s = x_0,x_1,\ldots,x_{k-1},x_k = t$ be the unique simple path from $s$ to $t$ in the final tree $T$. From the invariant we get that $f(x_i,x_{i+1}) = w(x_i,x_{i+1})$ for all $j$.

$\triangleright$ Then
\[
f_T(s,t) = \min_{i \in \{0,\ldots,k-1\}} \{ w(x_i,x_{i+1}) \} = \min_{i \in \{0,\ldots,k-1\}} \{ f(x_i,x_{i+1}) \} \leq f(s,t).
\]

$\triangleright$ Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.

$\triangleright$ Since by the invariant this edge induces an $s$-$t$ cut with capacity $f(x_j,x_{j+1})$ we get $f(s,t) \leq f(x_j,x_{j+1}) = f_T(s,t)$. Hence, $f_T(s,t) = f(s,t)$ (flow equivalence).

$\triangleright$ The edge $\{x_j, x_{j+1}\}$ is a mincut between $s$ and $t$ in $T$.

$\triangleright$ By invariant, it forms a cut with capacity $f(x_j,x_{j+1})$ in $G$ (which separates $s$ and $t$).

$\triangleright$ Since, we can send a flow of value $f(x_j,x_{j+1})$ btw. $s$ and $t$, this is an $s$-$t$ mincut (cut property).
Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let $S_i$ denote our selected cluster with nodes $a$ and $b$. Because of the invariant all edges leaving $\{S_i\}$ in $T$ correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. $a$ and $b$ due to Lemma 3.

After the split we have to choose representatives for all edges. For the new edge $\{S^a_i, S^b_i\}$ with capacity $w(S^a_i, S^b_i) = f_H(a, b)$ we can simply choose $a$ and $b$ as representatives.

For edges that are not incident to $S_i$ we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S^a_i\}$ in the new tree (the case when it is replaced by $\{X, S^b_i\}$ is analogous).

If $s \in S^a_i$ we can keep $x$ and $s$ as representatives.

Otherwise, we choose $x$ and $a$ as representatives. We need to show that $f(x,a) = f(x,s)$.

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in $G$ of capacity $f(x,s)$. Since, $x$ and $a$ are on opposite sides of this cut, we know that $f(x,a) \leq f(x,s)$.

The set $B$ forms a mincut separating $a$ from $b$. Contracting all nodes in this set gives a new graph $G'$ where the set $B$ is represented by node $v_B$. Because of Lemma 3 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

We further have $f'(x,a) \geq \min\{f'(x,v_B), f'(v_B,a)\}$.

Since $s \in B$ we have $f'(v_B,x) = f(s,x)$.

Also, $f'(a,v_B) \geq f(a,b) \geq f(x,s)$ since the $a$-$b$ cut that splits $S_i$ into $S^a_i$ and $S^b_i$ also separates $s$ and $x$. 

Analysis