16 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set $F$ and capacities $w$ that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s, t)$ in $G$ is equal to $f_T(s, t)$.

2. **Cut Property:** A minimum $s$-$t$ cut in $T$ is also a minimum cut in $G$.

Here, $f(s, t)$ is the value of a maximum $s$-$t$ flow in $G$, and $f_T(s, t)$ is the corresponding value in $T$. 
Overview of the Algorithm

The algorithm maintains a partition of $V$, (sets $S_1, \ldots, S_t$), and a spanning tree $T$ on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

- In each such split-operation it chooses a set $S_i$ with $|S_i| \geq 2$ and splits this set into two non-empty parts $X$ and $Y$.
- $S_i$ is then removed from $T$ and replaced by $X$ and $Y$.
- $X$ and $Y$ are connected by an edge, and the edges that before the split were incident to $S_i$ are attached to either $X$ or $Y$.

In the end this gives a tree on the vertex set $V$. 
Details of the Split-operation

- Select $S_i$ that contains at least two nodes $a$ and $b$.
- Compute the connected components of the forest obtained from the current tree $T$ after deleting $S_i$. Each of these components corresponds to a set of vertices from $V$.
- Consider the graph $H$ obtained from $G$ by contracting these connected components into single nodes.
- Compute a minimum $a$-$b$ cut in $H$. Let $A$, and $B$ denote the two sides of this cut.
- Split $S_i$ in $T$ into two sets/nodes $S_i^a := S_i \cap A$ and $S_i^b := S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a, b)$.
- Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$. 
Example: Gomory-Hu Construction
Lemma 1

For nodes $s, t, x \in V$ we have $f(s, t) \geq \min\{f(s, x), f(x, t)\}$

Lemma 2

For nodes $s, t, x_1, \ldots, x_k \in V$ we have

$f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k), f(x_k, t)\}$
Lemma 3
Let $S$ be some minimum $r$-$s$ cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum $v$-$w$-cut $T$ with $T \subset S$.

Proof: Let $X$ be a minimum $v$-$w$ cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$. Note that $S \setminus X$ and $S \cap X$ are $v$-$w$ cuts inside $S$. We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

$\quad \cap(X \setminus S) + \cap(S \setminus X) \leq \cap(S) + \cap(X)$.
$\quad \cap(X \setminus S) \geq \cap(S)$ because $X \setminus S$ is an $r$-$s$ cut.
$\quad$ This gives $\cap(S \setminus X) \leq \cap(X)$.

Second case $r \notin X$.

$\quad \cap(X \cup S) + \cap(S \cap X) \leq \cap(S) + \cap(X)$.
$\quad \cap(X \cup S) \geq \cap(S)$ because $X \cup S$ is an $r$-$s$ cut.
$\quad$ This gives $\cap(S \cap X) \leq \cap(X)$. 
\[ \text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X) \]
\[ \text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X) \]
Analysis

Lemma 3 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s, t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s, t) = f(s, t)$, where $f_H(s, t)$ is the value of a minimum $s$-$t$ mincut in graph $H$. 
Invariant [existence of representatives]:
For any edge \( \{S_i, S_j\} \) in \( T \), there are vertices \( a \in S_i \) and \( b \in S_j \) such that \( w(S_i, S_j) = f(a, b) \) and the cut defined by edge \( \{S_i, S_j\} \) is a minimum \( a-b \) cut in \( G \).
Analysis

We first show that the invariant implies that at the end of the algorithm $T$ is indeed a cut-tree.

- Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from $s$ to $t$ in the final tree $T$. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all $j$.
- Then

$$f_T(s, t) = \min_{i \in \{0, \ldots, k-1\}} \{w(x_i, x_{i+1})\}$$

$$= \min_{i \in \{0, \ldots, k-1\}} \{f(x_i, x_{i+1})\} \leq f(s, t).$$

- Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.
- Since by the invariant this edge induces an $s$-$t$ cut with capacity $f(x_j, x_{j+1})$ we get $f(s, t) \leq f(x_j, x_{j+1}) = f_T(s, t)$. 

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- Hence, $f_T(s, t) = f(s, t)$ (flow equivalence).
- The edge $\{x_j, x_{j+1}\}$ is a mincut between $s$ and $t$ in $T$.
- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in $G$ (which separates $s$ and $t$).
- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. $s$ and $t$, this is an $s$-$t$ mincut (cut property).
Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let $S_i$ denote our selected cluster with nodes $a$ and $b$. Because of the invariant all edges leaving $\{S_i\}$ in $T$ correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. $a$ and $b$ due to Lemma 3.

After the split we have to choose representatives for all edges. For the new edge $\{S_{i}^{a}, S_{i}^{b}\}$ with capacity $w(S_{i}^{a}, S_{i}^{b}) = f_H(a, b)$ we can simply choose $a$ and $b$ as representatives.
Proof of Invariant

For edges that are not incident to $S_i$ we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_{ai}\}$ in the new tree (the case when it is replaced by $\{X, S_{bi}\}$ is analogous).

If $s \in S_{ai}$ we can keep $x$ and $s$ as representatives.

Otherwise, we choose $x$ and $a$ as representatives. We need to show that $f(x, a) = f(x, s)$. 
Proof of Invariant

Because the invariant was true before the split we know that the edge \( \{X, S_i\} \) induces a cut in \( G \) of capacity \( f(x, s) \). Since, \( x \) and \( a \) are on opposite sides of this cut, we know that \( f'(x, a) \leq f(x, s) \).

The set \( B \) forms a mincut separating \( a \) from \( b \). Contracting all nodes in this set gives a new graph \( G' \) where the set \( B \) is represented by node \( v_B \). Because of Lemma 3 we know that \( f'(x, a) = f(x, a) \) as \( x, a \notin B \).

We further have \( f'(x, a) \geq \min\{f'(x, v_B), f'(v_B, a)\} \).

Since \( s \in B \) we have \( f'(v_B, x) \geq f(s, x) \).

Also, \( f'(a, v_B) \geq f(a, b) \geq f(x, s) \) since the \( a-b \) cut that splits \( S_i \) into \( S^a_i \) and \( S^b_i \) also separates \( s \) and \( x \).
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