10 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source $s$; target $t$
- no edges entering $s$ or leaving $t$
- at least for now: no parallel edges
Cuts

**Definition 1**
An \((s, t)\)-cut in the graph \(G\) is given by a set \(A \subset V\) with \(s \in A\) and \(t \in V \setminus A\).

**Definition 2**
The **capacity** of a cut \(A\) is defined as

\[
\operatorname{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e),
\]

where \(\text{out}(A)\) denotes the set of edges of the form \(A \times V \setminus A\) (i.e. edges leaving \(A\)).

**Minimum Cut Problem:** Find an \((s, t)\)-cut with minimum capacity.
The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$. 
Flows

Definition 4
An \((s, t)\)-flow is a function \(f : E \rightarrow \mathbb{R}^+\) that satisfies

1. For each edge \(e\)
   
   \[ 0 \leq f(e) \leq c(e) \, . \]
   
   (capacity constraints)

2. For each \(v \in V \setminus \{s, t\}\)
   
   \[ \sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \, . \]
   
   (flow conservation constraints)
**Definition 5**  
The value of an \((s,t)\)-flow \(f\) is defined as

\[
\text{val}(f) = \sum_{e \in \text{out}(s)} f(e).
\]

**Maximum Flow Problem:** Find an \((s,t)\)-flow with maximum value.
Flows

Example 6

The value of the flow is $\text{val}(f) = 24$. 
Lemma 7 (Flow value lemma)

Let $f$ be a flow, and let $A \subseteq V$ be an $(s,t)$-cut. Then the net-flow across the cut is equal to the amount of flow leaving $s$, i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e).$$
Proof.

\[
\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)
\]

\[
= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)
\]

\[
= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)
\]

The last equality holds since every edge with both end-points in \( A \) contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn’t cancel out are edges leaving or entering \( A \).
Example 8
Corollary 9

Let $f$ be an $(s, t)$-flow and let $A$ be an $(s, t)$-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then $f$ is a maximum flow.

Proof.
Suppose that there is a flow $f'$ with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$

$$= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e)$$

$$\leq \sum_{e \in \text{out}(A)} f'(e)$$

$$\leq \text{cap}(A, V \setminus A)$$

$\square$