10 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source $s$; target $t$
- no edges entering $s$ or leaving $t$
- at least for now: no parallel edges
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Cuts

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**Definition 2**
The capacity of a cut \(A\) is defined as

\[
\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e),
\]

where \(\text{out}(A)\) denotes the set of edges of the form \(A \times V \setminus A\) (i.e. edges leaving \(A\)).
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**Minimum Cut Problem:** Find an \((s, t)\)-cut with minimum capacity.
The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$. 
Definition 4
An \((s, t)\)-flow is a function \(f : E \rightarrow \mathbb{R}^+\) that satisfies

1. For each edge \(e\)
   \[
   0 \leq f(e) \leq c(e) .
   \]
   (capacity constraints)

2. For each \(v \in V \setminus \{s, t\}\)
   \[
   \sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .
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The value of an \((s,t)\)-flow \(f\) is defined as

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\text{val}(f) = \sum_{e \in \text{out}(s)} f(e).
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Maximum Flow Problem: Find an \((s,t)\)-flow with maximum value.
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Maximum Flow Problem: Find an \((s, t)\)-flow with maximum value.
The value of the flow is $\text{val}(f) = 24$. 
Lemma 7 (Flow value lemma)

Let $f$ be a flow, and let $A \subseteq V$ be an $(s, t)$-cut. Then the net-flow across the cut is equal to the amount of flow leaving $s$, i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e).$$
Proof.

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= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)
\]

The last equality holds since every edge with both end-points in \(A\) contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering \(A\).
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\( \square \)
Example 8
Corollary 9

Let $f$ be an $(s, t)$-flow and let $A$ be an $(s, t)$-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then $f$ is a maximum flow.
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$$\text{cap}(A, V \setminus A) < \text{val}(f')$$
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□