A Fast Matching Algorithm

Algorithm 28 Bimatch-Hopcroft-Karp($G$)

1: $M ← \emptyset$
2: repeat
3: let $P = \{P_1, \ldots, P_k\}$ be maximal set of vertex-disjoint, shortest augmenting path w.r.t. $M$.
4: $M ← M \oplus (P_1 \cup \cdots \cup P_k)$
5: until $P = \emptyset$
6: return $M$

We call one iteration of the repeat-loop a phase of the algorithm.
Analysis Hopcroft-Karp

**Lemma 1**

Given a matching $M$ and a maximal matching $M^*$ there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. $M$.

**Proof:**

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in $M$ and red if they are in $M^*$.
- The connected components of $G$ are cycles and paths.
- The graph contains $k \overset{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least $k$ components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. $M$. 
Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. $M$ (let $\ell = |P_i|$).

$M' \overset{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.

Let $P$ be an augmenting path in $M'$.

Lemma 2

The set $A \overset{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ contains at least $(k + 1)\ell$ edges.
Proof.

- The set describes exactly the symmetric difference between matchings $M$ and $M' \oplus P$.
- Hence, the set contains at least $k + 1$ vertex-disjoint augmenting paths w.r.t. $M$ as $|M'| = |M| + k + 1$.
- Each of these paths is of length at least $\ell$. 
Lemma 3

$P$ is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

▶ If $P$ does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.

▶ Otherwise, at least one edge from $P$ coincides with an edge from paths $\{P_1, \ldots, P_k\}$.

▶ This edge is not contained in $A$.

▶ Hence, $|A| \leq k\ell + |P| - 1$.

▶ The lower bound on $|A|$ gives $(k + 1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$. 
If the shortest augmenting path w.r.t. a matching $M$ has $\ell$ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

**Proof.**
The symmetric difference between $M$ and $M^*$ contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.
Lemma 4

The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Proof.

- After iteration $\lfloor \sqrt{|V|} \rfloor$ the length of a shortest augmenting path must be at least $\lfloor \sqrt{|V|} \rfloor + 1 \geq \sqrt{|V|}$.

- Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.
Analysis Hopcroft-Karp

Lemma 5

One phase of the Hopcroft-Karp algorithm can be implemented in time $O(m)$.

construct a “level graph” $G'$:

- construct Level 0 that includes all free vertices on left side $L$
- construct Level 1 containing all neighbors of Level 0
- construct Level 2 containing matching neighbors of Level 1
- construct Level 3 containing all neighbors of Level 2
- ...  
- stop when a level (apart from Level 0) contains a free vertex

can be done in time $O(m)$ by a modified BFS
Analysis Hopcroft-Karp

- a shortest augmenting path must go from Level 0 to the last layer constructed
- it can only use edges between layers
- construct a maximal set of vertex disjoint augmenting path connecting the layers
- for this, go forward until you either reach a free vertex or you reach a “dead end” \( v \)
- if you reach a free vertex delete the augmenting path and all incident edges from the graph
- if you reach a dead end backtrack and delete \( v \) together with its incident edges
Analysis Hopcroft-Karp

See lecture versions of the slides.
Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $O(mn)$
  ▶ a search (successful or unsuccessful) takes time $O(n)$
  ▶ a search deletes at least one edge from the level graph

there are at most $n$ phases

Time: $O(mn^2)$. 
Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$
  ▶ an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases
  ▶ after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph
  ▶ hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$. 