Baseball Elimination

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

<table>
<thead>
<tr>
<th>team</th>
<th>wins $w_i$</th>
<th>losses $\ell_i$</th>
<th>remaining games</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atlanta</td>
<td>83</td>
<td>71</td>
<td>−</td>
</tr>
<tr>
<td>Philadelphia</td>
<td>80</td>
<td>79</td>
<td>1</td>
</tr>
<tr>
<td>New York</td>
<td>78</td>
<td>78</td>
<td>6</td>
</tr>
<tr>
<td>Montreal</td>
<td>77</td>
<td>82</td>
<td>1 2</td>
</tr>
</tbody>
</table>

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Formal definition of the problem:

- Given a set $S$ of teams, and one specific team $z \in S$.
- Team $x$ has already won $w_x$ games.
- Team $x$ still has to play team $y$, $r_{xy}$ times.
- Does team $z$ still have a chance to finish with the most number of wins.
Baseball Elimination

Flow network for $z = 3$. $M$ is number of wins Team 3 can still obtain.

Idea. Distribute the results of remaining games in such a way that no team gets too many wins.
Certificate of Elimination

Let \( T \subseteq S \) be a subset of teams. Define

\[
\begin{align*}
w(T) &:= \sum_{i \in T} w_i, \\
r(T) &:= \sum_{i,j \in T, i < j} r_{ij}
\end{align*}
\]

wins of teams in \( T \)

remaining games among teams in \( T \)

If \( \frac{w(T) + r(T)}{|T|} > M \) then one of the teams in \( T \) will have more than \( M \) wins in the end. A team that can win at most \( M \) games is therefore eliminated.

12.2 Baseball Elimination
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A team \( z \) is eliminated if and only if the flow network for \( z \) does not allow a flow of value \( \sum_{i,j \in S \setminus \{z\}, i < j} r_{ij} \).
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Proof (\( \Leftarrow \))

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- This gives \( M < (w(T) + r(T))/|T| \), i.e., \( z \) is eliminated.
Baseball Elimination

Proof ($\Rightarrow$)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing $x$-$y$ it defines how many games team $x$ and team $y$ should win.
- The flow leaving the team-node $x$ can be interpreted as the additional number of wins that team $x$ will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than $M$ wins in total.
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