8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.
Additional implementation details:

- Every node $x$ stores its degree in a field $x.\text{degree}$. Note that this can be updated in constant time when adding a child to $x$.
- Every node stores a boolean value $x.\text{marked}$ that specifies whether $x$ is marked or not.
8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.

The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$. 
We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use $c$ to denote the amount of work that a unit of potential can pay for.
8.3 Fibonacci Heaps

$S. \text{ minimum()}$

- Access through the min-pointer.
- Actual cost $\Theta(1)$.
- No change in potential.
- Amortized cost $\Theta(1)$. 
8.3 Fibonacci Heaps

S. merge($S'$)

- Merge the root lists.
- Adjust the min-pointer

Running time:

- Actual cost $O(1)$.
- No change in potential.
- Hence, amortized cost is $O(1)$. 
8.3 Fibonacci Heaps

\textbf{S. merge(}\(S')\textbf{)}

- Merge the root lists.
- Adjust the min-pointer

\textbf{Running time:}

- Actual cost \(\mathcal{O}(1)\).
8.3 Fibonacci Heaps

S. merge($S'$)

- Merge the root lists.
- Adjust the min-pointer

Running time:

- Actual cost $O(1)$.
- No change in potential.
8.3 Fibonacci Heaps

S. merge($S'$)

- Merge the root lists.
- Adjust the min-pointer

Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$. 
8.3 Fibonacci Heaps

\textbf{S. insert}(x)

- Create a new tree containing \( x \).
- Insert \( x \) into the root-list.
- Update min-pointer, if necessary.

![Diagram of a Fibonacci heap](image)

\textbf{Running time:}

- Actual cost \( O(1) \).
- Change in potential is \(+1\).
- Amortized cost is \( c + O(1) = O(1) \).
8.3 Fibonacci Heaps

**S. insert(x)**

- Create a new tree containing \( x \).
- Insert \( x \) into the root-list.
- Update min-pointer, if necessary.
8.3 Fibonacci Heaps

S. insert(x)

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.

Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is +1.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$. 
8.3 Fibonacci Heaps

S. delete-min(x)

- **Delete minimum**: Add child-trees to heap; time: $D(\min) \cdot O(1)$.
- **Update min-pointer**: Time: $(t + D(\min)) \cdot O(1)$.

- **Consolidate root-list**: So that no roots have the same degree. Time $t \cdot O(1)$ (see next slide).
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\text{min}) \cdot O(1)$.

- Consolidate root-list so that no roots have the same degree. Time $t \cdot O(1)$ (see next slide).
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\text{min}) \cdot O(1)$.
- Update min-pointer; time: $(t + D(\text{min})) \cdot O(1)$. 

Diagram:

```
 min

7 18 41 52 23 24 17
39 44 35 46 30
18 39 44 26 24 17

min
```
8.3 Fibonacci Heaps

S. delete-min(\(x\))

- Delete minimum; add child-trees to heap; time: \(D(\text{min}) \cdot \Theta(1)\).
- Update min-pointer; time: \((t + D(\text{min})) \cdot \Theta(1)\).
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\text{min}) \cdot \Theta(1)$.
- Update min-pointer; time: $(t + D(\text{min})) \cdot \Theta(1)$.

- Consolidate root-list so that no roots have the same degree. Time $t \cdot \Theta(1)$ (see next slide).
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

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Consolidate:
8.3 Fibonacci Heaps

Consolidate:

![Diagram of Fibonacci Heaps]

- **current**
- **min**

Node values: 7, 18, 23, 24, 17, 52, 41, 39, 44, 26, 46, 35, 30
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:

![Diagram of Fibonacci Heaps]

- Current
- Min
- 7
- 18
- 23
- 17
- 52
- 24
- 46
- 39
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Actual cost for delete-min()

▪ At most $D_n + t$ elements in root-list before consolidate.
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $O(1) \cdot (D_n + t)$.
  Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$. 
8.3 Fibonacci Heaps

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Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
8.3 Fibonacci Heaps

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Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
8.3 Fibonacci Heaps

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Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
8.3 Fibonacci Heaps

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- The amortized cost is
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $\Theta(1) \cdot (D_n + t)$.
  Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is
  
  $$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
8.3 Fibonacci Heaps

Actual cost for delete-min()

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- The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \leq (c_1 + c)D_n + (c_1 - c)t + c$$
8.3 Fibonacci Heaps

Actual cost for `delete-min()`

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a `delete-min` is at most $O(1) \cdot (D_n + t)$.
  Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for `delete-min()`

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1)$$
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $\Theta(1) \cdot (D_n + t)$.
  Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \Theta(D_n)$$
### 8.3 Fibonacci Heaps

#### Actual cost for delete-min()
- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

#### Amortized cost for delete-min()
- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is
  
  $c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$
  
  $\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)$

  for $c \geq c_1$. 
  

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$. 
8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then

\[ D_n \leq \log n. \]
Fibonacci Heaps: decrease-key(handle $h, v$)

Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by $h$. Nothing else to do.
Fibonacci Heaps: decrease-key\((h, \nu)\)

**Case 1: decrease-key does not violate heap-property**

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Fibonacci Heaps: decrease-key(handle $h, v$)

Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by $h$. Nothing else to do.
Fibonacci Heaps: decrease-key(handle $h$, $v$)

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element $x$ reference by $h$.
- If the heap-property is violated, cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Mark the (previous) parent of $x$ (unless it’s a root).
Fibonacci Heaps: \texttt{decrease-key}((handle \( h, v \))

Case 2: heap-property is violated, but parent is not marked

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Fibonacci Heaps: `decrease-key`(handle $h$, $v$)

**Case 3: heap-property is violated, and parent is marked**

- Decrease key-value of element $x$ reference by $h$.
- Cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Continue cutting the parent until you arrive at an unmarked node.
**Fibonacci Heaps: decrease-key(handle \( h, v \))**

Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element \( x \) reference by \( h \).
- Cut the parent edge of \( x \), and make \( x \) into a root.
- Adjust min-pointers, if necessary.
- Continue cutting the parent until you arrive at an unmarked node.
Fibonacci Heaps: decrease-key(handle $h, v$)

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Fibonacci Heaps: decrease-key\((\text{handle } h, v)\)

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Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element $x$ reference by $h$.
- Cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Execute the following:
  
  ```
  p ← parent[x];
  while ($p$ is marked)
      pp ← parent[$p$];
      cut of $p$; make it into a root; unmark it;
      $p ← pp$;
  if $p$ is unmarked and not a root mark it;
  ```
Fibonacci Heaps: \texttt{decrease-key}(handle \( h \), \( v \))

**Actual cost:**
- Constant cost for decreasing the value.
- Constant cost for each of \( \ell \) cuts.
- Hence, cost is at most \( c_2 \cdot (\ell + 1) \), for some constant \( c_2 \).

**Amortized cost:**
- Initially, as every cut creates one new root.
- Amortized cost is at most \( c_2 \cdot (\ell + 1) \), since all but the first cut unmarks a node; the last cut may mark a node.
- Amortized cost is at most \( c_2 \cdot (\ell + 1) \).
Fibonacci Heaps: decrease-key(handle \( h, v \))

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- Hence, cost is at most \( c_2 \cdot (\ell + 1) \), for some constant \( c_2 \).

Amortized cost:

- \( t' = t + \ell \), as every cut creates one new root.
- \( m' \leq m - (\ell - 1) + 1 = m - \ell + 2 \), since all but the first cut unmarks a node; the last cut may mark a node.
- \( \Delta \Phi \leq \ell + 2(\ell - \ell + 2) = 4 \).

Amortized cost is at most

\[ c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = O(1), \] if \( c \geq c_2 \).
Fibonacci Heaps: \texttt{decrease-key}(\textit{handle} \ h, \ v)

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- Amortized cost is at most \( c_2 \cdot (\ell + 1) + c(4 - \ell) \leq (c_2 - c) \ell + 4c + c_2 = O(1), \) if \( c \geq c_2 \).
Fibonacci Heaps: decrease-key(handle $h$, $v$)

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- Amortized cost is at most $c_2 (\ell + 1) + c (4 - \ell) \leq (c_2 - c) \ell + 4c + c_2 = O(1)$, if $c \geq c_2$. 

8.3 Fibonacci Heaps

Ernst Mayr, Harald Räcke
Fibonacci Heaps: decrease-key(handle $h$, $v$)

Actual cost:
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- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta \Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- Amortized cost is at most
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Fibonacci Heaps: decrease-key\((\text{handle } h, v)\)

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Fibonacci Heaps: decrease-key(handle \( h, v \))

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- Amortized cost is at most

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c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \Theta(1),
\]

if \( c \geq c_2 \).
Fibonacci Heaps: \texttt{decrease-key(handle } h, v)\texttt{)

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- Constant cost for decreasing the value.
- Constant cost for each of $\ell$ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant $c_2$.

Amortized cost:

- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- Amortized cost is at most
  \[ c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1), \]
  if $c \geq c_2$. 

8.3 Fibonacci Heaps

Ernst Mayr, Harald Räcke
Delete node

\[ H. \, delete(x): \]
\begin{itemize}
  \item decrease value of \( x \) to \( -\infty \).
  \item delete-min.
\end{itemize}

Amortized cost: \( \Theta(D_n) \)
\begin{itemize}
  \item \( \Theta(1) \) for decrease-key.
  \item \( \Theta(D_n) \) for delete-min.
\end{itemize}
8.3 Fibonacci Heaps

Lemma 1

Let $x$ be a node with degree $k$ and let $y_1, \ldots, y_k$ denote the children of $x$ in the order that they were linked to $x$. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$
8.3 Fibonacci Heaps

Proof

- When $y_i$ was linked to $x$, at least $y_1, \ldots, y_{i-1}$ were already linked to $x$.
- Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- Since, then $y_i$ has lost at most one child.
- Therefore, $\text{degree}(y_i) \geq i - 2$. 

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8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

- Let $s_k$ be the minimum possible size of a sub-tree rooted at a node of degree $k$ that can occur in a Fibonacci heap.
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- $s_k$ monotonically increases with $k$. 
8.3 Fibonacci Heaps

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- $s_k$ monotonically increases with $k$.
- $s_0 = 1$ and $s_1 = 2$. 

\[ s_k = 2 + k \sum_{i=2}^{k} s_i - 2 \]
8.3 Fibonacci Heaps

- Let $s_k$ be the minimum possible size of a sub-tree rooted at a node of degree $k$ that can occur in a Fibonacci heap.
- $s_k$ monotonically increases with $k$.
- $s_0 = 1$ and $s_1 = 2$.

Let $x$ be a degree $k$ node of size $s_k$ and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$
8.3 Fibonacci Heaps

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s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i) \geq 2 + \sum_{i=2}^{k} s_{i-2}
\]
8.3 Fibonacci Heaps

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\[
\geq 2 + \sum_{i=2}^{k} s_{i-2}
\]

\[
= 2 + \sum_{i=0}^{k-2} s_i
\]
8.3 Fibonacci Heaps

Definition 2
Consider the following non-standard Fibonacci type sequence:

\[ F_k = \begin{cases} 
1 & \text{if } k = 0 \\
2 & \text{if } k = 1 \\
F_{k-1} + F_{k-2} & \text{if } k \geq 2 
\end{cases} \]

\[ \phi = \frac{1 + \sqrt{5}}{2} \]

denotes the golden ratio.

Note that \( \phi^2 = 1 + \phi \).

Facts:
1. \( F_k \geq \phi^k \).
2. For \( k \geq 2 \): \( F_k = 2 + \sum_{i=0}^{k-2} F_i \).

The above facts can be easily proved by induction. From this it follows that \( s_k \geq F_k \geq \phi^k \), which gives that the maximum degree in a Fibonacci heap is logarithmic.
k = 0: \[ 1 = F_0 \geq \Phi^0 = 1 \]

k = 1: \[ 2 = F_1 \geq \Phi^1 \approx 1.61 \]

k-2, k-1 \rightarrow k: \[ F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}(\Phi + 1) = \Phi^k \]

k = 2: \[ 3 = F_2 = 2 + 1 = 2 + F_0 \]

k-1 \rightarrow k: \[ F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i \]