The Inhomogeneous Case
If \( f(n) \) is a polynomial of degree \( r \) this method can be applied \( r + 1 \) times to obtain a homogeneous equation:

\[
T[n] = T[n - 1] + n^2
\]

Shift:

\[
T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1
\]

Difference:

\[
\]

\[
T[n] = 2T[n - 1] - T[n - 2] + 2n - 1
\]

and so on...

6.4 Generating Functions

Definition 4 (Generating Function)
Let \((a_n)_{n\geq 0}\) be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

\[
F(z) := \sum_{n\geq 0} a_n z^n;
\]

- exponential generating function (exponentielle Erzeugendenfunktion) is

\[
F(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n.
\]

Example 5
1. The generating function of the sequence \((1, 0, 0, \ldots)\) is

\[
F(z) = 1.
\]

2. The generating function of the sequence \((1, 1, 1, \ldots)\) is

\[
F(z) = \frac{1}{1 - z}.
\]
6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let \( f = \sum_{n \geq 0} a_n z^n \) and \( g = \sum_{n \geq 0} b_n z^n \).

- Equality: \( f \) and \( g \) are equal if \( a_n = b_n \) for all \( n \).
- Addition: \( f + g := \sum_{n \geq 0} (a_n + b_n) z^n \).
- Multiplication: \( f \cdot g := \sum_{n \geq 0} c_n z^n \) with \( c_n = \sum_{p=0}^{n} a_p b_{n-p} \).

There are no convergence issues here.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function \( f : \mathbb{C} \to \mathbb{C} \).

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

What does \( \sum_{n \geq 0} z^n = \frac{1}{1-z} \) mean in the algebraic view?

It means that the power series \( 1 - z \) and the power series \( \sum_{n \geq 0} z^n \) are invers, i.e.,

\[
(1 - z) \cdot \left( \sum_{n \geq 0} z^n \right) = 1.
\]

This is well-defined.
6.4 Generating Functions

We can repeat this
\[ \sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2}. \]

Derivative:
\[ \sum_{n \geq 1} n(n+1)z^{n-1} = \frac{2}{(1-z)^3} \sum_{n \geq 0} (n+1)(n+2)z^n \]

Hence, the generating function of the sequence \( a_n = (n+1)(n+2) \) is \( \frac{2}{(1-z)^3} \).

6.4 Generating Functions

Computing the \( k \)-th derivative of \( \sum z^n \).
\[ \sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1)z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \ldots \cdot (n+1)z^n \]
\[ = \frac{k!}{(1-z)^{k+1}}. \]

Hence:
\[ \sum_{n \geq 0} \binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}}. \]

The generating function of the sequence \( a_n = \binom{n+k}{k} \) is \( \frac{1}{(1-z)^{k+1}} \).

6.4 Generating Functions

We know
\[ \sum_{n \geq 0} y^n = \frac{1}{1-y}. \]

Hence,
\[ \sum_{n \geq 0} a^n z^n = \frac{1}{1-az}. \]

The generating function of the sequence \( f_n = a^n \) is \( \frac{1}{1-az} \).
Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= a_0 + \sum_{n=1}^{\infty} (a_{n-1} + 1) z^n$$

$$= 1 + z \sum_{n=1}^{\infty} a_{n-1} z^{n-1} + \sum_{n=1}^{\infty} z^n$$

$$= z \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} z^n$$

$$= zA(z) + \sum_{n=0}^{\infty} z^n$$

$$= zA(z) + \frac{1}{1-z}$$

Solving for $A(z)$ gives

$$\sum_{n=0}^{\infty} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$$

Hence, $a_n = n + 1$.

Some Generating Functions

<table>
<thead>
<tr>
<th>$n$-th sequence element</th>
<th>generating function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{1-z}$</td>
</tr>
<tr>
<td>$n + 1$</td>
<td>$\frac{1}{(1-z)^2}$</td>
</tr>
<tr>
<td>$\binom{n+k}{k}$</td>
<td>$\frac{1}{(1-z)^{k+1}}$</td>
</tr>
<tr>
<td>$n^2$</td>
<td>$\frac{z(1+z)}{(1-z)^3}$</td>
</tr>
<tr>
<td>$\frac{1}{n!}$</td>
<td>$e^z$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$-th sequence element</th>
<th>generating function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c f_n$</td>
<td>$c F$</td>
</tr>
<tr>
<td>$f_n + g_n$</td>
<td>$F + G$</td>
</tr>
<tr>
<td>$\sum_{i=0}^{n} f_i g_{n-i}$</td>
<td>$F \cdot G$</td>
</tr>
<tr>
<td>$f_{n-k}$ ($n \geq k$); 0 otw.</td>
<td>$z^k F$</td>
</tr>
<tr>
<td>$\sum_{i=0}^{n} f_i$</td>
<td>$\frac{F(z)}{1-z}$</td>
</tr>
<tr>
<td>$n f_n$</td>
<td>$z \frac{dF(z)}{dz}$</td>
</tr>
<tr>
<td>$c^n f_n$</td>
<td>$F(c z)$</td>
</tr>
</tbody>
</table>
Solving Recursions with Generating Functions

1. Set \( A(z) = \sum_{n \geq 0} a_n z^n \).
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by \( A(z) \).
4. Solving for \( A(z) \) gives an equation of the form \( A(z) = f(z) \), where hopefully \( f(z) \) is a simple function.
5. Write \( f(z) \) as a formal power series.
   Techniques:
     ▶ partial fraction decomposition (Partialbuckzerlegung)
     ▶ lookup in tables
6. The coefficients of the resulting power series are the \( a_n \).

Example: \( a_n = 2a_{n-1}, \ a_0 = 1 \)

3. Transform right hand side so that infinite sums can be replaced by \( A(z) \) or by simple function.
   \[
   A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n
   = 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1}
   = 1 + 2z \sum_{n \geq 0} a_n z^n
   = 1 + 2z \cdot A(z)
   \]
   4. Solve for \( A(z) \).
   \[
   A(z) = \frac{1}{1 - 2z}
   \]
Example: \( a_n = 3a_{n-1} + n, \ a_0 = 1 \)

1. Set up generating function:
\[
A(z) = \sum_{n \geq 0} a_n z^n
\]

Example: \( a_n = 3a_{n-1} + n, \ a_0 = 1 \)

2. Transform right hand side:
\[
A(z) = \sum_{n \geq 0} a_n z^n
= a_0 + \sum_{n \geq 1} a_n z^n
= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n
= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n
= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n
= 1 + 3zA(z) + \frac{z}{(1 - z)^2}
\]

Example: \( a_n = 3a_{n-1} + n, \ a_0 = 1 \)

4. Solve for \( A(z) \):
\[
A(z) = 1 + 3zA(z) + \frac{z}{(1 - z)^2}
\]
gives
\[
A(z) = \frac{(1 - z)^2 + z}{(1 - 3z)(1 - z)^2} = \frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2}
\]

Example: \( a_n = 3a_{n-1} + n, \ a_0 = 1 \)

5. Write \( f(z) \) as a formal power series:
We use partial fraction decomposition:
\[
\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} = \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}
\]
This gives
\[
z^2 - z + 1 = A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z)
= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z)
= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C)
\]
Example: \( a_n = 3a_{n-1} + n, a_0 = 1 \)

5. Write \( f(z) \) as a formal power series:

This leads to the following conditions:

\[
\begin{align*}
A + B + C &= 1 \\
2A + 4B + 3C &= 1 \\
A + 3B &= 1
\end{align*}
\]

which gives

\[
\begin{align*}
A &= \frac{7}{4} \\
B &= -\frac{1}{4} \\
C &= -\frac{1}{2}
\end{align*}
\]

6. This means \( a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}. \)

6.5 Transformation of the Recurrence

Example 6

\[
\begin{align*}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2
\end{align*}
\]

Define

\[g_n := \log f_n.\]

Then

\[
\begin{align*}
g_n &= g_{n-1} + g_{n-2} \text{ for } n \geq 2 \\
g_1 &= \log 2 = 1 \text{ (for } \log = \log_2\text{), } g_0 = 0 \\
g_n &= F_n \text{ (} n \text{-th Fibonacci number) } \\
f_n &= 2^{F_n}
\end{align*}
\]

Example 7

\[
\begin{align*}
f_1 &= 1 \\
f_n &= 3f_{n-1} + n \text{ for } n = 2^k, k \geq 1
\end{align*}
\]

Define

\[g_k := f_{2^k}.\]

Then

\[
\begin{align*}
g_0 &= 1 \\
g_k &= 3g_{k-1} + 2^k, \ k \geq 1
\end{align*}
\]