7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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Let $|L_1|$ denote the number of elements in the "express lane," and $|L_0| = n$ the number of all elements (ignoring dummy elements).

Worst case search time: $|L_1| + |L_0| = |L_1|$ (ignoring additive constants).

Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$. 
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Add more express lanes. Lane $L_i$ contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list $L_{i-1}$. 
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**Search \( x \) \( (k + 1 \) lists \( L_0, \ldots, L_k) \)**

- Find the largest item in list \( L_k \) that is smaller than \( x \). At most \( |L_k| + 2 \) steps.
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- \( \ldots \)
- At most \(|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k + 1)\) steps.
Choose ratios between list-lengths evenly, i.e., \( \frac{|L_{i-1}|}{|L_i|} = r \), and, hence, \( L_k \approx r^{-k} n \).
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Choosing \( k = \Theta(\log n) \) gives a logarithmic running time.
How to do insert and delete?

If we want that in \( L_i \) we always skip over roughly the same number of elements in \( L_{i-1} \), an insert or delete may require a lot of re-organisation.

Use randomization instead!
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Insert:

- A search operation gives you the insert position for element $x$ in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, \ldots\}$ of trials needed.
- Insert $x$ into lists $L_0, \ldots, L_{t-1}$.

Delete:

- You get all predecessors via backward pointers.
- Delete $x$ in all lists it actually appears in.

The time for both operations is dominated by the search time.
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High Probability

Definition 1 (High Probability)

We say a randomized algorithm has running time $O(\log n)$ with high probability if for any constant $\alpha$ the running time is at most $O(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the $O$-notation hides a constant that may depend on $\alpha$. 
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Here the $O$-notation hides a constant that may depend on $\alpha$. 
High Probability

Suppose there are a polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. $E_i$ may be the event that the $i$-th search in a skip list takes time at most $O(\log n)$).
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Then the probability that all $E_i$ hold is at least

$$\Pr[E_1 \land \cdots \land E_\ell]$$
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This means $\Pr[E_1 \land \cdots \land E_\ell]$ holds with high probability.
Lemma 2

A search (and, hence, also insert and delete) in a skip list with $n$ elements takes time $\Theta(\log n)$ with high probability (w. h. p.).
7.6 Skip Lists

Backward analysis:

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At each point the path goes up with probability $\frac{1}{2}$ and left with $\frac{1}{2}$.

We show that w.h.p:

▶ A "long" search path must also go very high.
▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.
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\[(\frac{n}{k})^k \leq \binom{n}{k} \leq (\frac{en}{k})^k\]
\[
\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n - k)!} = \frac{n \cdot \ldots \cdot (n - k + 1)}{k \cdot \ldots \cdot 1}
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left( \frac{n}{k} \right)^k
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
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\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left( \frac{n}{k} \right)^k
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\]

\[
\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!}
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left( \frac{n}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!}
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left( \frac{n}{k} \right)^k
\]

\[
\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}
\]
\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
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\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left( \frac{n}{k} \right)^k
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\[
\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}
\]

\[
= \left( \frac{n}{k} \right)^k \cdot \frac{k^k}{k!}
\]
\[
\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left(\frac{n}{k}\right)^k
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\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}
\]

\[
= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{en}{k}\right)^k
\]
Let $E_{z,k}$ denote the event that a search path is of length $z$ (number of edges) but does not visit a list above $L_k$. In particular, this means that during the construction in the backward analysis we see at most $k$ heads (i.e., coin flips that tell you to go up) in $z$ trials.
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In particular, this means that during the construction in the backward analysis we see at most $k$ heads (i.e., coin flips that tell you to go up) in $z$ trials.
7.6 Skip Lists

\[ \Pr[E_{z,k}] \]
Pr\[E_{z,k}\] \leq \text{Pr[at most } k \text{ heads in } z \text{ trials]}

Pr[$E_{z,k}$] ≤ Pr[at most $k$ heads in $z$ trials]

≤ \binom{z}{k} 2^{-(z-k)}
Pr[$E_{z,k}$] ≤ Pr[at most $k$ heads in $z$ trials]

\[ \leq \binom{z}{k} 2^{-(z-k)} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \]
7.6 Skip Lists

Pr[$E_{z,k}$] ≤ Pr[at most k heads in z trials]

\[ \leq \binom{z}{k} 2^{-(z-k)} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z} \]
7.6 Skip Lists

\[ \Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}] \]

\[ \leq \left( \frac{z}{k} \right) 2^{-(z-k)} \leq \left( \frac{e z}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z} \]

choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha) \gamma \log n \)
Pr\([E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]\) 

\[
\leq \binom{z}{k} 2^{-(z-k)} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z}
\]

choosing \(k = \gamma \log n\) with \(\gamma \geq 1\) and \(z = (\beta + \alpha) \gamma \log n\)

\[
\leq \left( \frac{2ez}{k} \right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha}
\]
7.6 Skip Lists

\[
\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]
\]

\[
\leq \left(\frac{z}{k}\right) 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}
\]

choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha)\gamma \log n \)

\[
\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2\beta k}\right)^k \cdot n^{-\alpha}
\]
Pr\[E_{z,k}\] \leq Pr[\text{at most } k \text{ heads in } z \text{ trials}]

\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}

choosing \(k = \gamma \log n\) with \(\gamma \geq 1\) and \(z = (\beta + \alpha) \gamma \log n\)

\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2\beta k}\right)^k \cdot n^{-\alpha}

\leq \left(\frac{2e(\beta + \alpha)}{2\beta}\right)^k n^{-\alpha}
Pr[$E_{z,k} \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$

\[ \leq \left( \frac{z}{k} \right)^{2^{-(z-k)}} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z} \]

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha) \gamma \log n$

\[ \leq \left( \frac{2ez}{k} \right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left( \frac{2ez}{2\beta k} \right)^k \cdot n^{-\alpha} \]

\[ \leq \left( \frac{2e(\beta + \alpha)}{2\beta} \right)^k n^{-\alpha} \]

now choosing $\beta = 6\alpha$ gives
Pr[$E_{z,k}$] $\leq$ Pr[at most $k$ heads in $z$ trials]

$$\leq \left(\frac{z}{k}\right)^{2^{-(z-k)}} \leq \left(\frac{ez}{k}\right)^{k} 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^{k} 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha) \gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^{k} 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^{k} \cdot n^{-\alpha}$$

$$\leq \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^{k} n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\leq \left(\frac{42\alpha}{64^\alpha}\right)^{k} n^{-\alpha}$$
7.6 Skip Lists

\[ \Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}] \]

\[ \leq \binom{z}{k} 2^{-(z-k)} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z} \]

choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha) \gamma \log n \)

\[ \leq \left( \frac{2ez}{k} \right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left( \frac{2ez}{2\beta k} \right)^k \cdot n^{-\alpha} \]

\[ \leq \left( \frac{2e(\beta + \alpha)}{2\beta} \right)^k n^{-\alpha} \]

now choosing \( \beta = 6\alpha \) gives

\[ \leq \left( \frac{42\alpha}{64\alpha} \right)^k n^{-\alpha} \leq n^{-\alpha} \]
7.6 Skip Lists

\[ \Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}] \]

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choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha) \gamma \log n \)

\[ \leq \left( \frac{2ez}{k} \right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left( \frac{2ez}{2^\beta k} \right)^k \cdot n^{-\alpha} \]

\[ \leq \left( \frac{2e(\beta + \alpha)}{2^\beta} \right)^k n^{-\alpha} \]

now choosing \( \beta = 6\alpha \) gives

\[ \leq \left( \frac{42\alpha}{64^\alpha} \right)^k n^{-\alpha} \leq n^{-\alpha} \]

for \( \alpha \geq 1 \).
So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7 \alpha \gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let $A_{k+1}$ denote the event that the list $L_{k+1}$ is non-empty. Then

$$\Pr[A_{k+1}] \leq n^2 - (k+1) \leq n - (\gamma - 1).$$

For the search to take at least $z = 7 \alpha \gamma \log n$ steps either the event $E_{z,k}$ or the event $A_{k+1}$ must hold. Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \leq n - \alpha + n - (\gamma - 1).$$

This means, the search requires at most $z$ steps, w.h.p.
7.6 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha \gamma \log n$, $\alpha \geq 1$. 
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This means that a search path of length \( \Omega(\log n) \) visits a list on a level \( \Omega(\log n) \), w.h.p.

Let \( A_{k+1} \) denote the event that the list \( L_{k+1} \) is non-empty. Then

\[
\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)}.
\]
7.6 Skip Lists

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$$\Pr[\text{search requires } z \text{ steps}]$$
7.6 Skip Lists

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$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$
So far we fixed \( k = \gamma \log n, \gamma \geq 1, \) and \( z = 7\alpha \gamma \log n, \alpha \geq 1. \)

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\[
\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]
\leq n^{-\alpha} + n^{-(\gamma-1)}
\]
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So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha \gamma \log n$, $\alpha \geq 1$.

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$$ \Pr[A_{k+1}] \leq n 2^{-(k+1)} \leq n^{-(\gamma-1)}.$$  

For the search to take at least $z = 7\alpha \gamma \log n$ steps either the event $E_{z,k}$ or the event $A_{k+1}$ must hold. Hence,

$$ \Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \leq n^{-\alpha} + n^{-(\gamma-1)}.$$  

This means, the search requires at most $z$ steps, w.h.p.