Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after access, an element is moved to the root; splay(x)
- repeated accesses are faster
- only amortized guarantee
- read-operations change the tree
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\textbf{find}(\textit{x})

- search for \textit{x} according to a search tree
- let \(\hat{x}\) be last element on search-path
- \textbf{splay}(\hat{x})
Splay Trees

insert(x)

▶ search for x; x̃ is last visited element during search (successer or predecessor of x)
▶ splay(x̃) moves x̃ to the root
▶ insert x as new root
**Splay Trees**

`delete(x)`

- search for `x`; splay(`x`); remove `x`
- search largest element $\tilde{x}$ in $A$
- splay($\tilde{x}$) (on subtree $A$)
- connect root of $B$ as right child of $\tilde{x}$

![Diagram of Splay Trees](attachment:image.png)
Move to Root

How to bring element to root?

▶ one (bad) option: moveToRoot(x)
▶ iteratively do rotation around parent of x until x is root
▶ if x is left child do right rotation otw. left rotation
better option \texttt{splay}(x):

- zig case: if $x$ is child of root do left rotation or right rotation around parent
### Splay: Zigzag Case

better option splay($x$):

- zigzag case: if $x$ is right child and parent of $x$ is left child (or $x$ left child parent of $x$ right child)
- do double right rotation around grand-parent (resp. double left left rotation)
Double Rotations

\[ \text{LeftRotate}(y) \]
\[ \text{RightRotate}(x) \]
\[ \text{DoubleRightRotate}(x) \]
Splay: Zigzig Case

better option splay($x$):

- zigzig case: if $x$ is left child and parent of $x$ is left child (or $x$ right child, parent of $x$ right child)
- do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)
Splay vs. Move to Root

7.3 Splay Trees
Splay vs. Move to Root

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Ernst Mayr, Harald Räcke
Splay vs. Move to Root

7.3 Splay Trees

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Static Optimality

Suppose we have a sequence of \( m \) find-operations. \( \text{find}(x) \) appears \( h_x \) times in this sequence.

The cost of a static search tree \( T \) is:

\[
\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)
\]

The total cost for processing the sequence on a splay-tree is \( \Theta(\text{cost}(T_{\text{min}})) \), where \( T_{\text{min}} \) is an optimal static search tree.
Dynamic Optimality

Let $S$ be a sequence with $m$ find-operations.

Let $A$ be a data-structure based on a search tree:
- the cost for accessing element $x$ is $1 + \text{depth}(x)$;
- after accessing $x$ the tree may be re-arranged through rotations;

**Conjecture:**
A splay tree that only contains elements from $S$ has cost $\Theta(\text{cost}(A, S))$, for processing $S$. 
Lemma 1
Splay Trees have an \textit{amortized} running time of $\mathcal{O} \left( \log n \right)$ for all operations.
Amortized Analysis

**Definition 2**
A data structure with operations \( \text{op}_1(), \ldots, \text{op}_k() \) has amortized running times \( t_1, \ldots, t_k \) for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most \( n \) elements, and let \( k_i \) denote the number of occurrences of \( \text{op}_i() \) within this sequence. Then the actual running time must be at most \( \sum_i k_i \cdot t_i(n) \).
Potential Method

Introduce a potential for the data structure.

$\Phi(D_i)$ is the potential after the $i$-th operation.

Amortized cost of the $i$-th operation is $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$.

Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then $k \sum_{i=1}^{k} c_i \leq k \sum_{i=1}^{k} \hat{c}_i = k \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0)$.

This means the amortized costs can be used to derive a bound on the total cost.
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Then

$$\sum_{i=1}^{k} c_i$$
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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

\[
\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0).
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Potential Method

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▷ $\Phi(D_i)$ is the potential after the $i$-th operation.
▷ Amortized cost of the $i$-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

▷ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.
Example: Stack

Stack

- **S. push()**
- **S. pop()**
- **S. multipop(k)**: removes \( k \) items from the stack. If the stack currently contains less than \( k \) items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- **S. push()**: cost 1.
- **S. pop()**: cost 1.
- **S. multipop(k)**: cost \( \min\{\text{size}, k\} = k \).
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Actual cost:

- \textit{S. push()}: cost 1.
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- \textit{S. multipop}(k): cost \( \min\{\text{size, } k\} = k \).
Example: Stack

Use potential function $\Phi(S) = \text{number of elements on the stack.}$

Amortized cost:

- $\hat{C}_{\text{push}} = C_{\text{push}} + \Delta \Phi = 1 + 1 \leq 2$.
- $\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta \Phi = 1 - 1 \leq 0$.
- $\hat{C}_{\text{multipop}}(k) = C_{\text{multipop}} + \Delta \Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0$. 

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Example: Stack

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Example: Binary Counter

**Incrementing a binary counter:**
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an $n$-bit binary counter may require to examine $n$-bits, and maybe change them.

**Actual cost:**

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k + 1$, where $k$ is the number of consecutive ones in the least significant bit-positions (e.g., 001101 has $k = 1$).
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Choose potential function $\Phi(x) = k$, where $k$ denotes the number of ones in the binary representation of $x$.

Amortized cost:

- **Changing bit from 0 to 1:**
  \[ \hat{C}_{0 \to 1} = C_{0 \to 1} + \Delta \Phi = 1 + 1 \leq 2. \]

- **Changing bit from 1 to 0:**
  \[ \hat{C}_{1 \to 0} = C_{1 \to 0} + \Delta \Phi = 1 - 1 \leq 0. \]

- **Increment:** Let $k$ denote the number of consecutive ones in the least significant bit-positions. An increment involves $k$ ($1 \to 0$)-operations, and one ($0 \to 1$)-operation.

Hence, the amortized cost is $k + 1$. For $k = 0$, we get $1$. For $k \geq 1$, we get $k + 1$. Thus, the amortized cost is at most $k + 1$. 

\[ \hat{C}_{\text{increment}} = k + 1 \leq 2. \]
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- Increment: Let $k$ denotes the number of consecutive ones in the least significant bit-positions. An increment involves $k$ ($1 \to 0$)-operations, and one ($0 \to 1$)-operation.

Hence, the amortized cost is $k\hat{C}_{1 \to 0} + \hat{C}_{0 \to 1} \leq 2$. 
Example: Binary Counter

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- **Changing bit from 0 to 1:**

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Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.  

Splay Trees

potential function for splay trees:

- size $s(x) = |T_x|$ 
- rank $r(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.
Splay: Zig Case

\[ \Delta \Phi = r'(x) + r'(p) - r(x) - r(p) \]

\[ = r'(p) - r(x) \]

\[ \leq r'(x) - r(x) \]

\[ \text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x)) \]
Splay: Zig Case

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\[ = r'(p) + r'(g) - r(x) - r(p) \]
\[ \leq r'(x) + r'(g) - r(x) - r(x) \]
\[ = r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \]
\[ = -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \]
\[ \leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
Splay: Zigzig Case

\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]
\[ = r'(p) + r'(g) - r(x) - r(p) \]
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Splay: Zigzig Case

ΔΦ = r′(x) + r′(p) + r′(g) − r(x) − r(p) − r(g)

= r′(p) + r′(g) − r(x) − r(p)

≤ r′(x) + r′(g) − r(x) − r(x)

= r′(x) + r′(g) + r(x) − 3r′(x) + 3r′(x) − r(x) − 2r(x)

= −2r′(x) + r′(g) + r(x) + 3(r′(x) − r(x))

≤ −2 + 3(r′(x) − r(x)) ⇒ cost_{zigzag} ≤ 3(r′(x) − r(x))
Splay: Zigzig Case

\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]

\[ = r'(p) + r'(g) - r(x) - r(p) \]

\[ \leq r'(x) + r'(g) - r(x) - r(x) \]

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\[ = -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \]

\[ \leq -2 + 3(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x)) \]
Splay: Zigzig Case

$$\Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$

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Splay: Zigzig Case

\[
\Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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ΔΦ = \( r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \)

= \( r'(p) + r'(g) - r(x) - r(p) \)

≤ \( r'(x) + r'(g) - r(x) - r(x) \)

= \( r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \)

= \(-2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \)

≤ \(-2 + 3(r'(x) - r(x)) \)  \( \Rightarrow \) cost_{zigzag} \leq 3(r'(x) - r(x))
Splay: Zigzig Case

\[ \frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \]

\[ = \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \]

\[ = \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \]

\[ \leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \]
\[
\frac{1}{2}\left(r(x) + r'(g) - 2r'(x)\right) \\
= \frac{1}{2}\left(\log(s(x)) + \log(s'(g)) - 2 \log(s'(x))\right) \\
= \frac{1}{2} \log \left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log \left(\frac{s'(g)}{s'(x)}\right) \\
\leq \log \left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log \left(\frac{1}{2}\right) = -1
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Splay: Zigzig Case

\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \\
= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
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\[ \leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
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Amortized cost of the whole splay operation:

\[
\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
= 2 + r(\text{root}) - r_0(x) \\
\leq \Theta(\log n)
\]