6.3 The Characteristic Polynomial

Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

This is the general form of a linear recurrence relation of order \( k \) with constant coefficients \((c_0, c_k \neq 0)\).

- \( T(n) \) only depends on the \( k \) preceding values. This means the recurrence relation is of order \( k \).
- The recurrence is linear as there are no products of \( T[n] \)'s.
- If \( f(n) = 0 \) then the recurrence relation becomes a linear, homogenous recurrence relation of order \( k \).

Note that we ignore boundary conditions for the moment.

### Observations:

- The solution \( T_1, T_2, T_3, \ldots \) is completely determined by a set of boundary conditions that specify values for \( T[1], \ldots, T[k] \).
- In fact, any \( k \) consecutive values completely determine the solution.
- \( k \) non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

### Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

#### The Homogenous Case

The solution space

\[ S = \left\{ T = T[1], T[2], T[3], \ldots \mid T \text{ fulfills recurrence relation} \right\} \]

is a vector space. This means that if \( T_1, T_2 \in S \), then also \( \alpha T_1 + \beta T_2 \in S \), for arbitrary constants \( \alpha, \beta \).

**How do we find a non-trivial solution?**

We guess that the solution is of the form \( \lambda^n \), \( \lambda \neq 0 \), and see what happens. In order for this guess to fulfill the recurrence we need

\[ c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_k \lambda^{n-k} = 0 \]

for all \( n \geq k \).

Dividing by \( \lambda^{n-k} \) gives that all these constraints are identical to

\[ c_0 \lambda^k + c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_k = 0 \]

characteristic polynomial \( P[\lambda] \)

This means that if \( \lambda_i \) is a root (Nullstelle) of \( P[\lambda] \) then \( T[n] = \lambda_i^n \) is a solution to the recurrence relation.

Let \( \lambda_1, \ldots, \lambda_k \) be the \( k \) (complex) roots of \( P[\lambda] \). Then, because of the vector space property

\[ \alpha_1 \lambda_i^n + \alpha_2 \lambda_i^n + \cdots + \alpha_k \lambda_i^n \]

is a solution for arbitrary values \( \alpha_i \).
The Homogenous Case

Lemma 1
Assume that the characteristic polynomial has \( k \) distinct roots \( \lambda_1, \ldots, \lambda_k \). Then all solutions to the recurrence relation are of the form
\[
\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n.
\]

Proof.
There is one solution for every possible choice of boundary conditions for \( T[1], \ldots, T[k] \).
We show that the above set of solutions contains one solution for every choice of boundary conditions.

Proof (cont.).
Suppose I am given boundary conditions \( T[i] \) and I want to see whether I can choose the \( \alpha_i \)'s such that these conditions are met:
\[
\begin{align*}
\alpha_1 \lambda_1^1 &+ \alpha_2 \lambda_2^1 + \cdots + \alpha_k \lambda_k^1 = T[1] \\
\alpha_1 \lambda_1^2 &+ \alpha_2 \lambda_2^2 + \cdots + \alpha_k \lambda_k^2 = T[2] \\
& \vdots \\
\alpha_1 \lambda_1^k &+ \alpha_2 \lambda_2^k + \cdots + \alpha_k \lambda_k^k = T[k]
\end{align*}
\]
We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant
\[
\left| \begin{array}{ccccccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \\
\end{array} \right| = k \prod_{i=1}^{k} \lambda_i \cdot
\left| \begin{array}{ccccccc}
1 & 1 & \cdots & 1 & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \\
\end{array} \right|
\]
\[
= k \prod_{i=1}^{k} \lambda_i \cdot
\left| \begin{array}{cccc}
1 & \lambda_1 & \cdots & \lambda_{k-2} \\
1 & \lambda_2 & \cdots & \lambda_{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_k & \cdots & \lambda_{k-2} \\
\end{array} \right|
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 & \ldots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \ldots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \ldots & \lambda_k^{k-2} & \lambda_k^{k-1} \\
\end{vmatrix} = \\
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 & 1 & \ldots & \lambda_1^{k-2} - \lambda_1 & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 & 1 & \ldots & \lambda_2^{k-2} - \lambda_1 & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 & 1 & \ldots & \lambda_k^{k-2} - \lambda_1 & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \\
\end{vmatrix} \\
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 & 1 & \ldots & \lambda_1^{k-2} - \lambda_1 & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 & 1 & \ldots & \lambda_2^{k-2} - \lambda_1 & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 & 1 & \ldots & \lambda_k^{k-2} - \lambda_1 & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \\
\end{vmatrix} = \\
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 & 1 & \ldots & \lambda_1^{k-2} - \lambda_1 & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 & 1 & \ldots & \lambda_2^{k-2} - \lambda_1 & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 & 1 & \ldots & \lambda_k^{k-2} - \lambda_1 & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \\
\end{vmatrix}
\]

Hence, if all \(\lambda_i\)'s are different, then the determinant is non-zero.

Repeating the above steps gives:

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \ldots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i \neq \ell} (\lambda_i - \lambda_\ell)
\]

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Ernst Mayr, Harald Räcke
The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root $\lambda_i$ with multiplicity (Vielfachheit) at least 2. Then not only is $\lambda_i^n$ a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since $\lambda_i$ is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$.

Calculating the derivative gives a polynomial that still has root $\lambda_i$.

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This means

$$c_0n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\frac{c_0n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k}}{T(n)} = 0$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$’s is a solution to the recurrence.

The Homogeneous Case

Suppose $\lambda_i$ has multiplicity $j$. We know that

$$c_0n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

$$c_0n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j - 1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$.

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Lemma 2

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let $\lambda_i$, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities $\ell_i$. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j\lambda_i^n)$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$’s is a solution to the recurrence.
Example: Fibonacci Sequence

\[ T[0] = 0 \]
\[ T[1] = 1 \]
\[ T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2 \]

The characteristic polynomial is

\[ \lambda^2 - \lambda - 1 \]

Finding the roots, gives

\[ \lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1 - \frac{1}{2}(1 + \sqrt{5})} \]

Hence, the solution is of the form

\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[ T[0] = 0 \text{ gives } \alpha + \beta = 0. \]
\[ T[1] = 1 \text{ gives } \alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \Rightarrow \alpha - \beta = \frac{2}{\sqrt{5}} \]

Example: Fibonacci Sequence

Hence, the solution is

\[ \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \]

The Inhomogeneous Case

Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

with \( f(n) \neq 0. \)

While we have a fairly general technique for solving \textbf{homogeneous}, linear recurrence relations the inhomogeneous case is different.
The Inhomogeneous Case

The general solution of the recurrence relation is

\[ T(n) = T_h(n) + T_p(n) , \]

where \( T_h \) is any solution to the homogeneous equation, and \( T_p \) is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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Example: Characteristic polynomial:

\[ \lambda^2 - 2\lambda + 1 = 0 \]

Then the solution is of the form

\[ T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n \]

\( T[0] = 1 \) gives \( \alpha = 1 \).

\( T[1] = 2 \) gives \( 1 + \beta = 2 \Rightarrow \beta = 1 \).

The Inhomogeneous Case

Example: \( T[n] = T[n - 1] + 1 \quad T[0] = 1 \)

Then, \( T[n - 1] = T[n - 2] + 1 \quad (n \geq 2) \)

Subtracting the first from the second equation gives,

\[ T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2) \]

or

\[ T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2) \]

I get a completely determined recurrence if I add \( T[0] = 1 \) and \( T[1] = 2 \).

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The Inhomogeneous Case

If \( f(n) \) is a polynomial of degree \( r \) this method can be applied \( r + 1 \) times to obtain a homogeneous equation:

\[ T[n] = T[n - 1] + n^2 \]

Shift:

\[ T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1 \]

Difference:


\[ T[n] = 2T[n - 1] - T[n - 2] + 2n - 1 \]
\[ T[n] = 2T[n - 1] - T[n - 2] + 2n - 1 \]

Shift:
\[ T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \]
\[ = 2T[n - 2] - T[n - 3] + 2n - 3 \]

Difference:
\[ T[n] - T[n - 1] = 2T[n - 1] - T[n - 2] + 2n - 1 \]
\[ - 2T[n - 2] + T[n - 3] - 2n + 3 \]

\[ T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2 \]

and so on...