# 7.5 $(a, b)$-trees

**Definition 1**

For $b \geq 2a - 1$ an $(a, b)$-tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex $v$ has at least $a$ and at most $b$ children
3. the root has degree at least $2$ if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
5. there is a special dummy leaf node with key-value $\infty$

# 7.5 $(a, b)$-trees

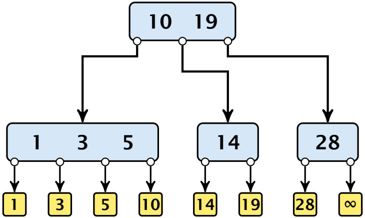
Each internal node $v$ with $d(v)$ children stores $d - 1$ keys $k_1, \dots, k_{d-1}$. The $i$-th subtree of $v$ fulfills

$$k_{i-1} < \text{ key in } i\text{-th sub-tree } \leq k_i \ ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

# 7.5 $(a, b)$-trees

## Example 2



# 7.5 $(a, b)$-trees

## Variants

- The dummy leaf element may not exist; it only makes implementation more convenient.
- Variants in which $b = 2a$ are commonly referred to as $B$-trees.
- A $B$-tree usually refers to the variant in which keys and data are stored at internal nodes.
- A $B^+$ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- A $B^*$ tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a $B$-tree).

## Lemma 3

*Let $T$ be an $(a,b)$-tree for $n > 0$ elements (i.e., $n+1$ leaf nodes) and height $h$ (number of edges from root to a leaf vertex). Then*

1. $2a^{h-1} \le n+1 \le b^h$
2. $\log_b(n+1) \le h \le 1 + \log_a(\frac{n+1}{2})$

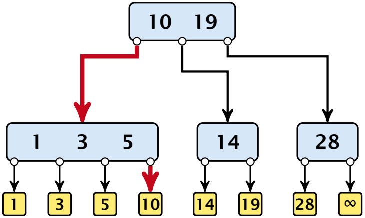
## Proof.

- If $n > 0$ the root has degree at least $2$ and all other nodes have degree at least $a$. This gives that the number of leaf nodes is at least $2a^{h-1}$.
- Analogously, the degree of any node is at most $b$ and, hence, the number of leaf nodes at most $b^h$.

□

# Search

**Search(8)**

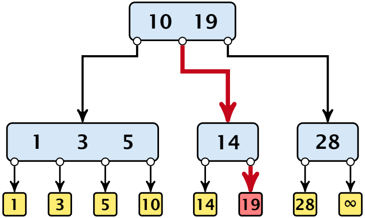



The search is straightforward. It is only important that you need to go all the way to the leaf.

Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

# Search

**Search(19)**




The search is straightforward. It is only important that you need to go all the way to the leaf.

Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

# Insert

Insert element $x$:

- Follow the path as if searching for $\mathrm{key}[x]$.
- If this search ends in leaf $\ell$, insert $x$ before this leaf.
- For this add $\mathrm{key}[x]$ to the key-list of the last internal node $v$ on the path.
- If after the insert $v$ contains $b$ nodes, do Rebalance($v$).

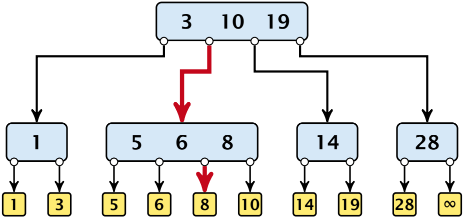
# Insert

Rebalance($v$):

- Let $k_i$, $i = 1, \dots, b$ denote the keys stored in $v$.
- Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- Create two nodes $v_1$, and $v_2$. $v_1$ gets all keys $k_1, \dots, k_{j-1}$ and $v_2$ gets keys $k_{j+1}, \dots, k_b$.
- Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
- They get at most $\lceil \frac{b-1}{2} \rceil$ keys, and have therefore degree at most $\lceil \frac{b-1}{2} \rceil + 1 \leq b$ (since $b \geq 2$).
- The key $k_j$ is promoted to the parent of $v$. The current pointer to $v$ is altered to point to $v_1$, and a new pointer (to the right of $k_j$) in the parent is added to point to $v_2$.
- Then, re-balance the parent.

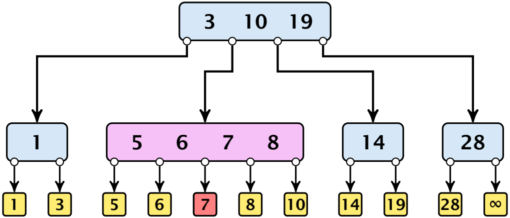
# Insert

**Insert(7)**



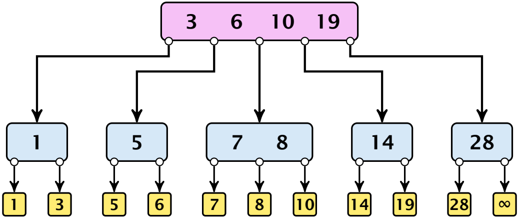
# Insert

## Insert(7)



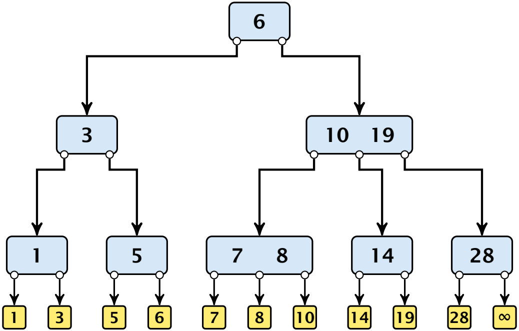
# Insert

**Insert(7)**



# Insert

**Insert(7)**



# Delete

Delete element $x$ (pointer to leaf vertex):

- Let $v$ denote the parent of $x$. If $\mathrm{key}[x]$ is contained in $v$, remove the key from $v$, and delete the leaf vertex.
- Otherwise delete the key of the predecessor of $x$ from $v$; delete the leaf vertex; and replace the occurrence of $\mathrm{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- If now the number of keys in $v$ is below $a - 1$ perform Rebalance′$(v)$.

# Delete

Rebalance$'(v)$:

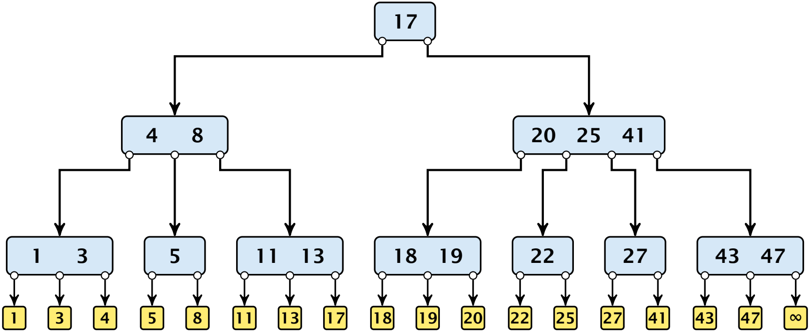
- If there is a neighbour of $v$ that has at least $a$ keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- If not: merge $v$ with one of its neighbours.
- The merged node contains at most $(a-2)+(a-1)+1$ keys, and has therefore at most $2a-1 \le b$ successors.
- Then rebalance the parent.
- During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

# Delete

Animation for deleting in an $(a, b)$-tree is only available in the lecture version of the slides.

# $(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:




First make it into an internal search tree by moving the satellite-data from the leaves to internal nodes. Add dummy leaves.

# $(2,4)$-trees and red black trees

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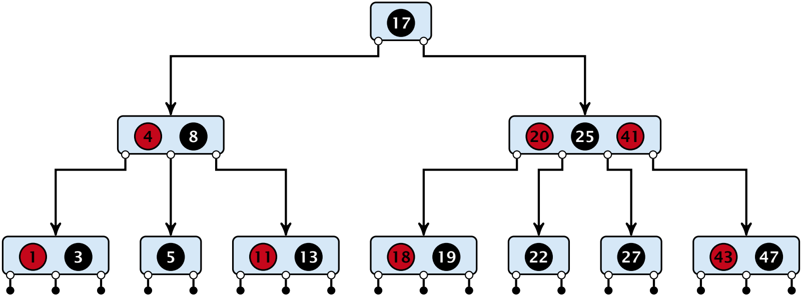


Then, color one key in each internal node $v$ black. If $v$ contains $3$ keys you need to select the middle key otherwise choose a black key arbitrarily. The other keys are colored red.

# $(2,4)$-trees and red black trees

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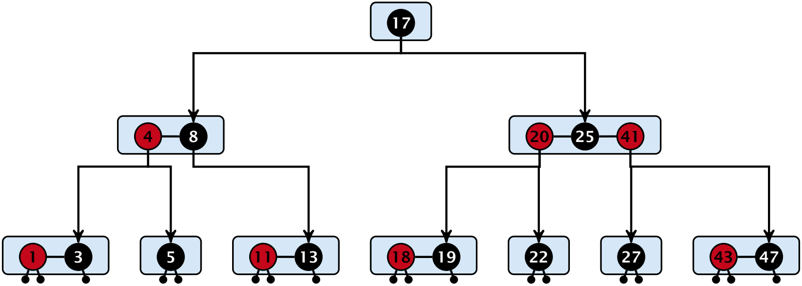
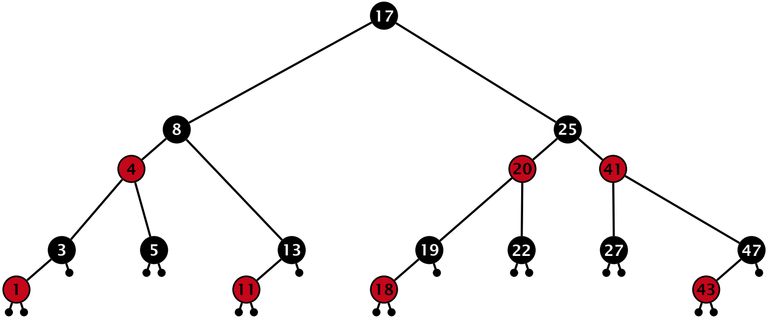



Re-attach the pointers to individual keys. A pointer that is between two keys is attached as a child of the red key. The incoming pointer, points to the black key.

# $(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:




Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same $(2,4)$-tree.

# Augmenting Data Structures


**Bibliography**

[MS08] Kurt Mehlhorn, Peter Sanders:
*Algorithms and Data Structures — The Basic Toolbox*,
Springer, 2008

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
*Introduction to algorithms (3rd ed.)*,
MIT Press and McGraw-Hill, 2009


A description of B-trees (a specific variant of $(a, b)$-trees) can be found in Chapter 18 of [CLRS90]. Chapter 7.2 of [MS08] discusses $(a, b)$-trees as discussed in the lecture.