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Winter term 2016 / 2017
Assignment 4
November 14, 2016

Efficient Algorithms and Data Structures I

*Deadline: November 21, 10:15 am in the **Efficient Algorithms** mailbox.*

Homework 1 (5 Points)

Solve the following recurrence relations using generating functions:

1. $a_n = a_{n-1} + 2^{n-1}$ for $n \geq 1$ with $a_0 = 2$.
2. $a_n = -2a_{n-1} - a_{n-2}$ for $n \geq 2$ with $a_0 = 1$ and $a_1 = -1$.

Homework 2 (4 Points)

Solve the following recurrence

$$\begin{aligned} f_0 &= 2 \\ f_1 &= 4 \\ f_n &= (f_{n-1})^{\log(f_{n-2})} \qquad \text{for } n \geq 2 \ . \end{aligned}$$

Recall that $\log n$ denotes the binary logarithm.

Homework 3 (7 Points)

The mathematical collector Anita Binaros updates her collection of full rooted binary search trees. In these trees, a node is either of leaf or it has two children. While admiring her collection, she muses about the number of full binary search trees with $n+1$ leaves ....

1. Let b_n be the number of full rooted binary search trees with $n+1$ leaves (conveniently named $0, 1, \ldots, n$). We set $b_0 = 1$. Show that for $n \geq 1$,

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k} \ .$$

2. Let $B(z) = \sum_{n \geq 0} b_n z^n$. Show that $B(z) = zB(z)^2 + 1$.
3. Show that $B(z) = \frac{1-\sqrt{1-4z}}{2z}$ is a solution to $B(z)$. We will only consider this solution in the remainder of the exercise.
4. Show that the number of full rooted binary search trees with $n+1$ leaves is $\frac{1}{n+1}\binom{2n}{n}$.

   Hint: Use the equality

$$\sqrt{1-4z} = -2\left(-\frac{1}{2} + \sum_{n \geq 1} \frac{1}{n}\binom{2(n-1)}{n-1} z^n\right) \ .$$

Homework 4 (4 Points)

The *depth* of a node v in a binary search tree is the number of edges on the shortest path from v to the root of the tree.
Show that there exists a binary search tree with n nodes with height in $\omega(\log(n))$ and average depth in $\mathcal{O}(\log(n))$.

Tutorial Exercise 1

Solve the following mutual recursion using generating functions:

$$\begin{aligned} a_n &= a_{n-1} + 4b_{n-1} && \text{with } a_0 = 1 \\ b_n &= a_{n-1} + b_{n-1} && \text{with } b_0 = 0\ . \end{aligned}$$

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A generating function is a device somewhat
similar to a bag.  Instead of carrying many
little objects detachedly, which could be em-
barrassing, we put them all in a bag, and then
we have only one object to carry, the bag.
                                   - G. Pòlya
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