Part II

Linear Programming

Brewery Problem

Brewery brews ale and beer.

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

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- only brew ale: 34 barrels of ale $\Longrightarrow$ 442 €
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- Choose the variables in such a way that the objective function (profit) is maximized.
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Standard Form LPs

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Standard Form LPs

LP in standard form:

- input: numbers a_{ij}, c_j, b_i
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Standard Form LPs

Original LP

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Standard Form

Add a slack variable to every constraint.

$$
\begin{array}{lrcrcccl}
\max & 13a & + & 23b & & & & \\
\text{s.t.} & 5a & + & 15b & + s_c & & & = 480 \\
 & 4a & + & 4b & & + s_h & & = 160 \\
 & 35a & + & 20b & & & + s_m & = 1190 \\
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Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

standard maximization form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

standard minimization form

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Standard Form LPs

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Standard Form

Add a slack variable to every constraint.

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Standard Form LPs

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$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

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$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

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Standard Form LPs

Original LP

$$\begin{array}{lrcrl} \max & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a, b & \geq 0 \end{array}$$

Standard Form

Add a slack variable to every constraint.

$$\begin{array}{lrcrcccccl} \max & 13a & + & 23b & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & = 480 \\ & 4a & + & 4b & & & + & s_h & & = 160 \\ & 35a & + & 20b & & & & & + \; s_m & = 1190 \\ & a & , & b & , & s_c & , & s_h & , \; s_m & \geq 0 \end{array}$$

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Observations:

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Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP?
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Input size:

- n number of variables, m constraints, L number of bits to encode the input

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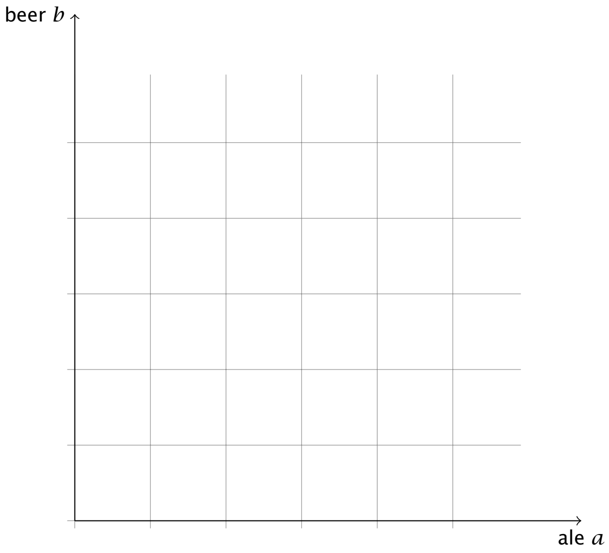
Geometry of Linear Programming



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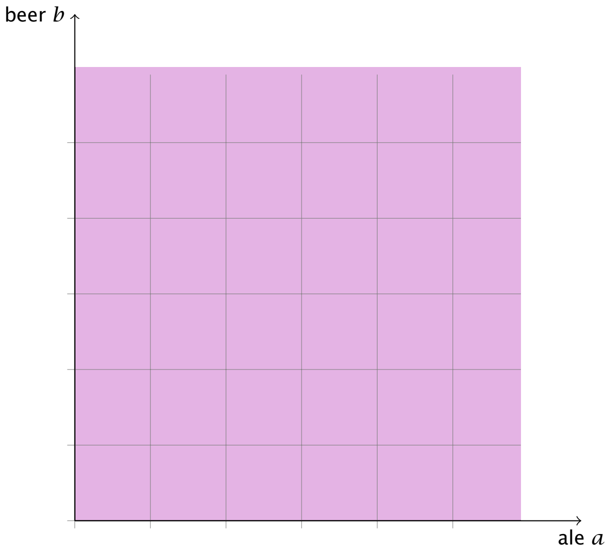

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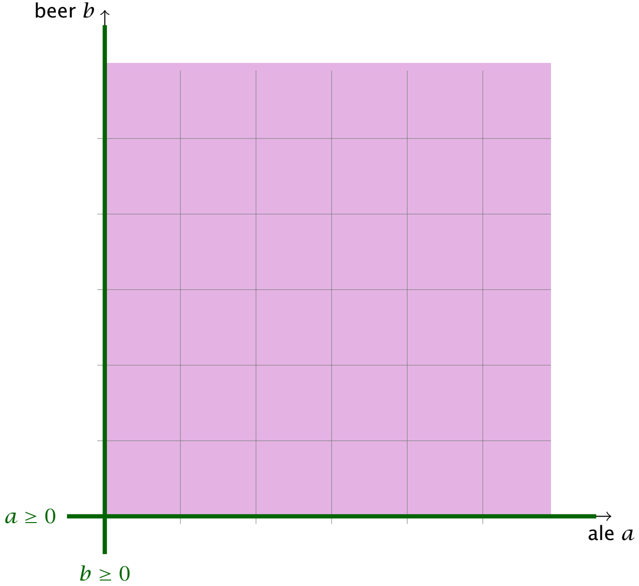

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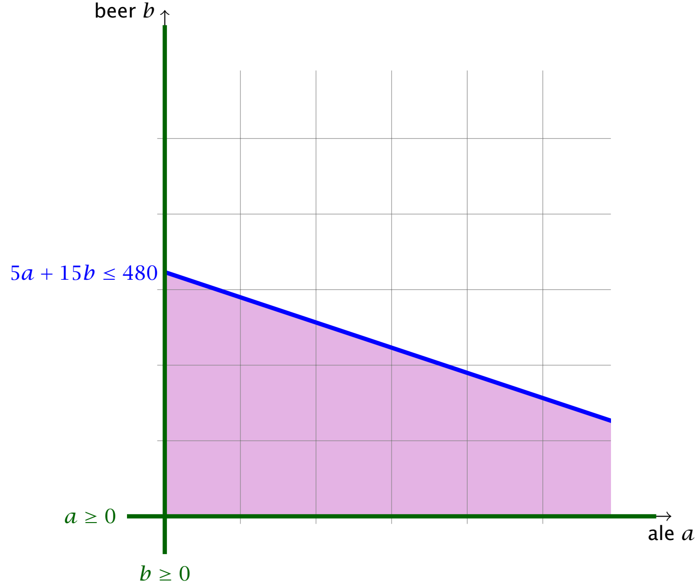

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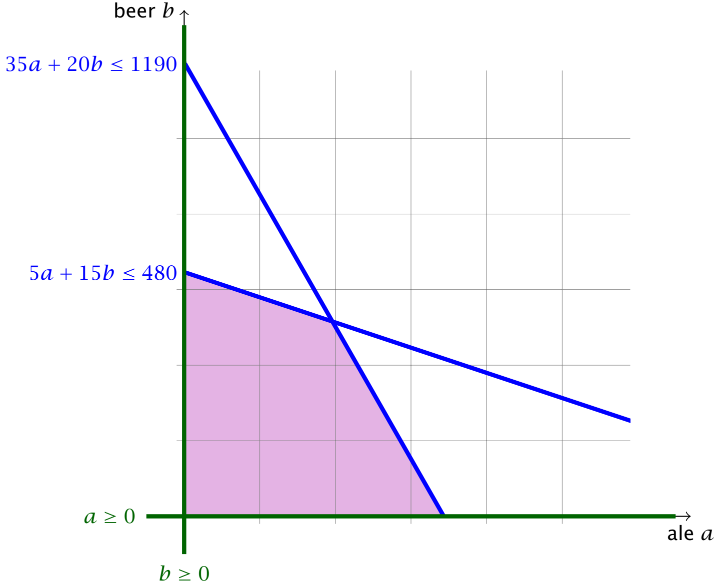

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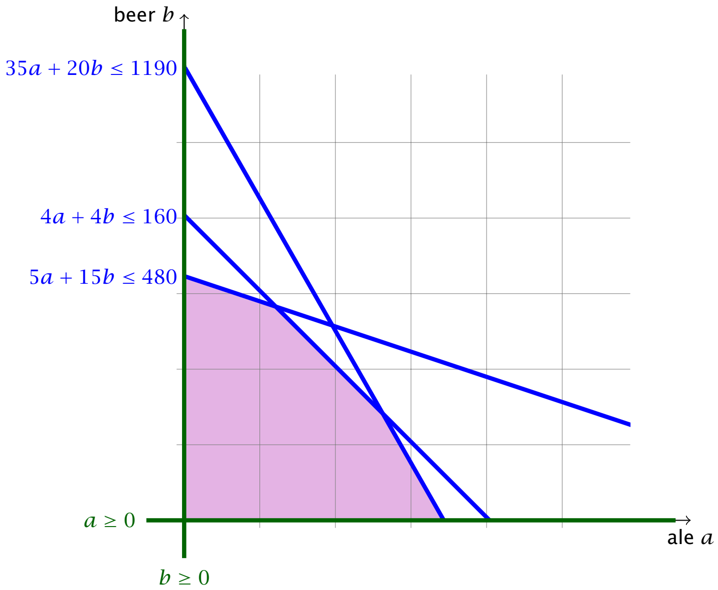

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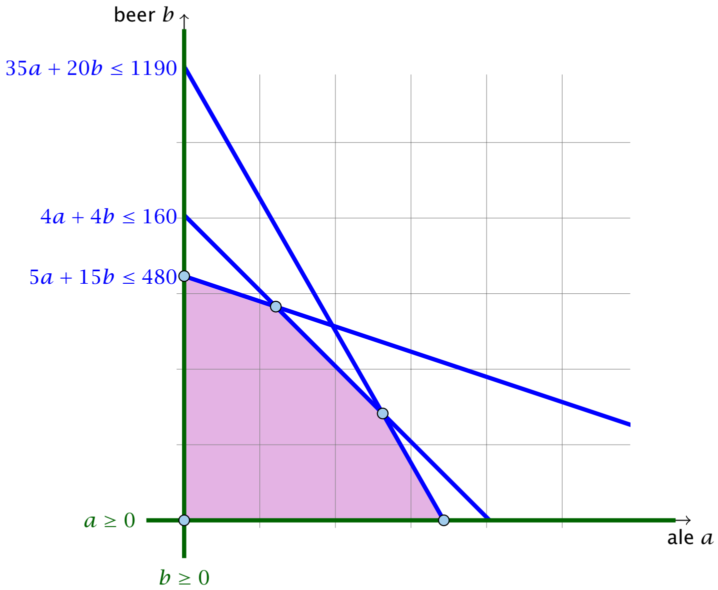

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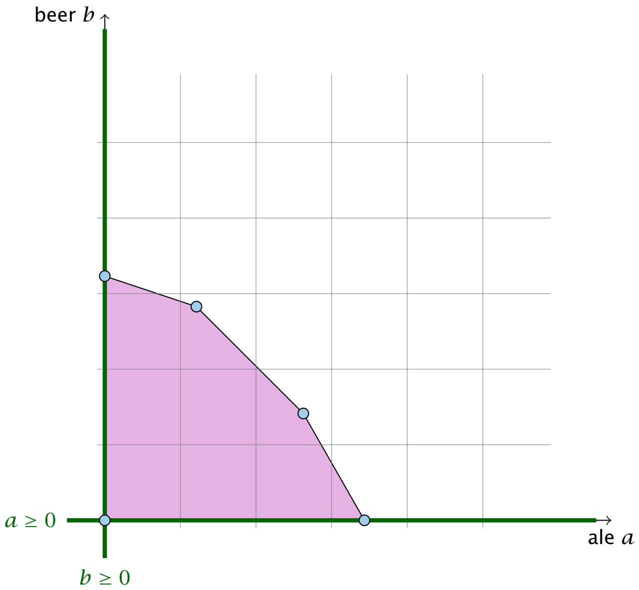

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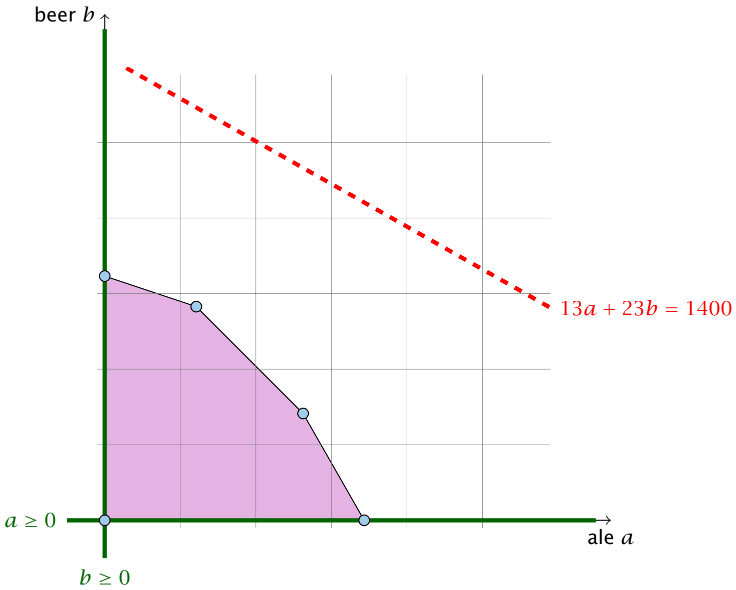

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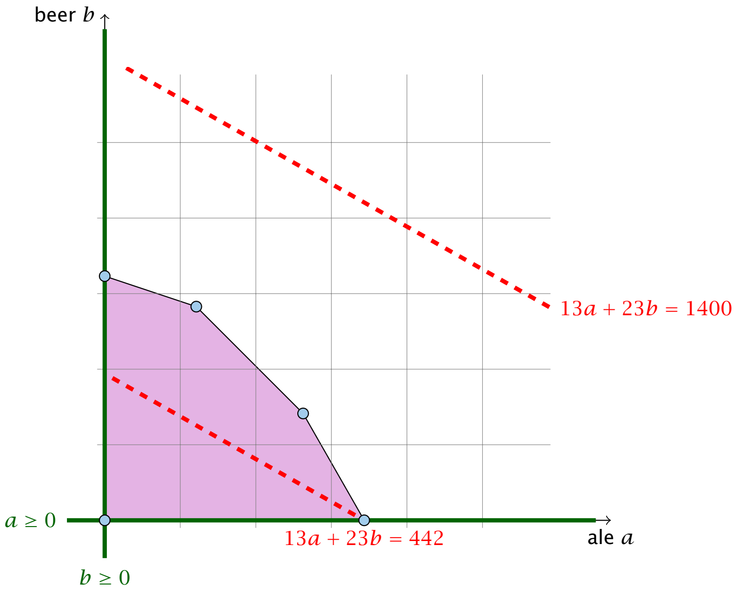

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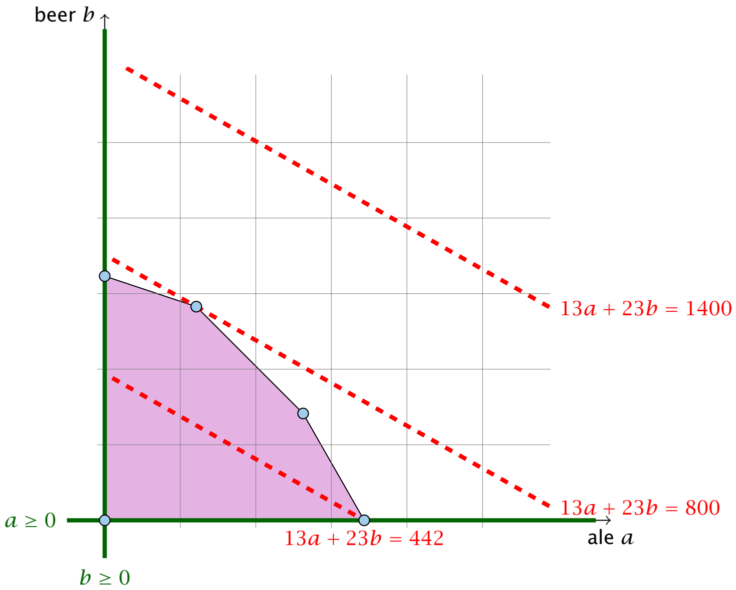

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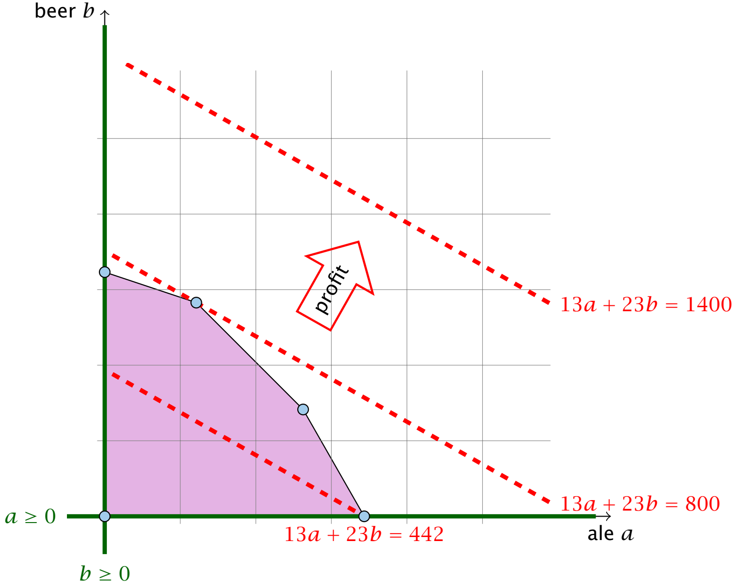

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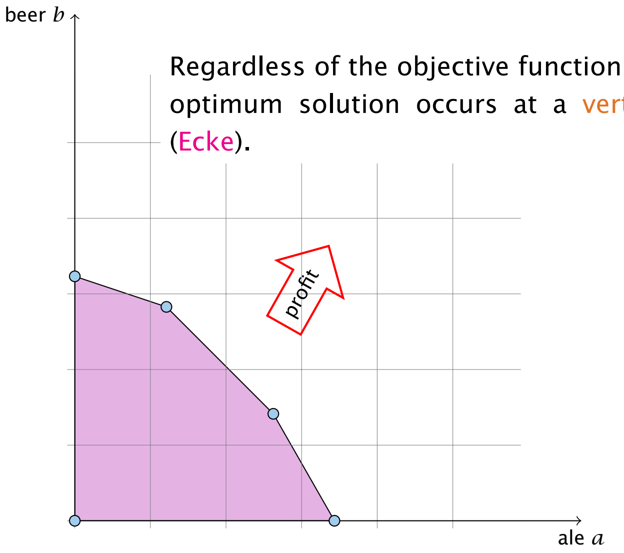

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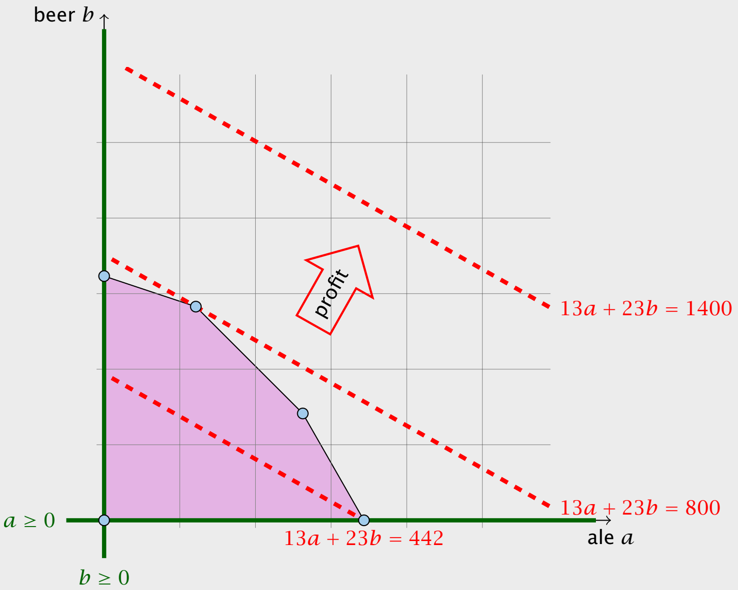

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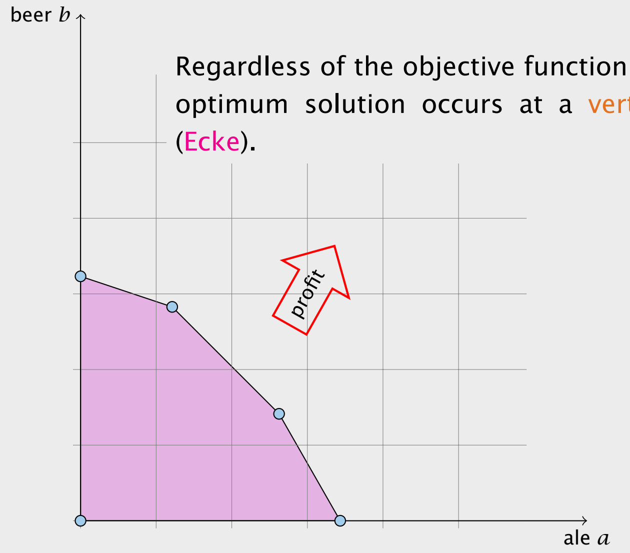

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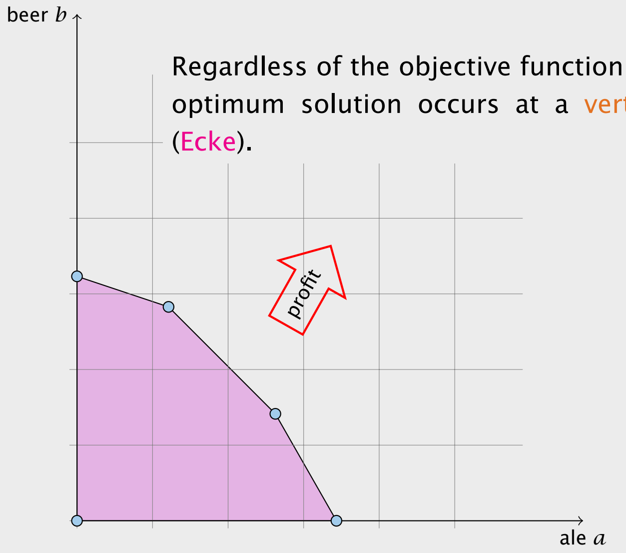

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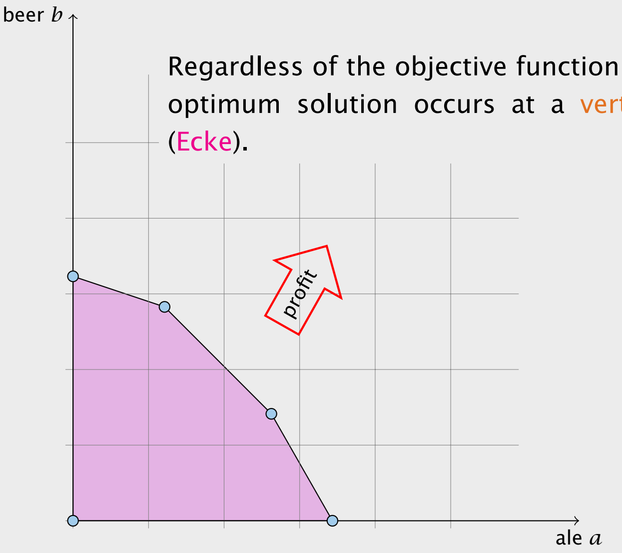

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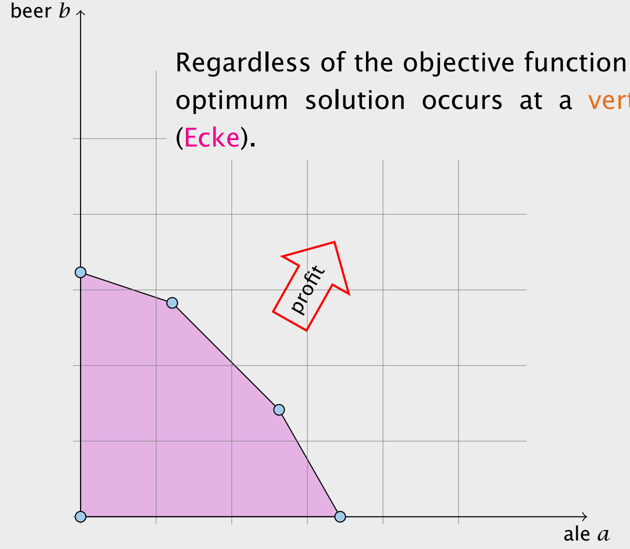

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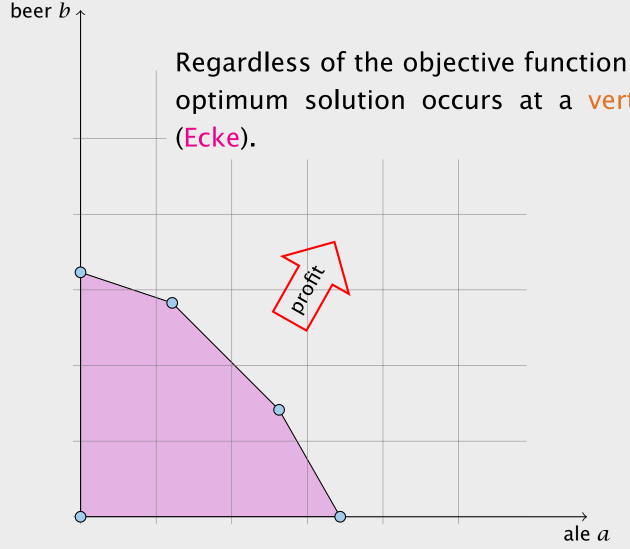

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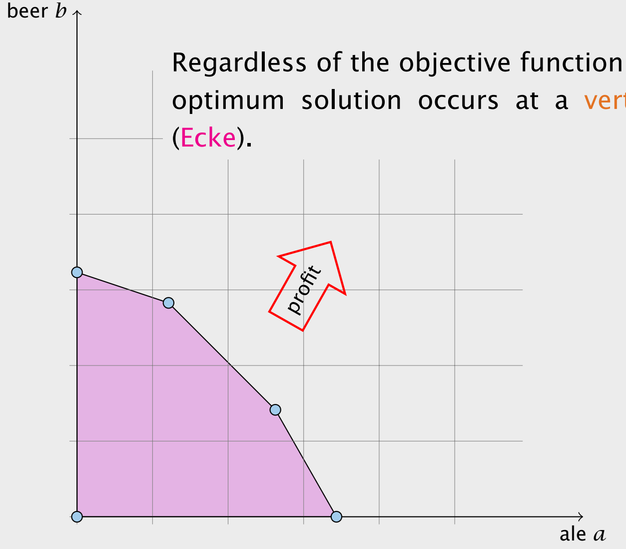

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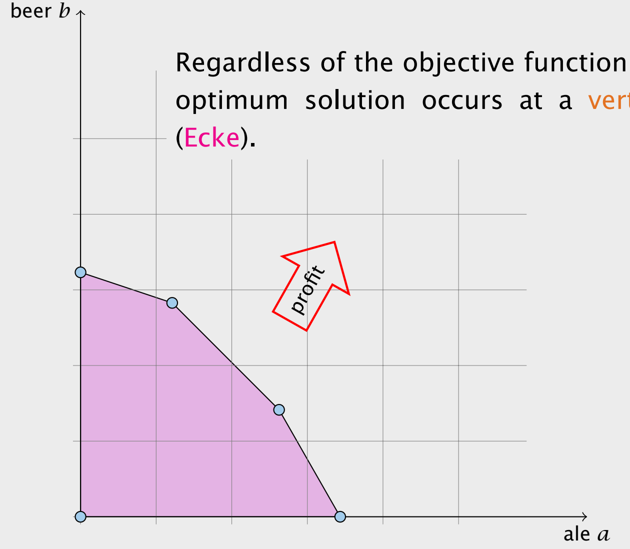

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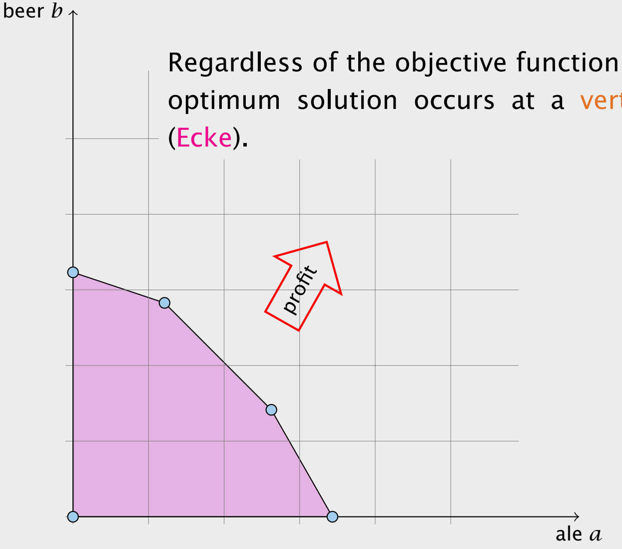

Definition 2
Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- linear combination if $\lambda_i \in \mathbb{R}$.
- affine combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- convex combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
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- a linear subspace if it is closed under linear combinations.
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Note that an affine subspace is **not** a vector space

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Given a set $X \subseteq \mathbb{R}^n$.

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If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ convex then also

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The dimension $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

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Definition 16
Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

Definition 17
Let $P \subseteq \mathbb{R}^n$.

- a face v is a vertex of P if $\{v\}$ is a face of P.
- a face e is an edge of P if e is a face and $\dim(e) = 1$.
- a face F is a facet of P if F is a face and $\dim(F) = \dim(P) - 1$.

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P is a bounded polyhedron iff P is a polytop.

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Equivalent definition for vertex:

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Given polyhedron P. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

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The feasible region of an LP is a Polyhedron.

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- suppose x is optimal solution that is not extreme point
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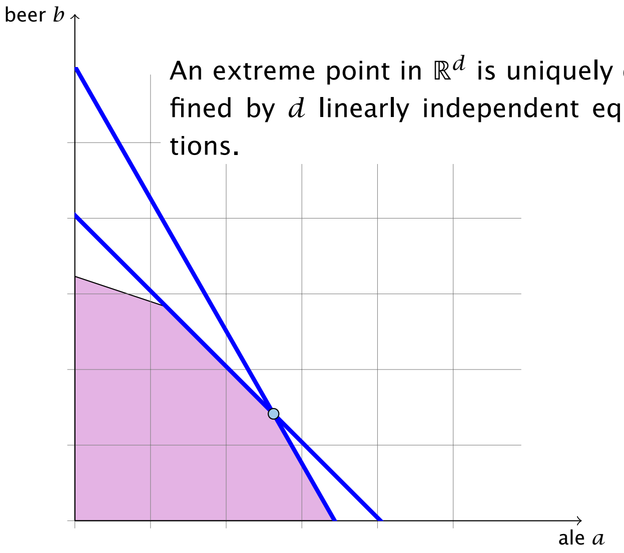
Algebraic View



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Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 22

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point iff A_B has linearly independent columns.

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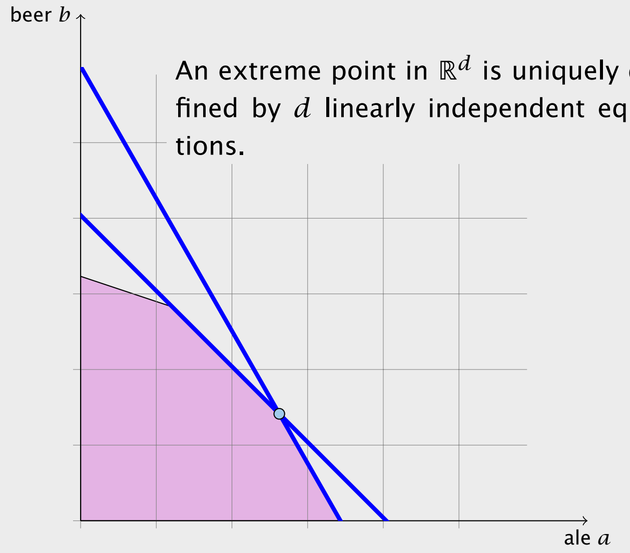

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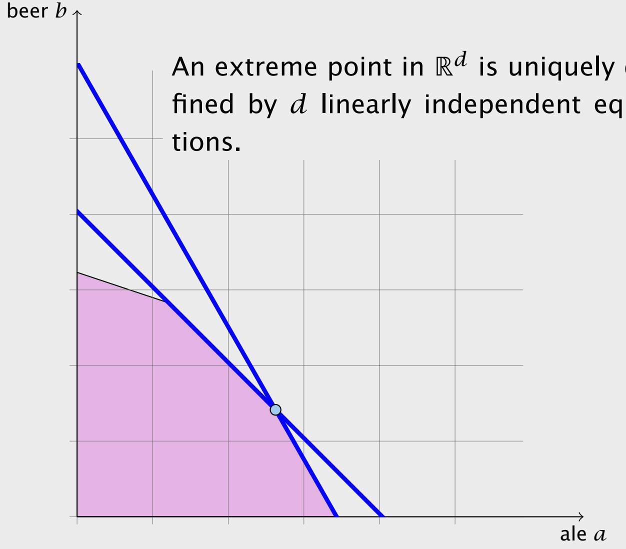

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Notation

Suppose $B \subseteq \{1 \ldots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

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*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

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For an LP we can assume wlog. that the matrix A has full row-rank. This means $\operatorname{rank}(A) = m$.

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$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i \text{, for suitable } \lambda_i$$

C1 if now $b_1 = \sum_{i=2}^m \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous

C2 if $b_1 \neq \sum_{i=2}^m \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \dots, A_m$ we have

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Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

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Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- A_B is non-singular
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where $N = \{1, \dots, n\} \setminus B$.

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A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.

Fact:
In a BFS at least n constraints are fulfilled with equality.

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For a general LP ($\max\{c^T x \mid Ax \le b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

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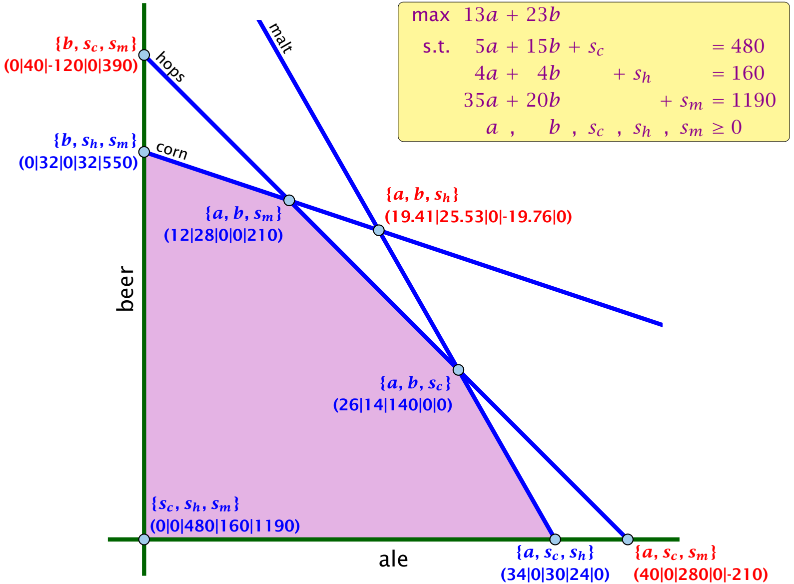
Algebraic View



$$\begin{array}{lrcrcrcrcrcl}
\max & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
& 4a & + & 4b & & & + & s_h & & & = & 160 \\
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Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:

- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

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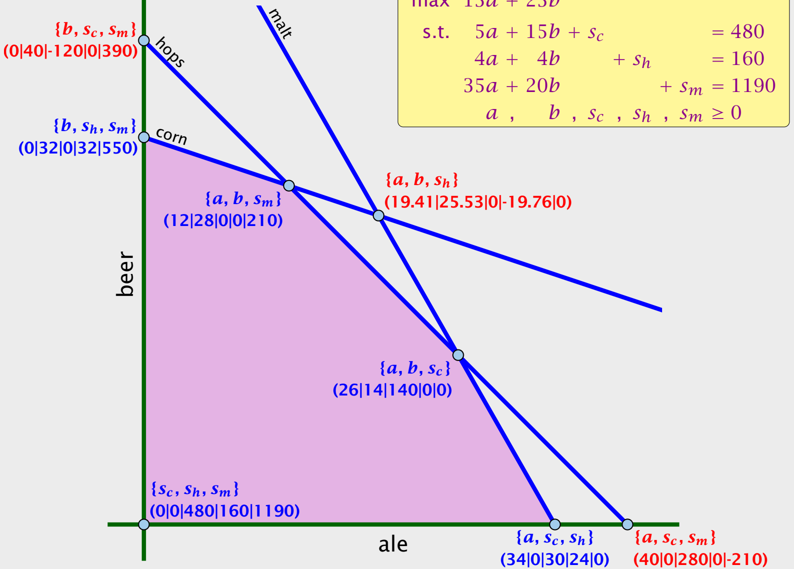

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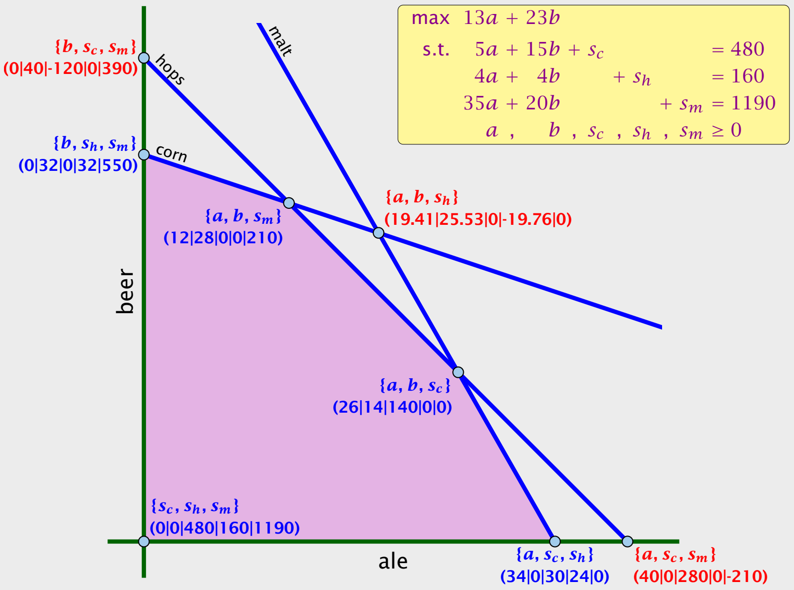

Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \mathrm{poly}(n,m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?

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4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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$$
\begin{array}{lrcl}
\max Z & & & \\
 & 13a + 23b - Z & = & 0 \\
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basis = $\{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$
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$s_h = 160$
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 & 5a & + & 15b & + & s_c & & & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$A = B = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

Pivoting Step

$$\begin{array}{l}\max\ Z\\ \begin{array}{rcrcrcrcrcrcl} 13a &+& 23b & & & & & & & -& Z &=& 0\\ 5a &+& 15b &+& s_c & & & & & & &=& 480\\ 4a &+& 4b & & &+& s_h & & & & &=& 160\\ 35a &+& 20b & & & & &+& s_m & & &=& 1190\\ a &,& b &,& s_c &,& s_h &,& s_m & & &\geq& 0\end{array}\end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\}\\ a = b = 0\\ Z = 0\\ s_c = 480\\ s_h = 160\\ s_m = 1190\end{array}$$

4 Simplex Algorithm

$$\begin{array}{llcrcrcrcrcrcl} \max & 13a &+& 23b & & & & & & & & \\ \text{s.t.} & 5a &+& 15b &+& s_c & & & & &=& 480\\ & 4a &+& 4b & & &+& s_h & & &=& 160\\ & 35a &+& 20b & & & & &+& s_m &=& 1190\\ & a &,& b &,& s_c &,& s_h &,& s_m &\geq& 0\end{array}$$

$$\begin{array}{l}\max\ Z\\ \begin{array}{rcrcrcrcrcrcl} 13a &+& 23b & & & & & & & -& Z &=& 0\\ 5a &+& 15b &+& s_c & & & & & & &=& 480\\ 4a &+& 4b & & &+& s_h & & & & &=& 160\\ 35a &+& 20b & & & & &+& s_m & & &=& 1190\\ a &,& b &,& s_c &,& s_h &,& s_m & & &\geq& 0\end{array}\end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\}\\ A = B = 0\\ Z = 0\\ s_c = 480\\ s_h = 160\\ s_m = 1190\end{array}$$

Pivoting Step

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad\qquad - Z = 0 \\
\quad\; 5a + 15b + s_c \qquad\qquad\qquad\quad = 480 \\
\quad\; 4a + \;\;4b \qquad + s_h \qquad\qquad = 160 \\
\quad 35a + 20b \qquad\qquad\quad + s_m \quad = 1190 \\
\quad\;\; a\;, \quad b\;,\; s_c\;,\; s_h\;,\; s_m \qquad\quad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a \;= b = 0 \\
Z \;= 0 \\[6pt]
s_c \;= 480 \\
s_h \;= 160 \\
s_m = 1190
\end{array}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

4 Simplex Algorithm

$$
\begin{array}{lrcl}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & & = 480 \\
 & 4a + \;\;4b \qquad + s_h & & = 160 \\
 & 35a + 20b \qquad\qquad + s_m & & = 1190 \\
 & a\;, \quad b\;,\; s_c\;,\; s_h\;,\; s_m & & \geq 0
\end{array}
$$

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad\qquad - Z = 0 \\
\quad\; 5a + 15b + s_c \qquad\qquad\qquad\quad = 480 \\
\quad\; 4a + \;\;4b \qquad + s_h \qquad\qquad = 160 \\
\quad 35a + 20b \qquad\qquad\quad + s_m \quad = 1190 \\
\quad\;\; a\;, \quad b\;,\; s_c\;,\; s_h\;,\; s_m \qquad\quad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
A \;= B = 0 \\
Z \;= 0 \\[6pt]
s_c \;= 480 \\
s_h \;= 160 \\
s_m = 1190
\end{array}
$$

Pivoting Step

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad\quad - Z = 0 \\
\quad\ 5a + 15b + s_c \qquad\qquad\quad = 480 \\
\quad\ 4a + \ \ 4b \qquad + s_h \qquad\quad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \ = 1190 \\
\quad\ a\ ,\ \ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

4 Simplex Algorithm

$$
\begin{array}{rl}
\max & 13a + 23b \\
\text{s.t.} & 5a + 15b + s_c = 480 \\
& 4a + 4b + s_h = 160 \\
& 35a + 20b + s_m = 1190 \\
& a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \geq 0
\end{array}
$$

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad\quad - Z = 0 \\
\quad\ 5a + 15b + s_c \qquad\qquad\quad = 480 \\
\quad\ 4a + \ \ 4b \qquad + s_h \qquad\quad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \ = 1190 \\
\quad\ a\ ,\ \ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
A = B = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

Pivoting Step

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

4 Simplex Algorithm

$$
\begin{array}{llcrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

$$\begin{array}{llllll}
\max Z & & & & & \\
13a + 23b & & & & -Z & = 0 \\
5a + 15b & + s_c & & & & = 480 \\
4a + 4b & & + s_h & & & = 160 \\
35a + 20b & & & + s_m & & = 1190 \\
a\ ,\quad b & ,\ s_c & ,\ s_h & ,\ s_m & & \geq 0
\end{array}$$

$\text{basis} = \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

4 Simplex Algorithm

$$\begin{array}{lllllll}
\max & 13a + 23b & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = 480 \\
& 4a + 4b & & + s_h & & = 160 \\
& 35a + 20b & & & + s_m & = 1190 \\
& a\ ,\quad b & ,\ s_c & ,\ s_h & ,\ s_m & \geq 0
\end{array}$$

$$\begin{array}{llllll}
\max Z & & & & & \\
13a + 23b & & & & -Z & = 0 \\
5a + 15b & + s_c & & & & = 480 \\
4a + 4b & & + s_h & & & = 160 \\
35a + 20b & & & + s_m & & = 1190 \\
a\ ,\quad b & ,\ s_c & ,\ s_h & ,\ s_m & & \geq 0
\end{array}$$

$\text{basis} = \{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

4 Simplex Algorithm

$$
\begin{array}{ll}
\text{max} & 13a + 23b \\
\text{s.t.} &
\begin{array}{rcrcrcrcrcl}
5a & + & 15b & + & s_c & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

$$
\begin{array}{l}
\max\ Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

$$
\begin{array}{l}
\max\ Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{l} \max Z \\ \quad 13a + 23\boldsymbol{b} \qquad\qquad - Z = 0 \\ \quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad = 480 \\ \quad 4a + 4\boldsymbol{b} \quad + s_h \qquad = 160 \\ \quad 35a + 20\boldsymbol{b} \qquad + s_m \quad = 1190 \\ \quad a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \quad \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- Choose variable with coefficient > 0 as entering variable.

Pivoting Step

$$\begin{array}{l} \max Z \\ \quad 13a + 23b \qquad\qquad - Z = 0 \\ \quad 5a + 15b + s_c \qquad\qquad = 480 \\ \quad 4a + 4b \quad + s_h \qquad = 160 \\ \quad 35a + 20b \qquad + s_m \quad = 1190 \\ \quad a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \quad \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$
\begin{array}{l}
\max\ Z \\
\quad 13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad\quad = 480 \\
\quad 4a + 4\boldsymbol{b} \qquad + s_h \qquad\quad = 160 \\
\quad 35a + 20\boldsymbol{b} \qquad\qquad + s_m \quad = 1190 \\
\quad a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \ge 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

Pivoting Step

$$
\begin{array}{l}
\max\ Z \\
\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15b + s_c \qquad\qquad\quad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad\quad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \quad = 1190 \\
\quad a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \ge 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{l} \max Z \\ \quad 13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\ \quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad\quad = 480 \\ \quad 4a + 4\boldsymbol{b} \quad + s_h \qquad\quad = 160 \\ \quad 35a + 20\boldsymbol{b} \qquad + s_m \quad = 1190 \\ \quad a \;,\; \boldsymbol{b} \;,\; s_c \;,\; s_h \;,\; s_m \quad \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

Pivoting Step

$$\begin{array}{l} \max Z \\ \quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\ \quad 5a + 15b + s_c \qquad\qquad\quad = 480 \\ \quad 4a + 4b \quad + s_h \qquad\quad = 160 \\ \quad 35a + 20b \qquad + s_m \quad = 1190 \\ \quad a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m \quad \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{ll} \max Z & \\ 13a + 23b & - Z = 0 \\ 5a + 15\mathbf{b} + s_c & = 480 \\ 4a + 4b + s_h & = 160 \\ 35a + 20b + s_m & = 1190 \\ a, b, s_c, s_h, s_m & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

Pivoting Step

$$\begin{array}{ll} \max Z & \\ 13a + 23b & - Z = 0 \\ 5a + 15b + s_c & = 480 \\ 4a + 4b + s_h & = 160 \\ 35a + 20b + s_m & = 1190 \\ a, b, s_c, s_h, s_m & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{l}\max Z\\ \quad 13a + 23b \qquad\qquad\qquad - Z = 0\\ \quad 5a + 15\mathbf{b} + \mathbf{s_c} \qquad\qquad = 480\\ \quad 4a + \;\;4b \qquad + s_h \qquad\quad = 160\\ \quad 35a + 20b \qquad\qquad + s_m \quad = 1190\\ \quad a\;,\quad b\;,\; s_c\;,\; s_h\;,\; s_m \quad \geq 0\end{array}$$

$$\begin{array}{l}\text{basis} = \{s_c, s_h, s_m\}\\ a = b = 0\\ Z = 0\\ s_c = 480\\ s_h = 160\\ s_m = 1190\end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

Pivoting Step

$$\begin{array}{l}\max Z\\ \quad 13a + 23b \qquad\qquad\qquad - Z = 0\\ \quad 5a + 15b + s_c \qquad\qquad = 480\\ \quad 4a + \;\;4b \qquad + s_h \qquad\quad = 160\\ \quad 35a + 20b \qquad\qquad + s_m \quad = 1190\\ \quad a\;,\quad b\;,\; s_c\;,\; s_h\;,\; s_m \quad \geq 0\end{array}$$

$$\begin{array}{l}\text{basis} = \{s_c, s_h, s_m\}\\ a = b = 0\\ Z = 0\\ s_c = 480\\ s_h = 160\\ s_m = 1190\end{array}$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15\mathbf{b} & + & \mathbf{s_c} & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$\begin{array}{l}
\max Z \\
13a + 23\boldsymbol{b} \qquad\qquad - Z = 0 \\
5a + 15\boldsymbol{b} + s_c \qquad\qquad = 480 \\
4a + 4\boldsymbol{b} \qquad + s_h \qquad = 160 \\
35a + 20\boldsymbol{b} \qquad\qquad + s_m = 1190 \\
a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \ \geq 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l}
\max Z \\
13a + 23b \qquad\qquad - Z = 0 \\
5a + 15\boldsymbol{b} + \boldsymbol{s_c} \qquad\qquad = 480 \\
4a + 4b \qquad + s_h \qquad = 160 \\
35a + 20b \qquad\qquad + s_m = 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \ \geq 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$
\begin{array}{l}
\max Z \\
13a + 23b \qquad\qquad - Z = 0 \\
5a + 15b + s_c \qquad\qquad = 480 \\
4a + 4b \qquad + s_h \qquad = 160 \\
35a + 20b \qquad\qquad + s_m = 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \ge 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\frac{16}{3}a - \frac{23}{15}s_c \qquad\qquad - Z = -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c \qquad\qquad = 32 \\
\frac{8}{3}a - \frac{4}{15}s_c + s_h \qquad\qquad = 32 \\
\frac{85}{3}a - \frac{4}{3}s_c \qquad + s_m \qquad = 550 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \ge 0
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

$$
\begin{array}{l}
\max Z \\
13a + 23b \qquad\qquad - Z = 0 \\
5a + 15b + s_c \qquad\qquad = 480 \\
4a + 4b \qquad + s_h \qquad = 160 \\
35a + 20b \qquad\qquad + s_m = 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \ge 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \ge 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$
\begin{array}{l}
\max Z \\
\begin{alignedat}{6}
\tfrac{16}{3}a & & & {}-{} & \tfrac{23}{15}s_c & & & & & {}-Z & {}={} & -736 \\
\tfrac{1}{3}a & {}+{} & b & {}+{} & \tfrac{1}{15}s_c & & & & & & {}={} & 32 \\
\tfrac{8}{3}a & & & {}-{} & \tfrac{4}{15}s_c & {}+{} & s_h & & & & {}={} & 32 \\
\tfrac{85}{3}a & & & {}-{} & \tfrac{4}{3}s_c & & & {}+{} & s_m & & {}={} & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & {}\geq{} & 0
\end{alignedat}
\end{array}
$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{alignedat}{6}
13a & {}+{} & 23\boldsymbol{b} & & & & & & & {}-Z & {}={} & 0 \\
5a & {}+{} & 15\boldsymbol{b} & {}+{} & s_c & & & & & & {}={} & 480 \\
4a & {}+{} & 4\boldsymbol{b} & & & {}+{} & s_h & & & & {}={} & 160 \\
35a & {}+{} & 20\boldsymbol{b} & & & & & {}+{} & s_m & & {}={} & 1190 \\
a & , & \boldsymbol{b} & , & s_c & , & s_h & , & s_m & & {}\geq{} & 0
\end{alignedat}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{alignedat}{6}
\tfrac{16}{3}a & & & {}-{} & \tfrac{23}{15}s_c & & & & & {}-Z & {}={} & -736 \\
\tfrac{1}{3}a & {}+{} & b & {}+{} & \tfrac{1}{15}s_c & & & & & & {}={} & 32 \\
\tfrac{8}{3}a & & & {}-{} & \tfrac{4}{15}s_c & {}+{} & s_h & & & & {}={} & 32 \\
\tfrac{85}{3}a & & & {}-{} & \tfrac{4}{3}s_c & & & {}+{} & s_m & & {}={} & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & {}\geq{} & 0
\end{alignedat}
\end{array}
$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{aligned}
\tfrac{16}{3}\boldsymbol{a} \qquad\quad - \tfrac{23}{15}s_c \qquad\qquad\qquad - Z &= -736 \\
\tfrac{1}{3}\boldsymbol{a} + b + \tfrac{1}{15}s_c \qquad\qquad\qquad\qquad &= 32 \\
\tfrac{8}{3}\boldsymbol{a} \qquad\quad - \tfrac{4}{15}s_c + s_h \qquad\qquad\qquad &= 32 \\
\tfrac{85}{3}\boldsymbol{a} \qquad\quad - \tfrac{4}{3}s_c \qquad + s_m \qquad\quad &= 550 \\
\boldsymbol{a}\ ,\ b\ ,\quad s_c\ ,\ s_h\ ,\ s_m \qquad\quad &\geq 0
\end{aligned}
\end{array}
$$

$$
\begin{aligned}
\text{basis} &= \{b, s_h, s_m\} \\
a &= s_c = 0 \\
Z &= 736 \\
b &= 32 \\
s_h &= 32 \\
s_m &= 550
\end{aligned}
$$

Choose variable a to bring into basis.

$$
\begin{array}{l}
\max Z \\
\begin{aligned}
13a + 23\boldsymbol{b} \qquad\qquad\qquad\qquad - Z &= 0 \\
5a + 15\boldsymbol{b} + s_c \qquad\qquad\qquad &= 480 \\
4a + \ \ 4\boldsymbol{b} \qquad + s_h \qquad\qquad &= 160 \\
35a + 20\boldsymbol{b} \qquad\qquad + s_m \qquad &= 1190 \\
a\ ,\ \ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad &\geq 0
\end{aligned}
\end{array}
$$

$$
\begin{aligned}
\text{basis} &= \{s_c, s_h, s_m\} \\
a &= b = 0 \\
Z &= 0 \\
s_c &= 480 \\
s_h &= 160 \\
s_m &= 1190
\end{aligned}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{aligned}
\tfrac{16}{3}a \qquad\quad - \tfrac{23}{15}s_c \qquad\qquad\qquad - Z &= -736 \\
\tfrac{1}{3}a + b + \tfrac{1}{15}s_c \qquad\qquad\qquad\qquad &= 32 \\
\tfrac{8}{3}a \qquad\quad - \tfrac{4}{15}s_c + s_h \qquad\qquad\qquad &= 32 \\
\tfrac{85}{3}a \qquad\quad - \tfrac{4}{3}s_c \qquad + s_m \qquad\quad &= 550 \\
a\ ,\ b\ ,\quad s_c\ ,\ s_h\ ,\ s_m \qquad\quad &\geq 0
\end{aligned}
\end{array}
$$

$$
\begin{aligned}
\text{basis} &= \{b, s_h, s_m\} \\
a &= s_c = 0 \\
Z &= 736 \\
b &= 32 \\
s_h &= 32 \\
s_m &= 550
\end{aligned}
$$

$$\max Z$$
$$\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

$$\max Z$$
$$\begin{array}{rcrcrcrcrcrcl}
13a & + & 23\boldsymbol{b} & & & & & & & - & Z & = & 0 \\
5a & + & 15\boldsymbol{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4\boldsymbol{b} & & & + & s_h & & & & & = & 160 \\
35a & + & 20\boldsymbol{b} & & & & & + & s_m & & & = & 1190 \\
a & , & \boldsymbol{b} & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\max Z$$
$$\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\mathbf{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcl}
13a & + & 23\mathbf{b} & & & & & & - & Z = & 0 \\
5a & + & 15\mathbf{b} & + & s_c & & & & & = & 480 \\
4a & + & 4\mathbf{b} & & & + & s_h & & & = & 160 \\
35a & + & 20\mathbf{b} & & & & & + s_m & & = & 1190 \\
a & , & \mathbf{b} & , & s_c & , & s_h & , s_m & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3 \cdot 32}/_{8}, {}^{3 \cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} = \{a, b, s_m\} \\
s_c = s_h = 0 \\
Z = 800 \\
b = 28 \\
a = 12 \\
s_m = 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{a, b, s_m\} \\
s_c = s_h = 0 \\
Z = 800 \\
b = 28 \\
a = 12 \\
s_m = 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h,\ s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3\cdot32/8, 3\cdot550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

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$$\begin{array}{l}
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\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}$$

$$\begin{array}{rcl}
\text{basis} & = & \{b, s_h, s_m\} \\
a & = & s_c = 0 \\
Z & = & 736 \\
b & = & 32 \\
s_h & = & 32 \\
s_m & = & 550
\end{array}$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3\cdot 32/8, 3\cdot 550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}$$

$$\begin{array}{rcl}
\text{basis} & = & \{a, b, s_m\} \\
s_c & = & s_h = 0 \\
Z & = & 800 \\
b & = & 28 \\
a & = & 12 \\
s_m & = & 210
\end{array}$$

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- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
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$$
\begin{array}{l}
\max Z \\
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\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
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\end{array}
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Computing $\min\{3 \cdot 32, 3\cdot32/8, 3\cdot550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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Solution is optimal:

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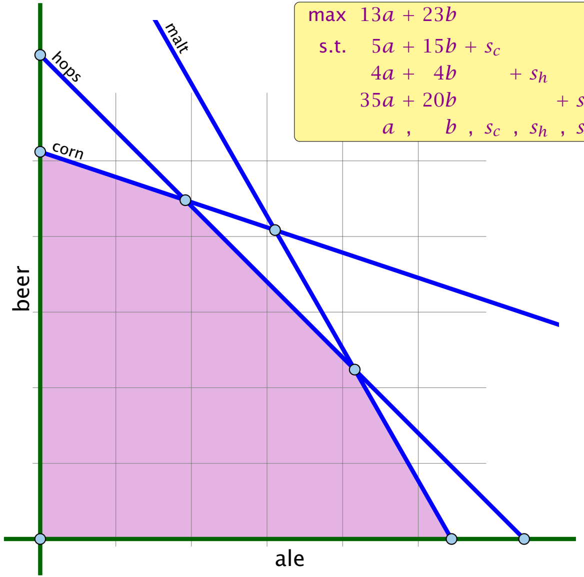
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Geometric View of Pivoting

$$\begin{array}{lrcrcrcrcrcl} \text{max} & 13a & + & 23b & & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\ & 4a & + & 4b & & & + & s_h & & & = & 160 \\ & 35a & + & 20b & & & & & + & s_m & = & 1190 \\ & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0 \end{array}$$



Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

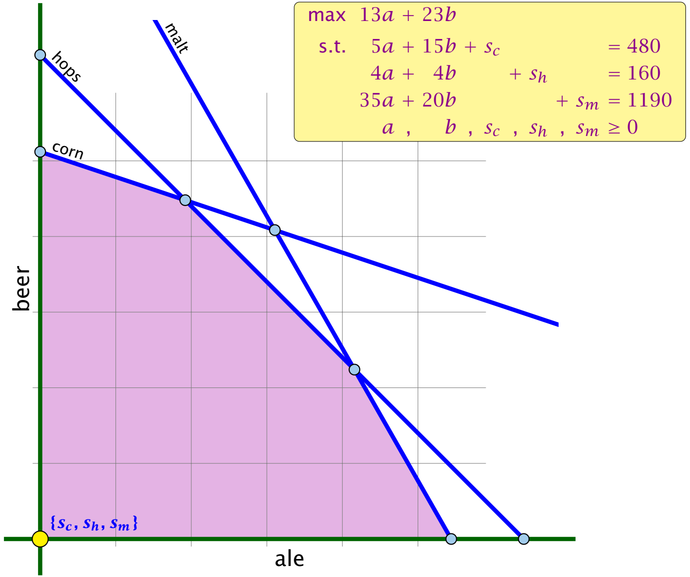
The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



$$
\begin{array}{lrl}
\max & 13a + 23b & \\
\text{s.t.} & 5a + 15b + s_c & = 480 \\
& 4a + 4b \quad + s_h & = 160 \\
& 35a + 20b \quad\quad + s_m & = 1190 \\
& a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m & \geq 0
\end{array}
$$

Matrix View

Let our linear program be

$$
\begin{array}{rcrcl}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

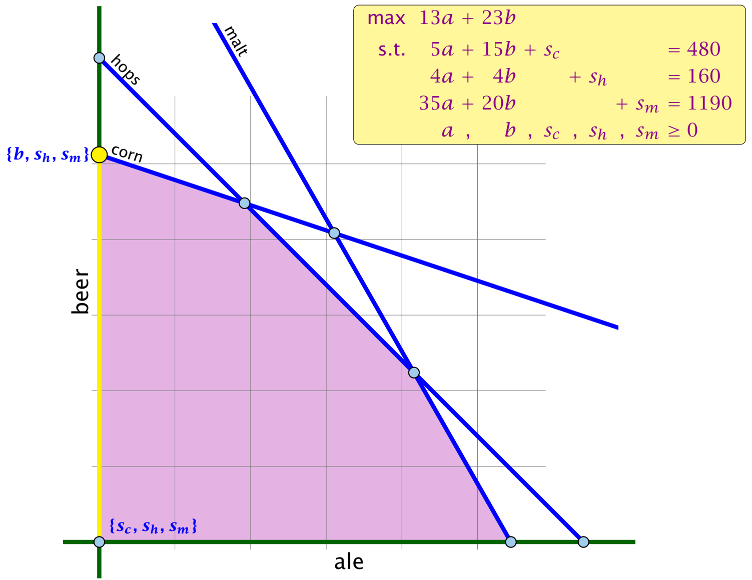
The simplex tableaux for basis B is

$$
\begin{array}{rcrcl}
& & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



Matrix View

Let our linear program be

$$
\begin{array}{ccccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

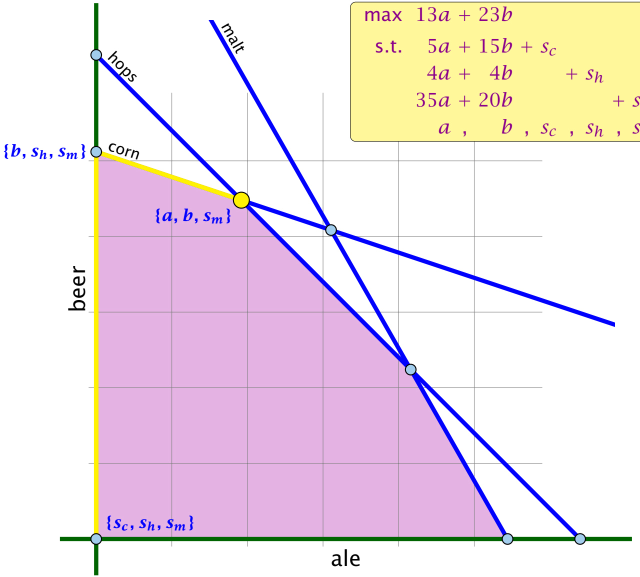
$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting

$$
\begin{array}{lrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$



Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

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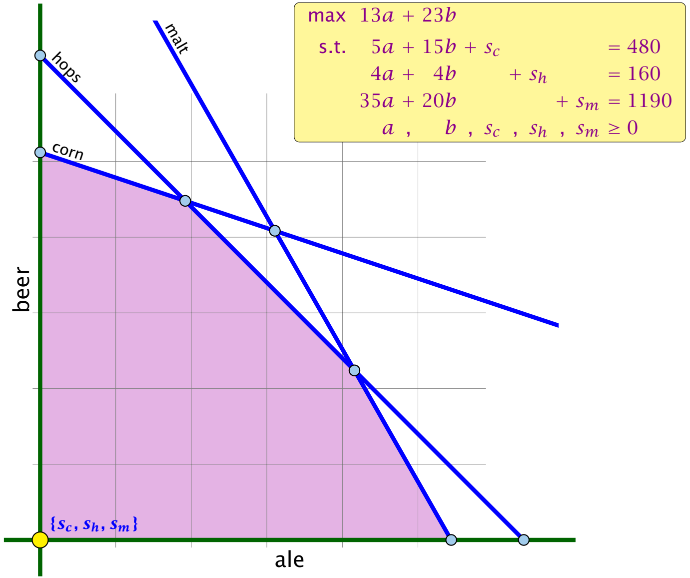
Geometric View of Pivoting



Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

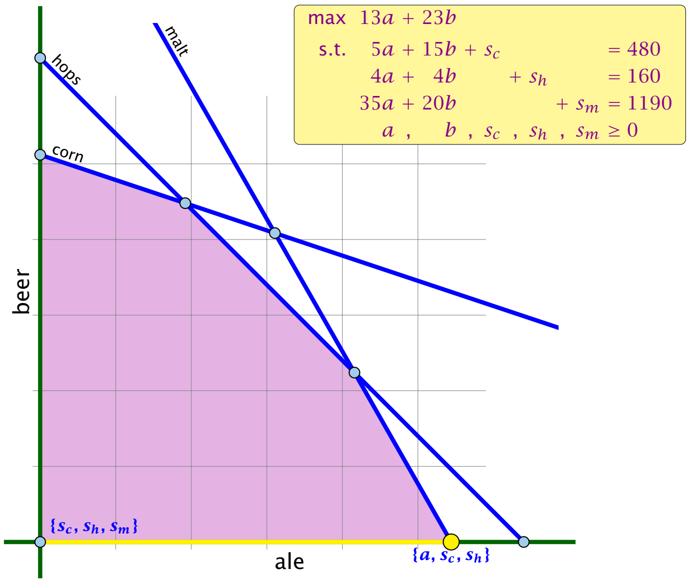
The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

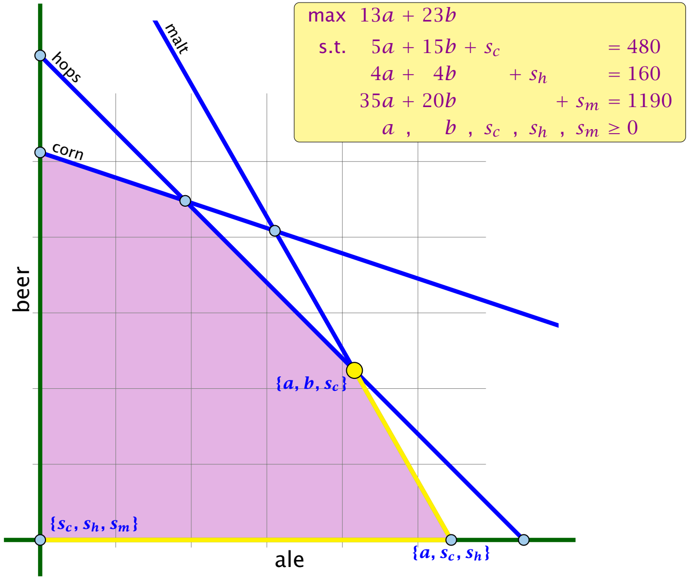
The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

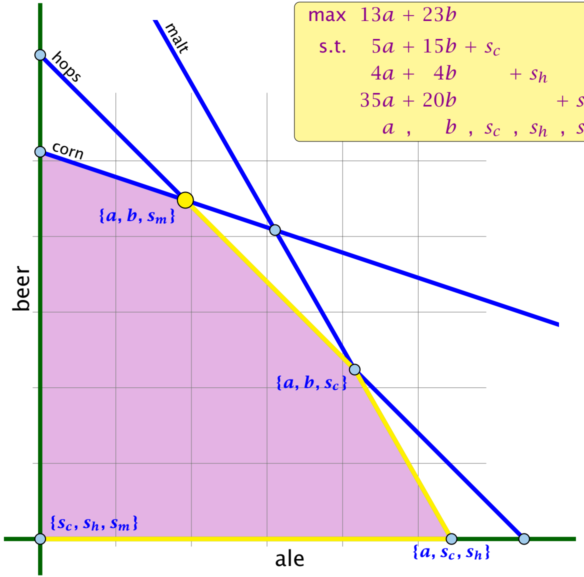
$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting

$$
\begin{array}{lllllll}
\max & 13a + 23b & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
 & 4a + 4b & & + s_h & & = & 160 \\
 & 35a + 20b & & & + s_m & = & 1190 \\
 & a\ ,\ b & ,\ s_c & ,\ s_h & ,\ s_m & \geq & 0
\end{array}
$$



Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
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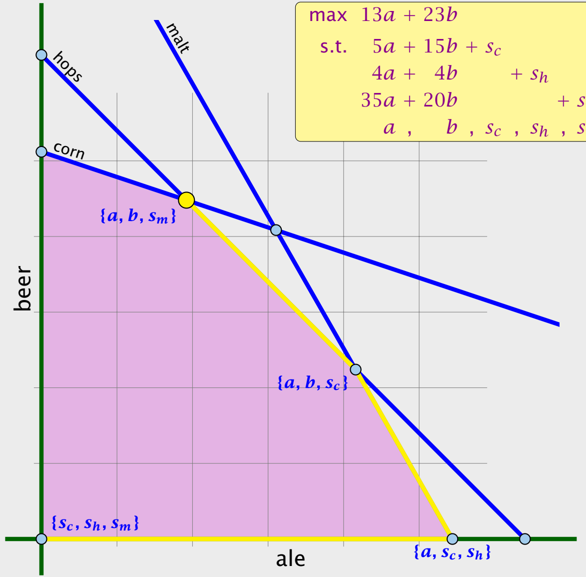
Requirements for d:

Geometric View of Pivoting

$$
\begin{array}{lrcl}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 4a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0
\end{array}
$$



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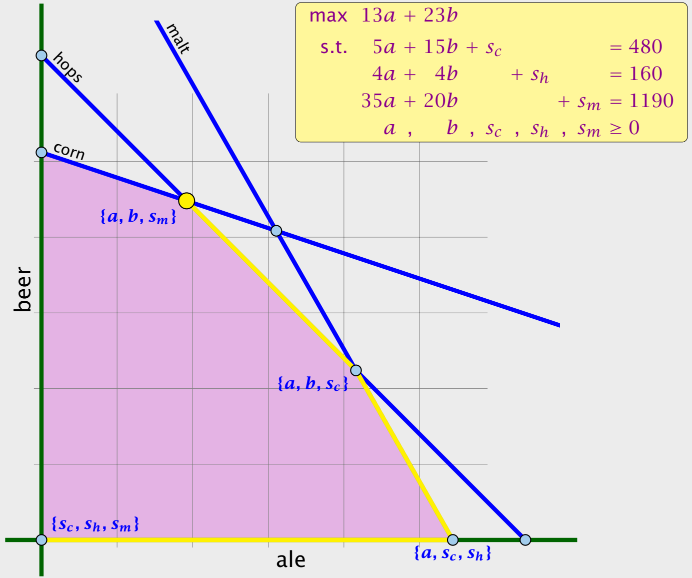

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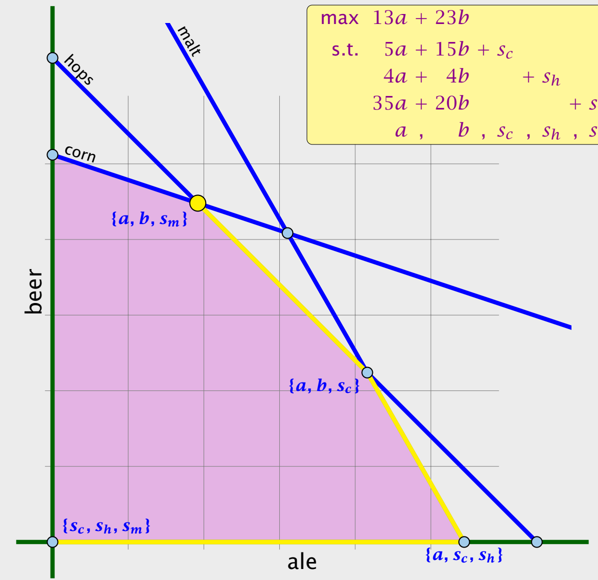

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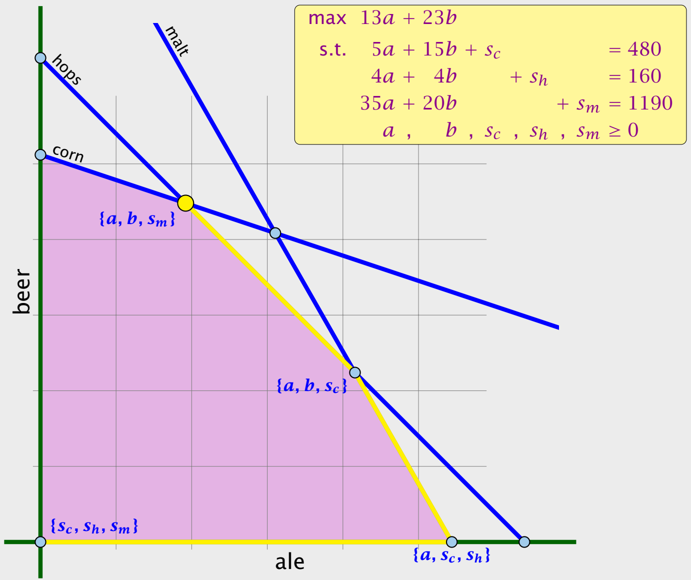

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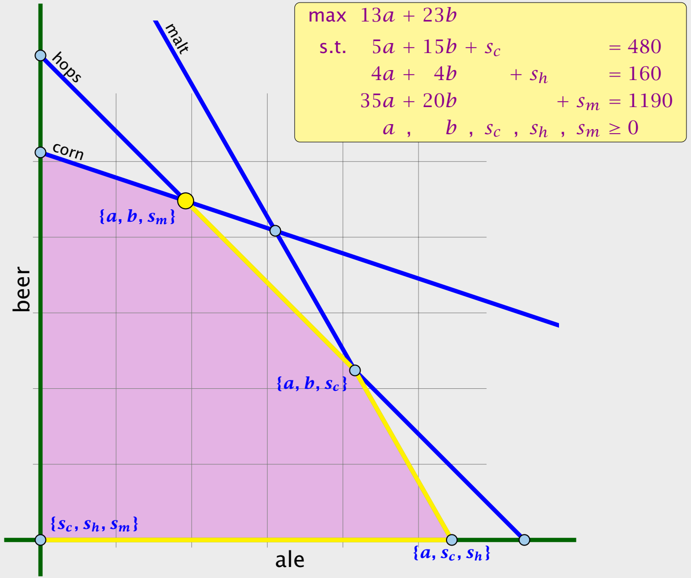

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- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

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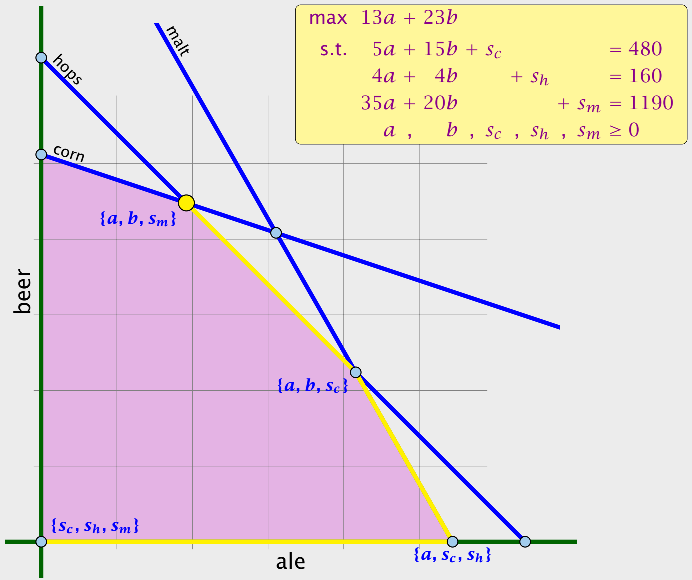

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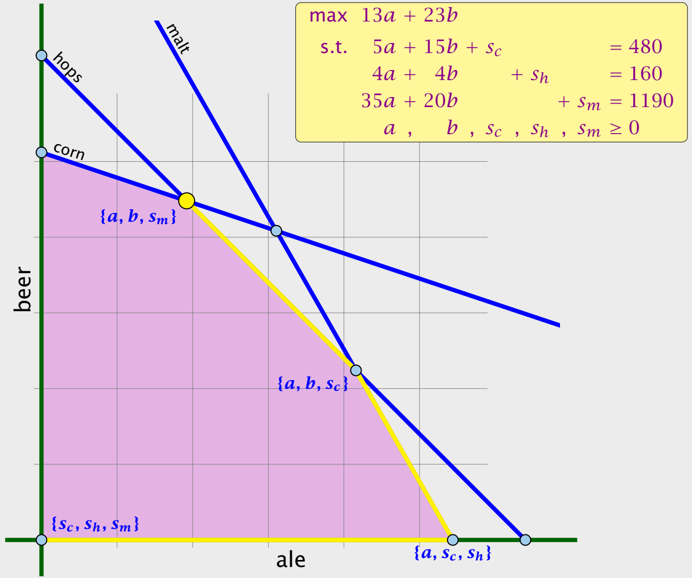

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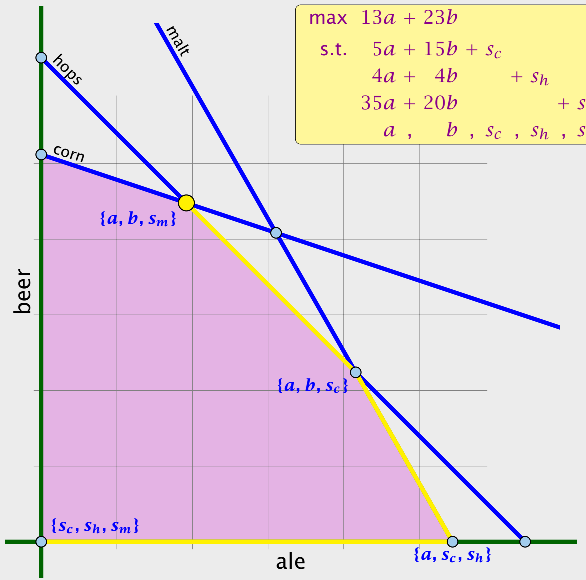

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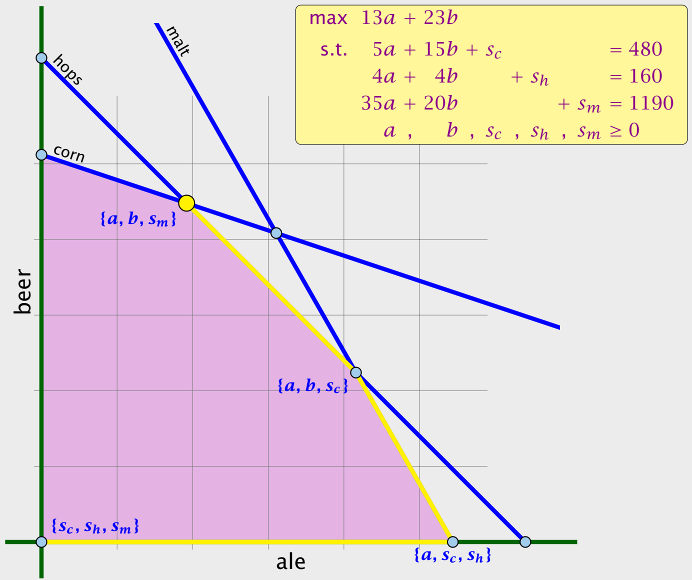

Algebraic Definition of Pivoting

Definition 26 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta (c_j - c_B^T A_B^{-1} A_{*j})$$

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Definition 27 (Reduced Cost)
For a basis B the value

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is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

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Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
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Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

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  Then we can terminate because we know that the solution is optimal.
- If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

4 Simplex Algorithm

Questions:

- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \;\; ?$$

  Then we can terminate because we know that the solution is optimal.
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Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

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The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

Definition 28 (Degeneracy)
A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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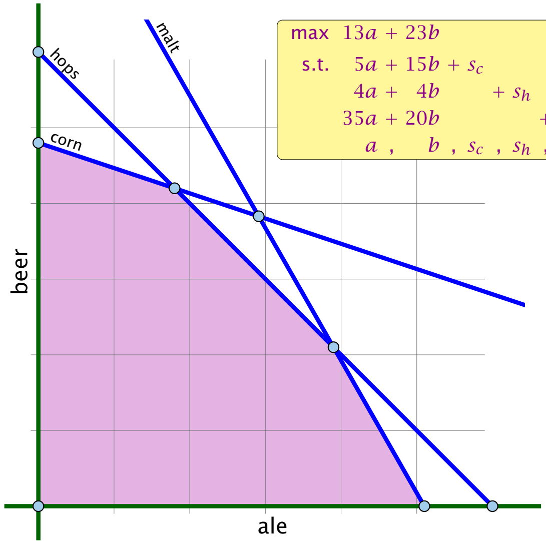

$$\begin{array}{lrcrcccccl}
\text{max} & 13a & + & 23b \\
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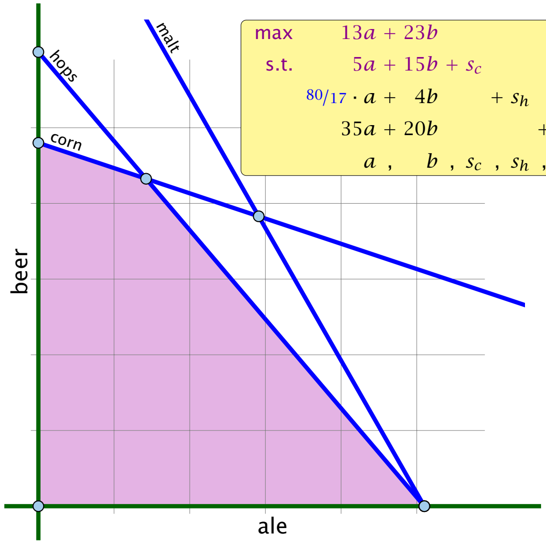

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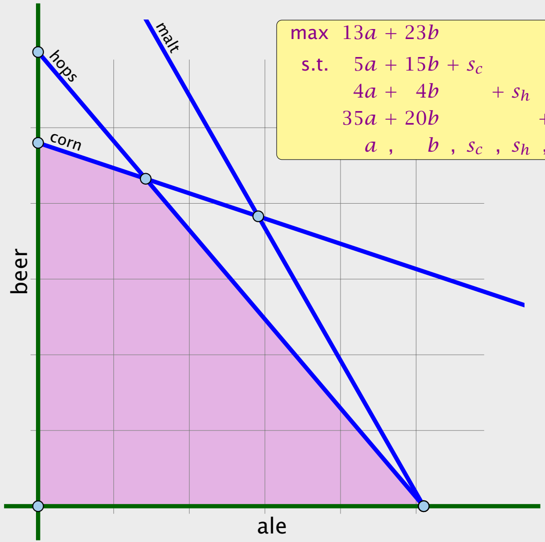

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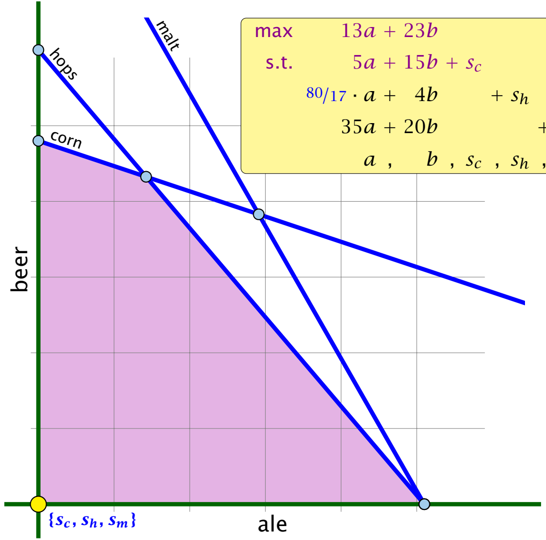

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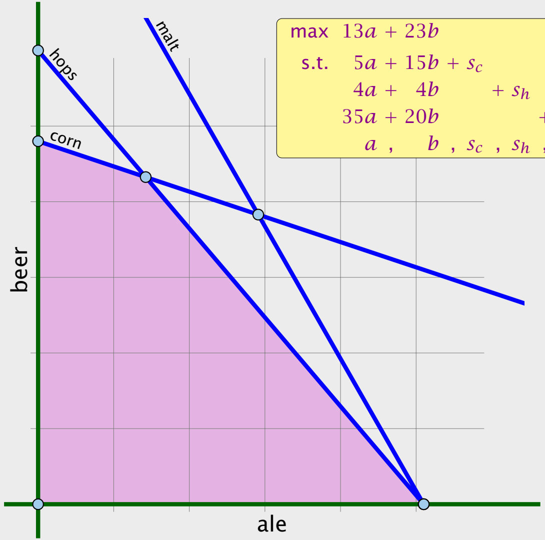

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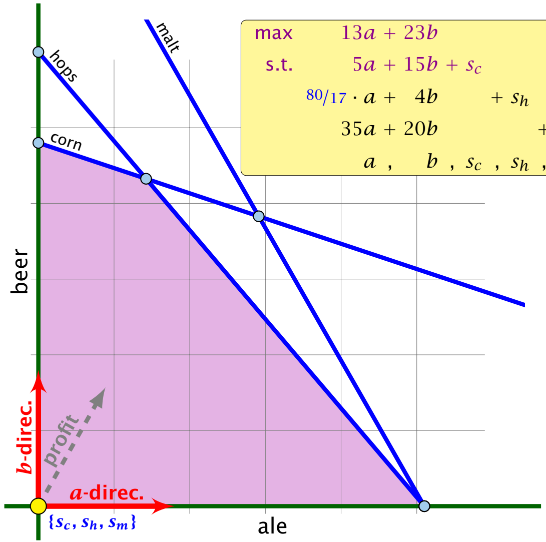

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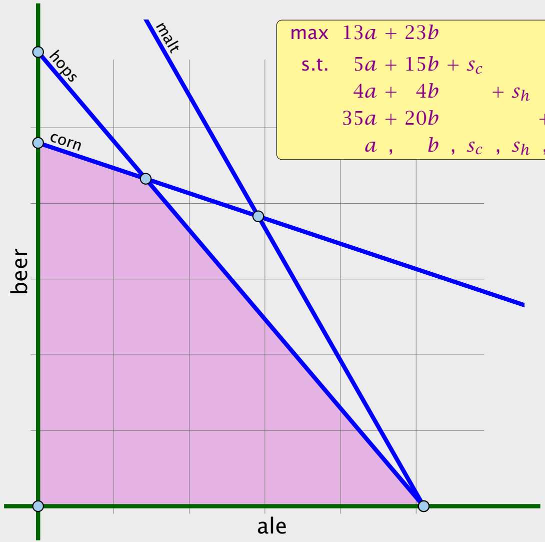

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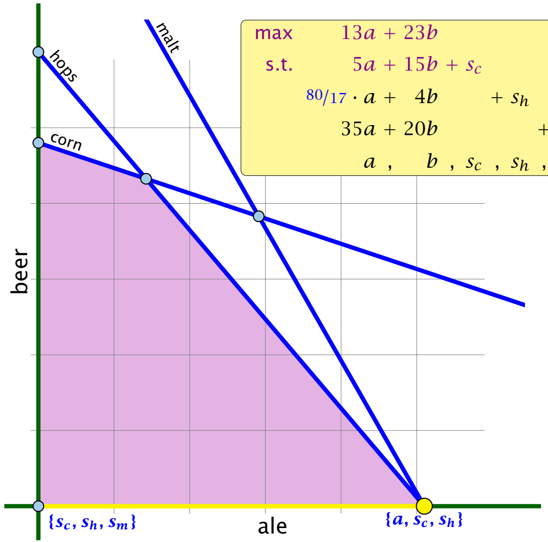

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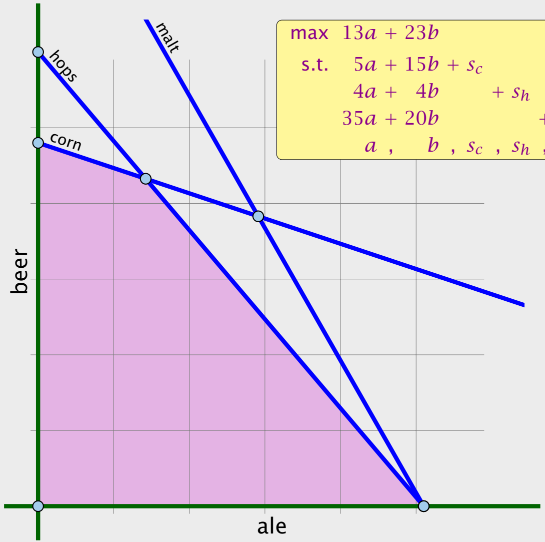

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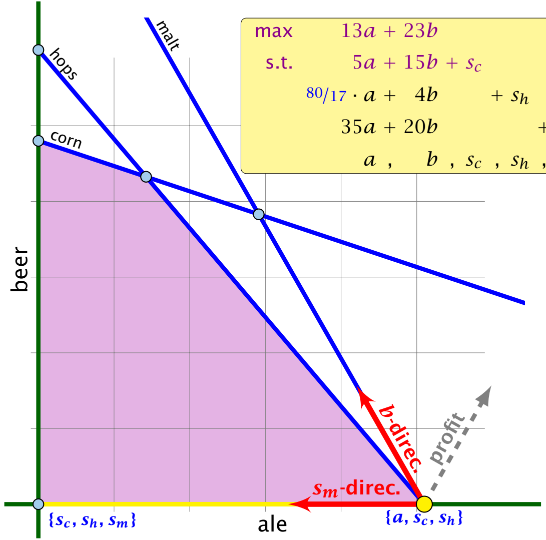

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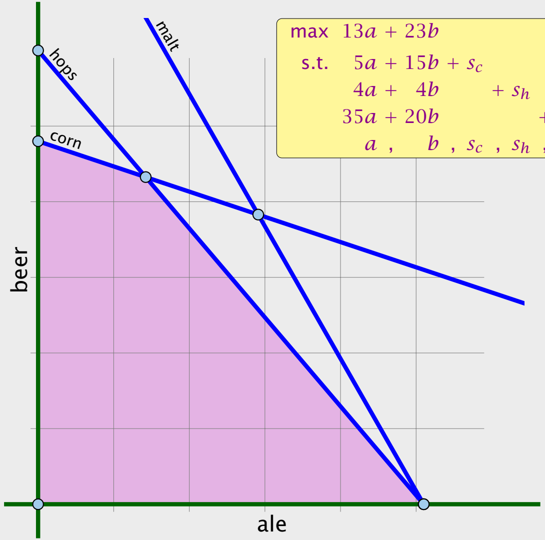

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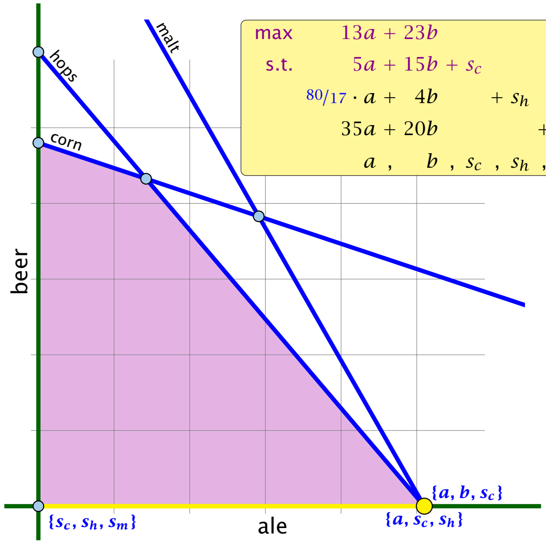

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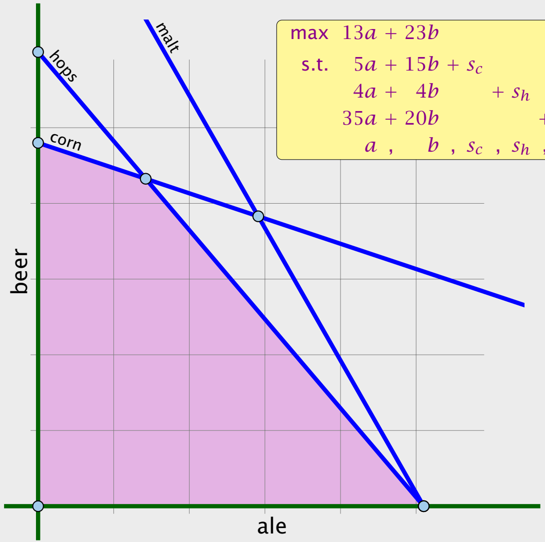

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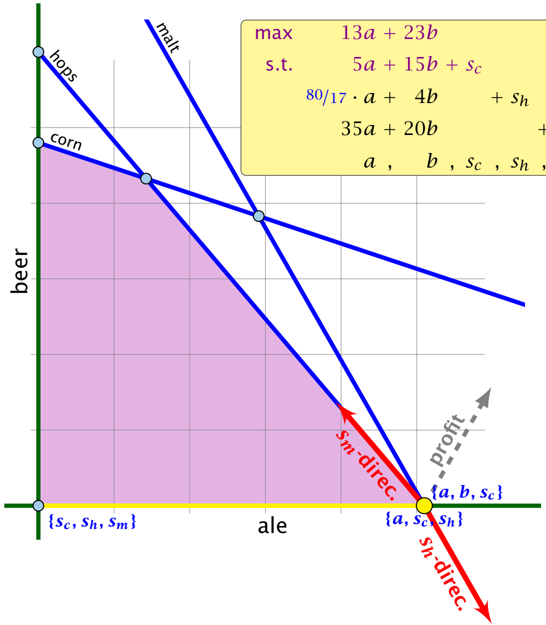

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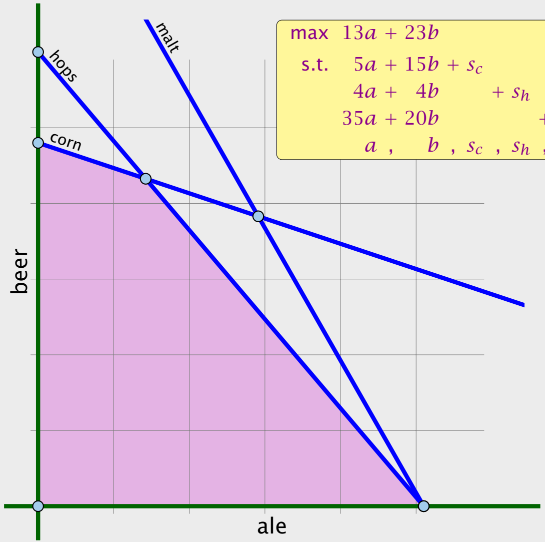

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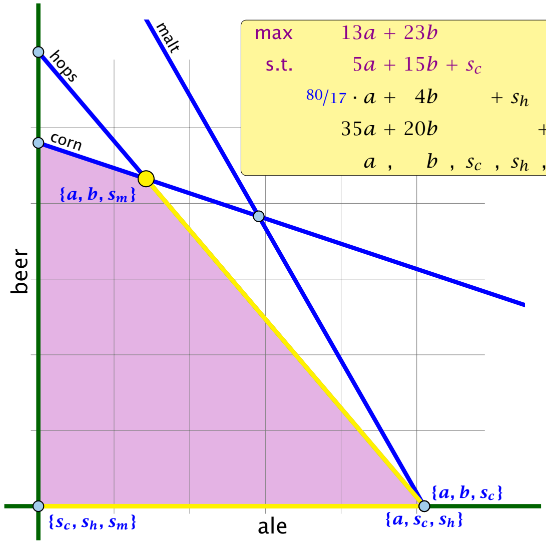

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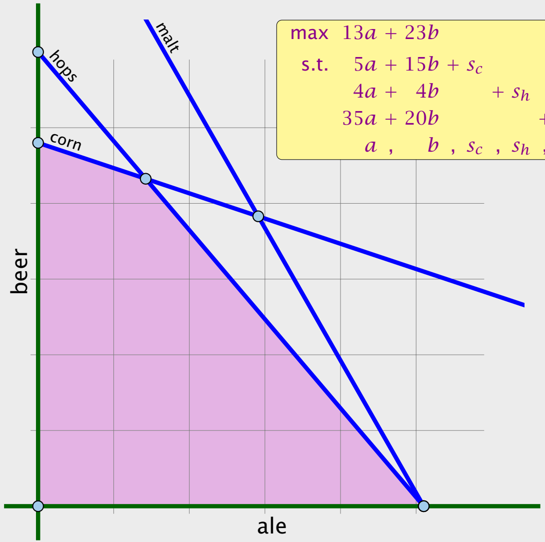

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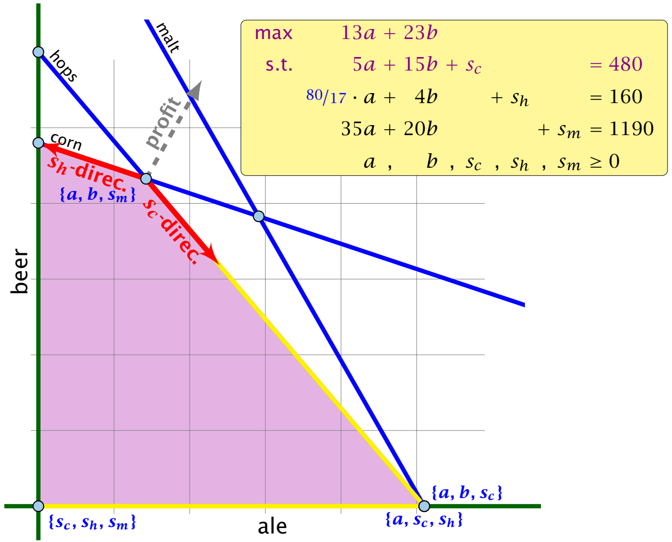

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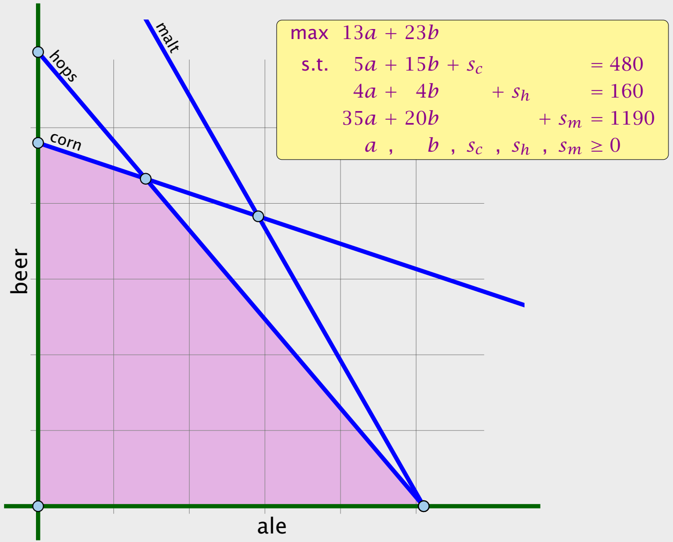

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Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \le 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
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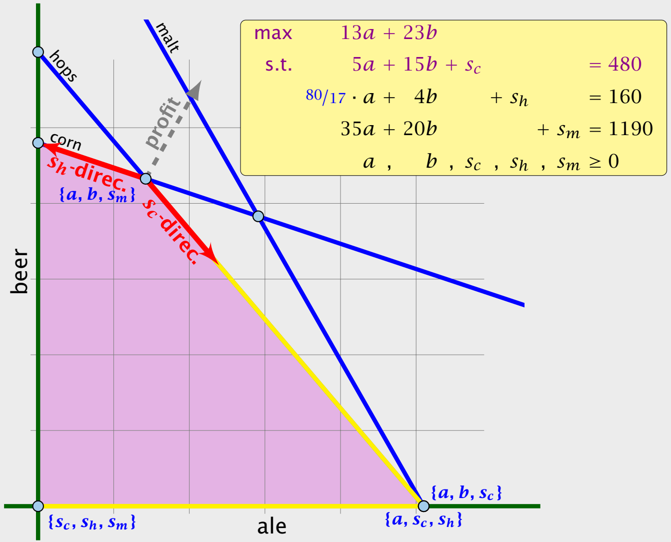

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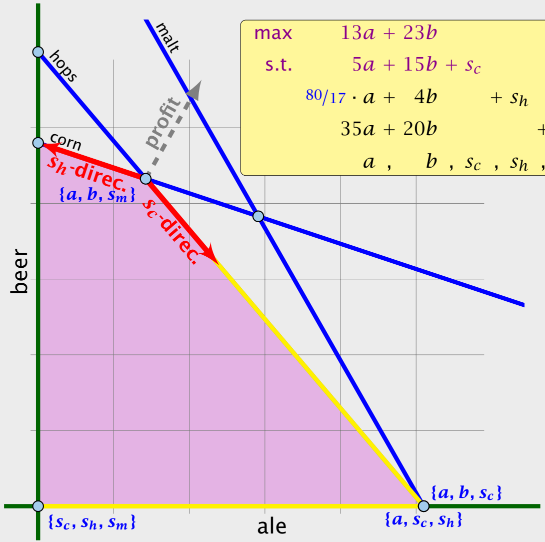

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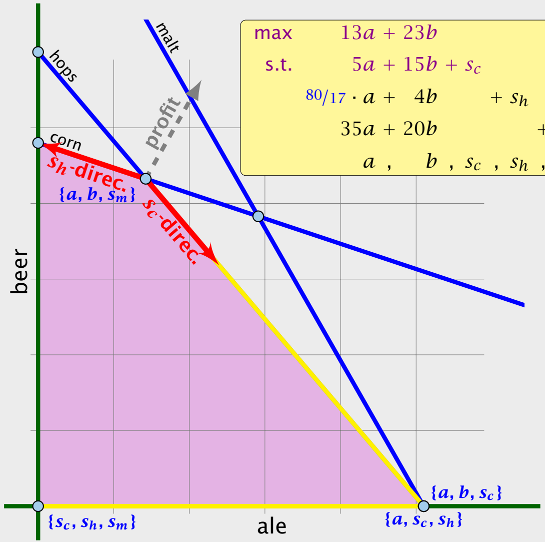

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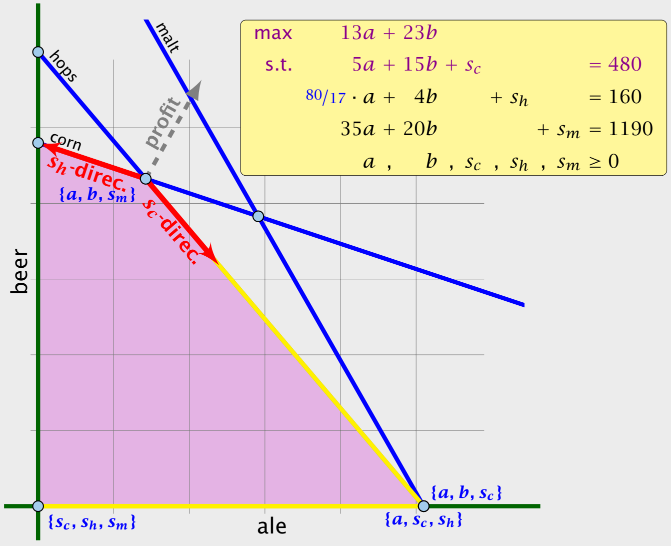

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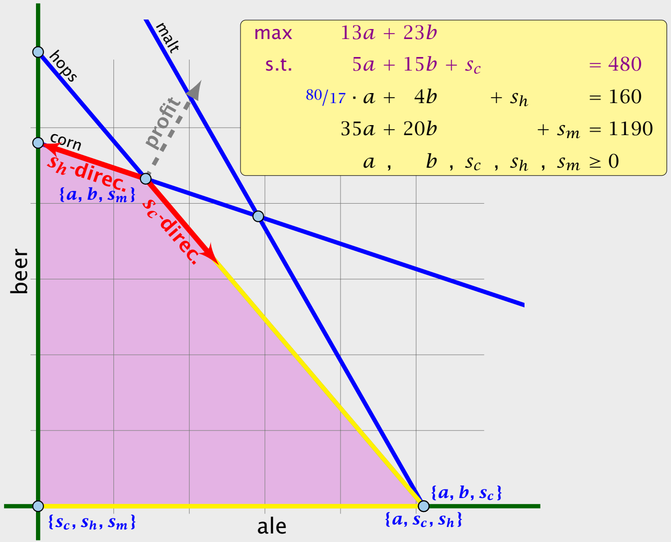

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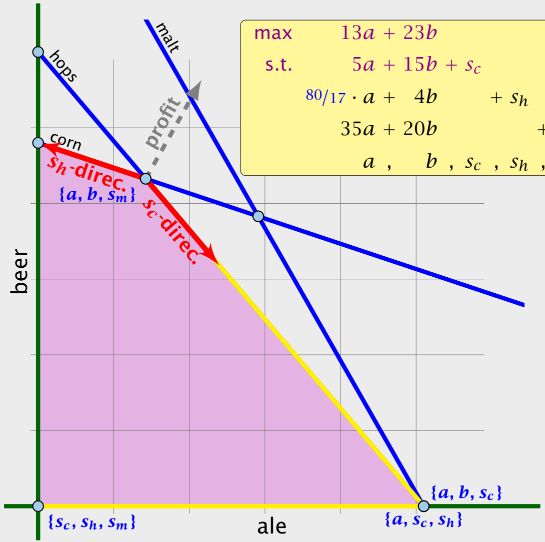
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
- Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Degenerate Example



$$
\begin{array}{llcl}
\text{max} & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & {}^{80}/_{17} \cdot a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0
\end{array}
$$

Termination

What do we have so far?
Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
- Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \mathbf{0}$.
- The standard slack from for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

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Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

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Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
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3. If $\sum_i v_i > 0$ then the original problem is infeasible.
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Two phase algorithm

Suppose we want to maximize c^Tx s.t. $Ax = b, x \geq 0$.

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Optimality

Lemma 29
Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.

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Duality

How do we get an upper bound to a maximization LP?

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 30
Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$$

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The dual of the dual problem is the primal problem.

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Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and
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x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

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Theorem 32 (Weak Duality)
Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

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Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^T y = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

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Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^Ty = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

Let $z = \max\{c^Tx \mid Ax \leq b, x \geq 0\}$ and
$w = \min\{b^Ty \mid A^Ty \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

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Theorem 32 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T\hat{x} \leq z \leq w \leq b^T\hat{y} \; .$$

Weak Duality

$$A^T\hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \;(\hat{x} \geq 0)$$

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5.2 Simplex and Duality

The following linear programs form a primal dual pair:

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$$\begin{aligned}&\max\{c^Tx \mid Ax = b, x \geq 0\}\\&\quad = \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\end{aligned}$$

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Dual:

$$\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}$$

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Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

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Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \ge c, y \ge 0\} \\ &\quad = \min\left\{ [b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &\quad = \min\left\{ b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &\quad = \min\left\{ b^T y' \;\middle|\; A^T y' \ge c \right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

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5.3 Strong Duality

$P = \max\{c^T x \mid Ax \le b, x \ge 0\}$
n_A: number of variables, m_A: number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \le \bar{b}\}$
$n_{\bar{A}} = n_A$, $m_{\bar{A}} = m_A + n_A$

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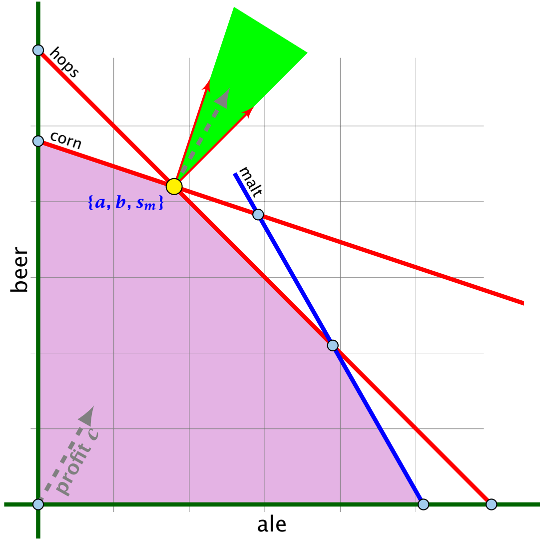

The profit vector c lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

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Strong Duality

Theorem 33 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

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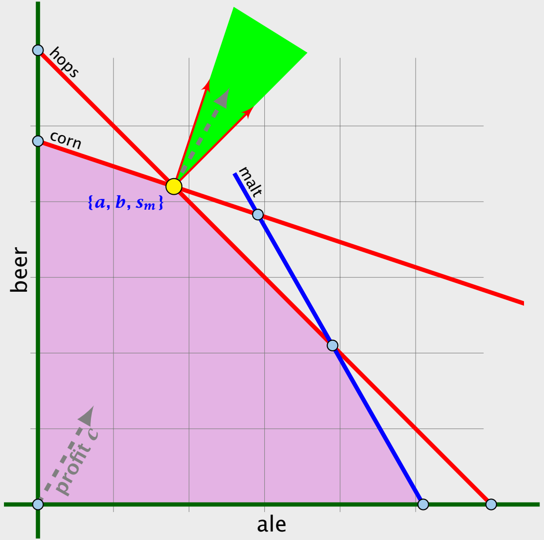

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Lemma 34 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

(without proof)

Strong Duality

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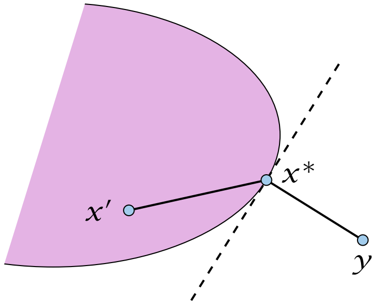
Lemma 35 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.



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- Define $f(x) = \|y - x\|$.
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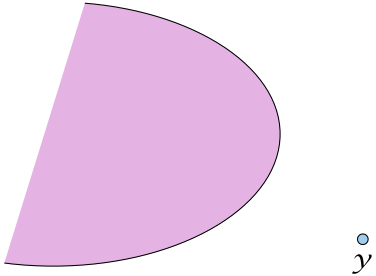

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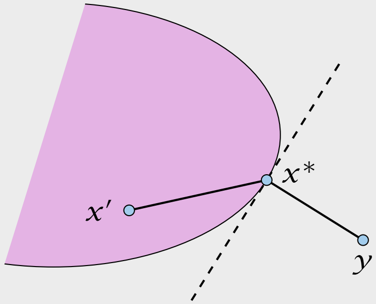

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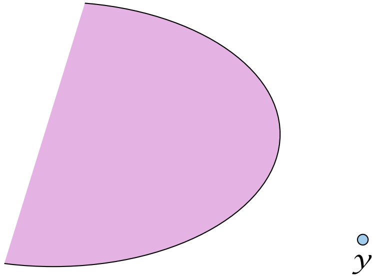

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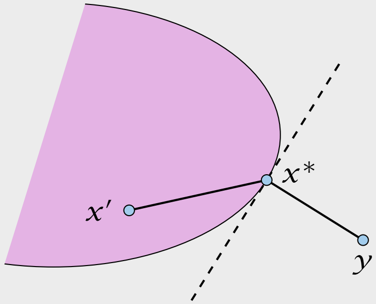

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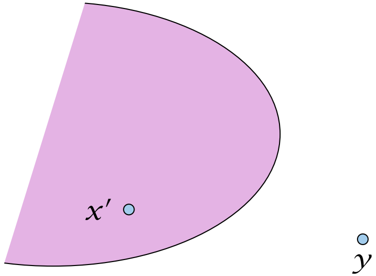

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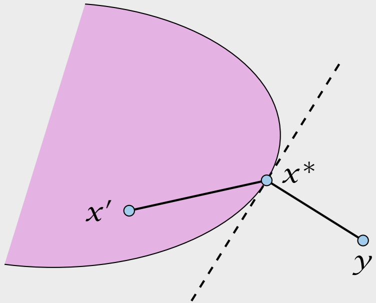

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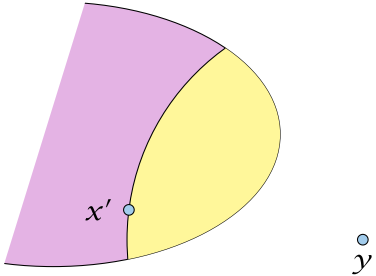

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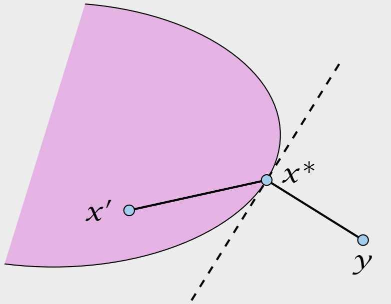

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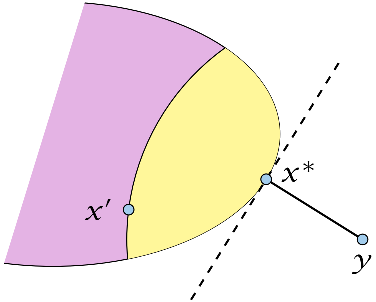

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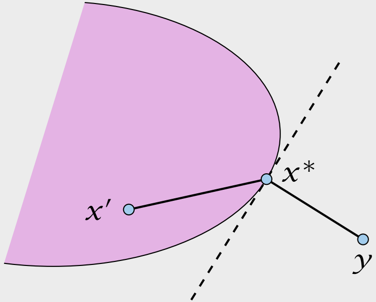

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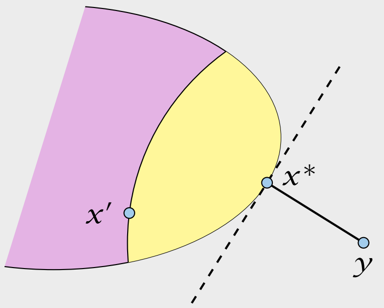

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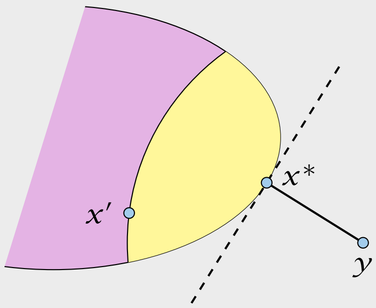

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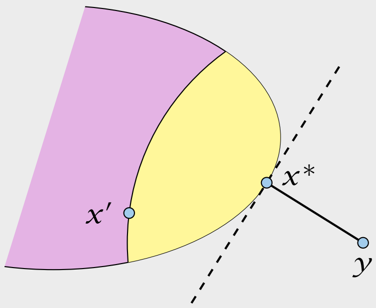

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$\|y - x^*\|^2$

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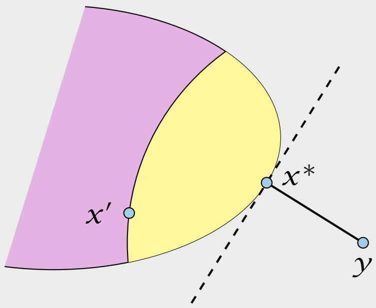

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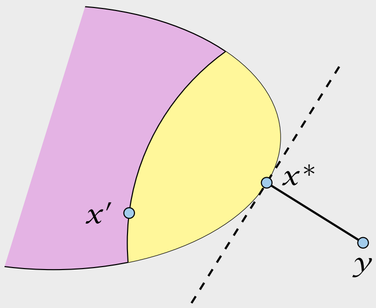

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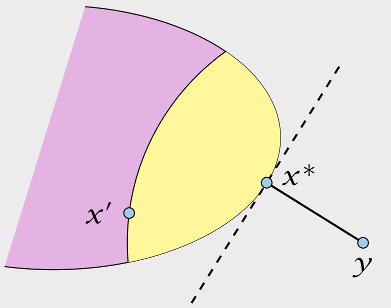

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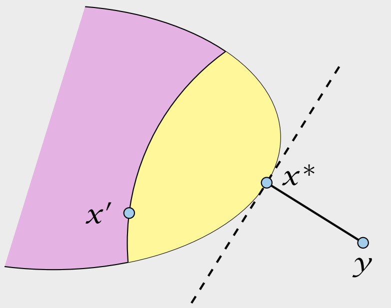

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Letting $\epsilon \to 0$ gives the result.

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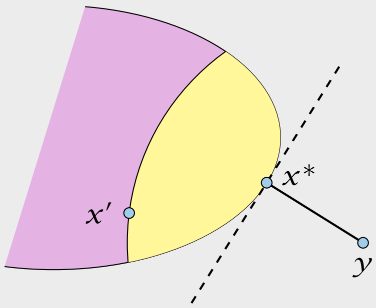

Theorem 36 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

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- Let $x^* \in X$ be closest point to y in X.
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- For $x \in X : a^T(x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.
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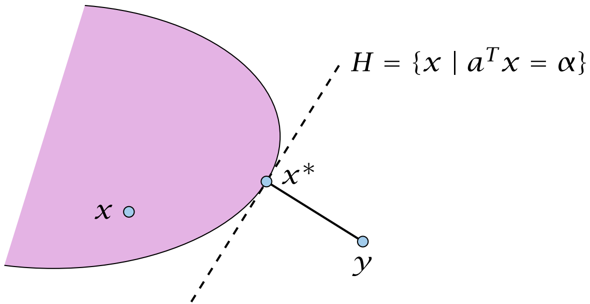

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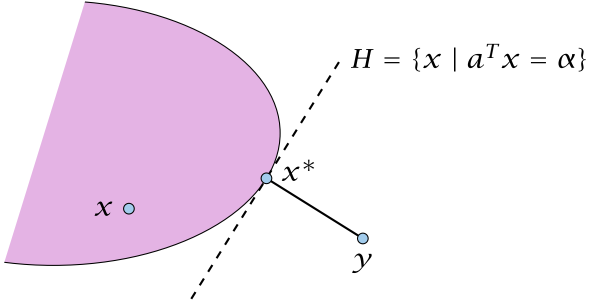

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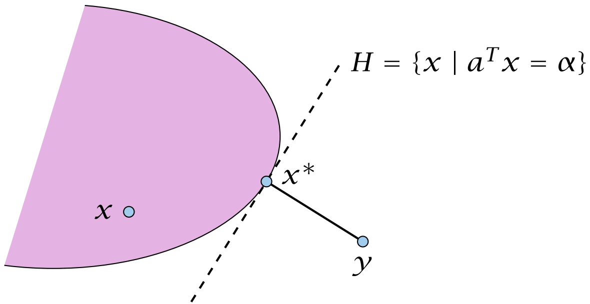

Theorem 36 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

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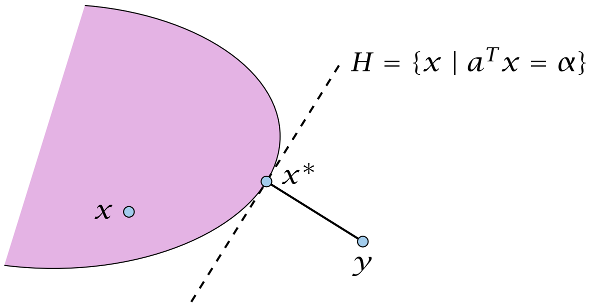

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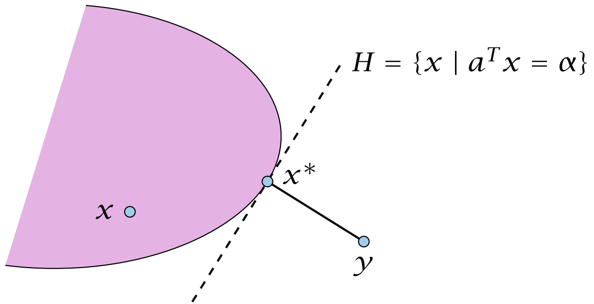

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Hence, at most one of the statements can hold.

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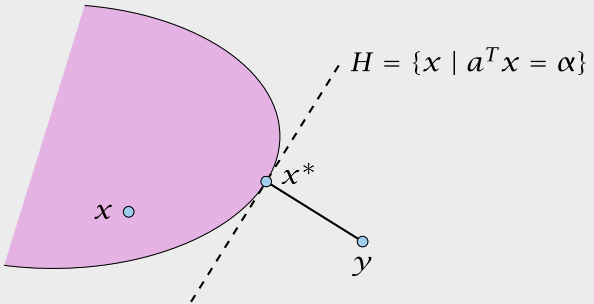

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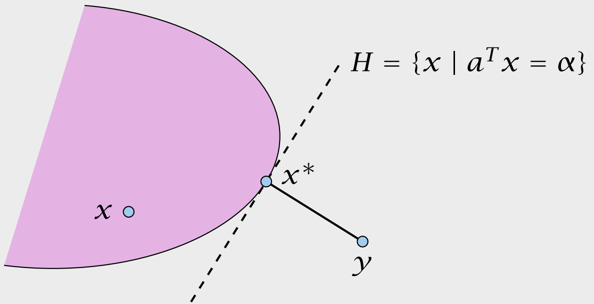

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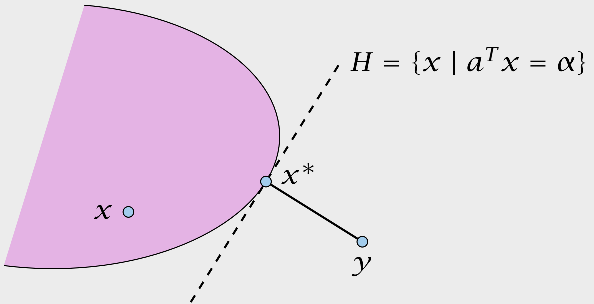

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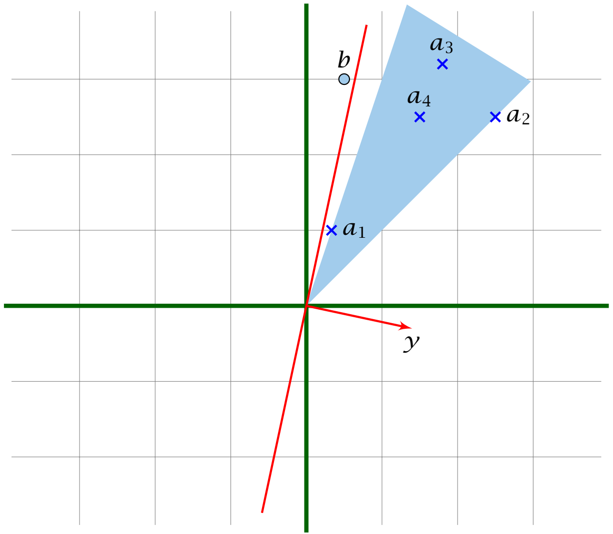

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$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

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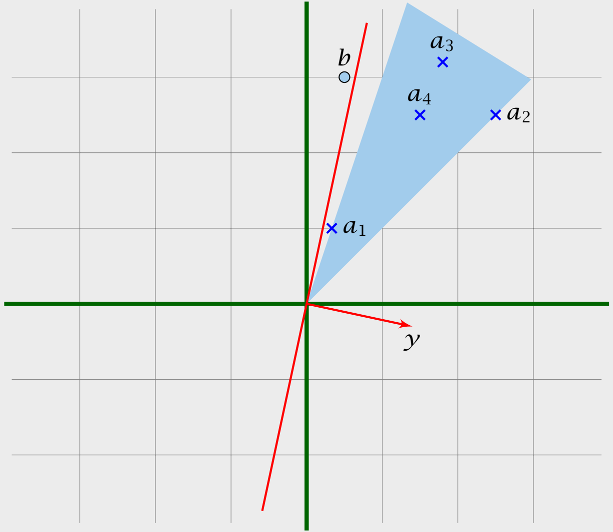

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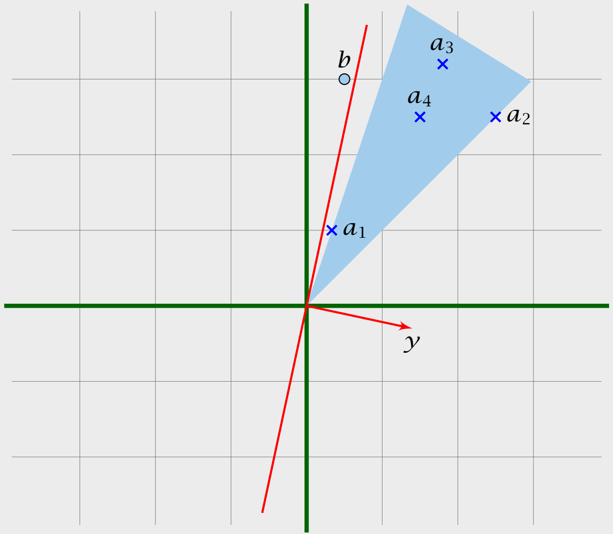

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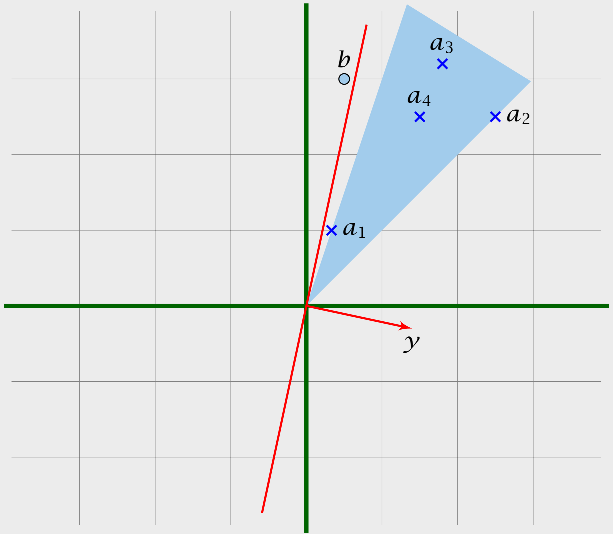

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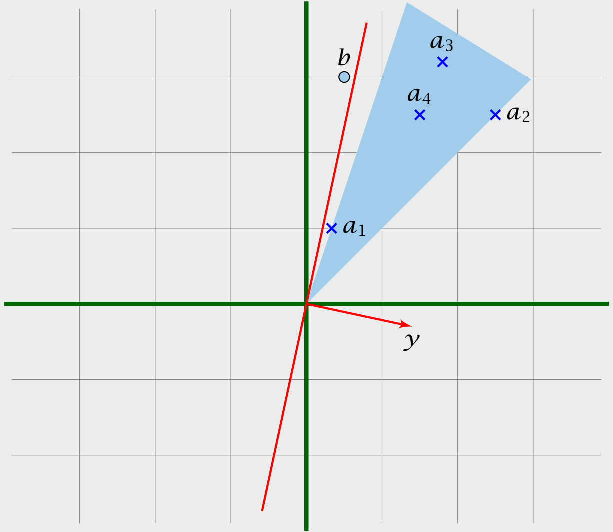

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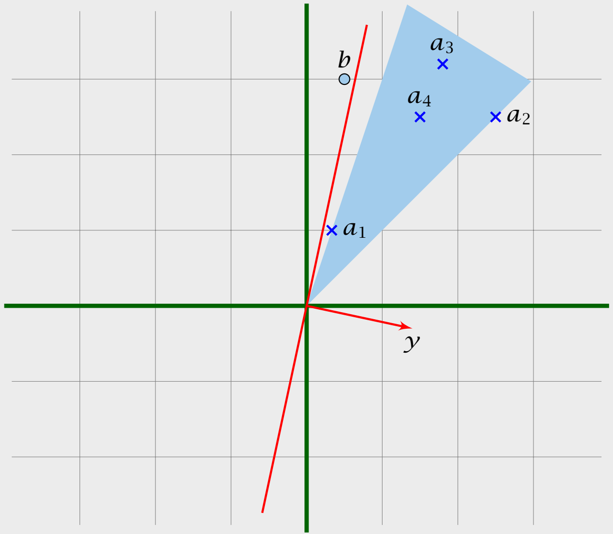

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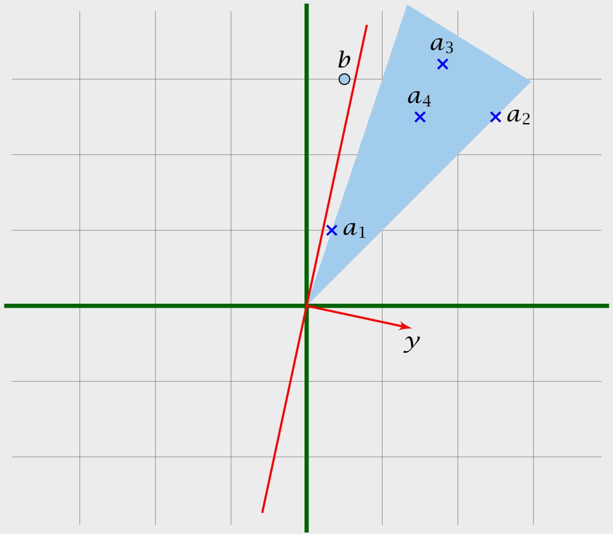

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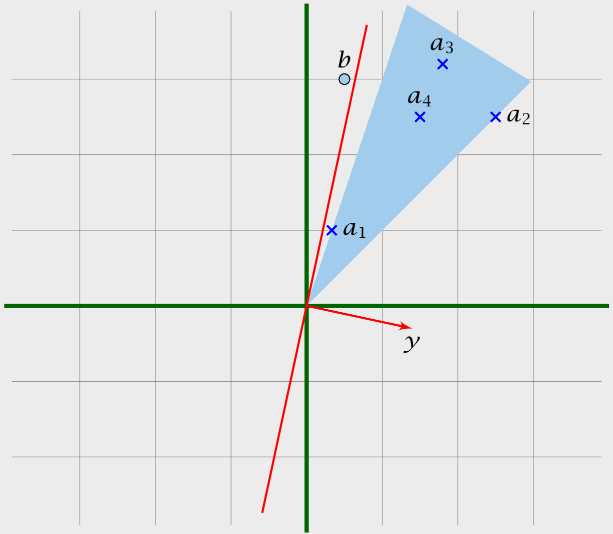

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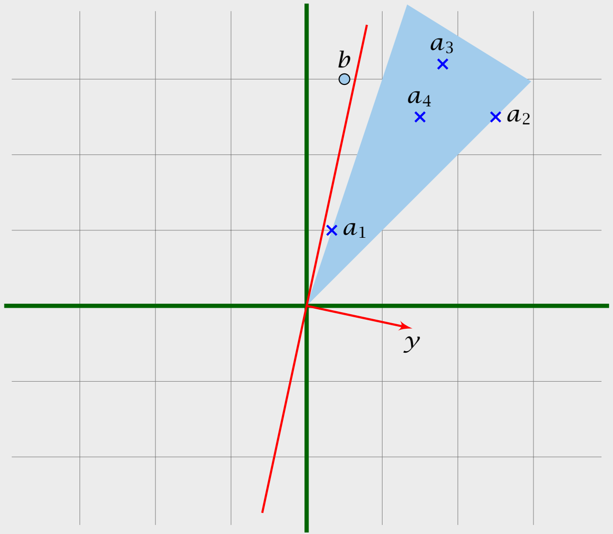

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Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
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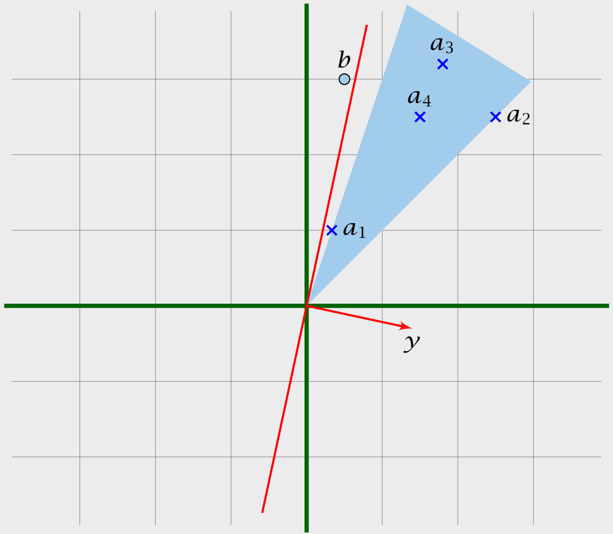

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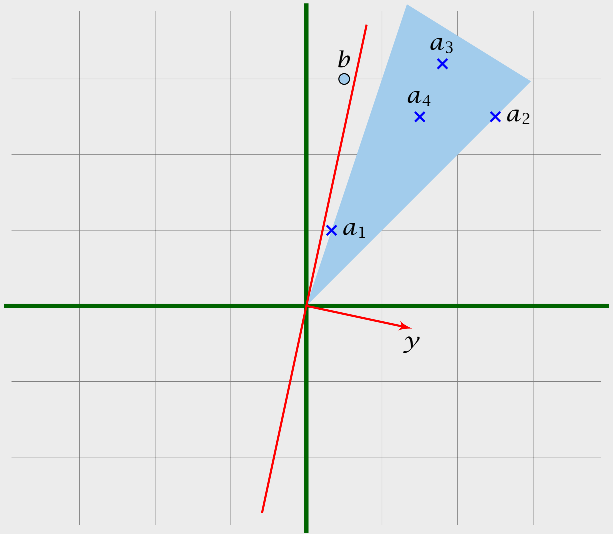

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P: $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

D: $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$

Theorem 39 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

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If the solution y, v has $v = 0$ we have that

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Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

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Fundamental Questions

Definition 40 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
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Complementary Slackness

Lemma 41

Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ *has solution* x^* *and its dual* $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ *has solution* y^*.

1. *If* $x_j^* > 0$ *then the* j*-th constraint in* D *is tight.*
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Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

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Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

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\begin{array}{lrcrl}
\text{max} & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \le 480 \\
 & 4a & + & 4b & \le 160 \\
 & 35a & + & 20b & \le 1190 \\
 & & & a, b & \ge 0
\end{array}
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- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

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\begin{array}{lrcrcrl}
\text{min} & 480C & + & 160H & + & 1190M & \\
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Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ε_C, ε_H, and ε_M, respectively.

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If ϵ is “small” enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

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- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C, \varepsilon_H$, and ε_M, respectively.

The profit increases to $\max\{c^Tx \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \min & (b^T + \epsilon^T)y & & \\ \text{s.t.} & A^Ty & \ge & c \\ & y & \ge & 0 \end{array}$$

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If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

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Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
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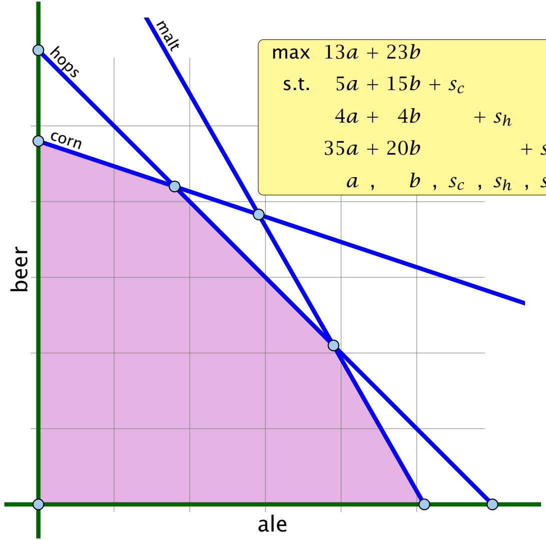
Example



$$\begin{array}{llllll} \max & 13a + 23b & & & & \\ \text{s.t.} & 5a + 15b + s_c & & & = & 480 \\ & 4a + 4b & + s_h & & = & 160 \\ & 35a + 20b & & + s_m & = & 1190 \\ & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & & \geq & 0 \end{array}$$

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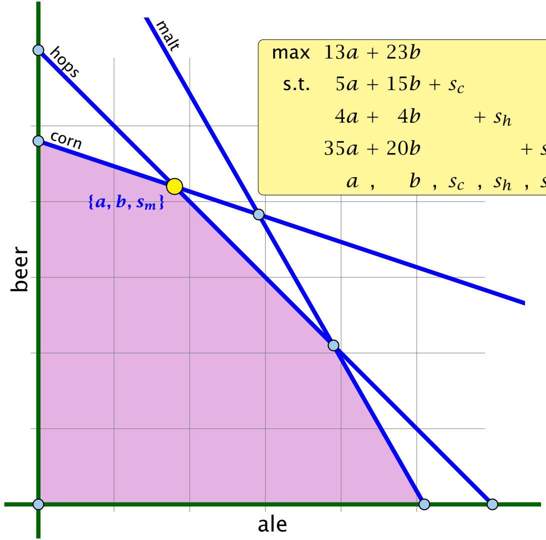

$$\begin{array}{lllllll} \max & 13a & + 23b & & & & \\ \text{s.t.} & 5a & + 15b & + s_c & & & = 480 \\ & 4a & + 4b & & + s_h & & = 160 \\ & 35a & + 20b & & & + s_m & = 1190 \\ & a & , b & , s_c & , s_h & , s_m & \geq 0 \end{array}$$

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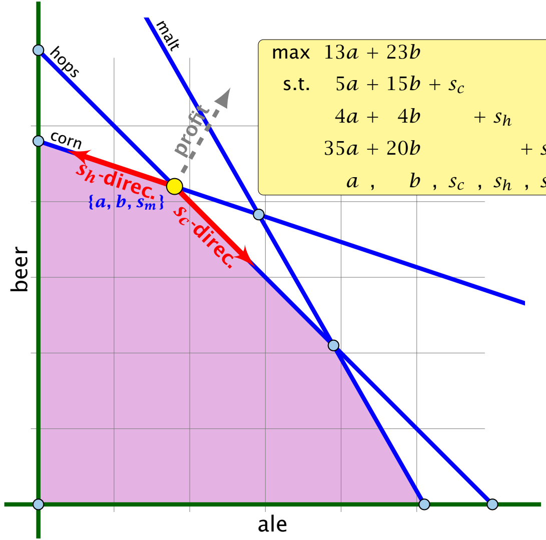

$$
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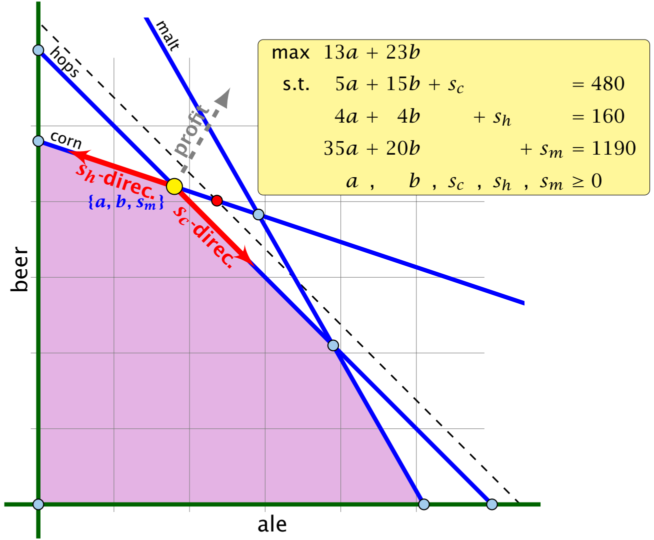

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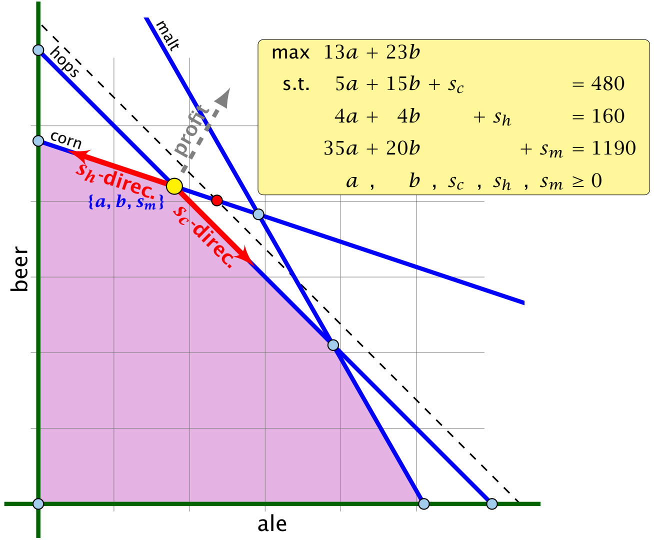

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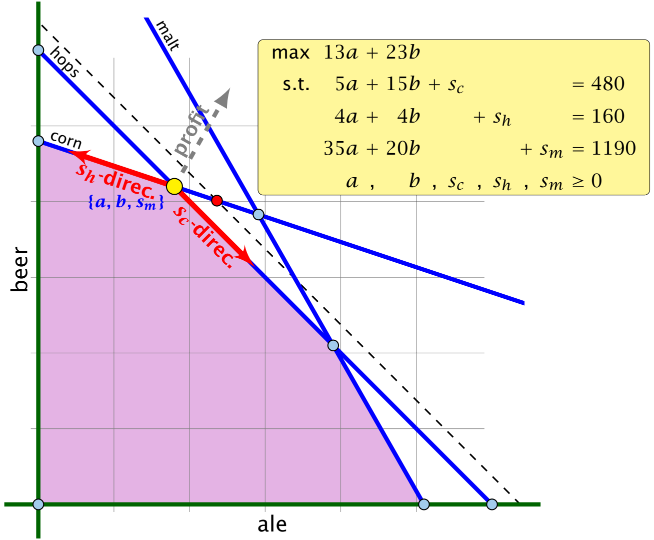

The change in profit when increasing hops by one unit is $= c_B^T A_B^{-1} e_h$.

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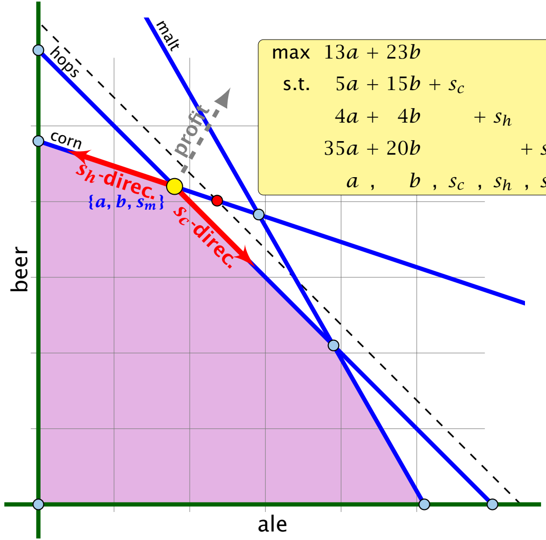

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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

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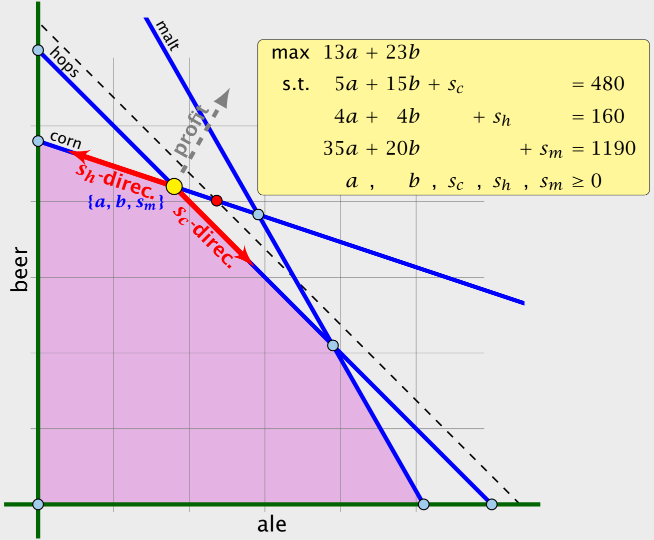

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Flows

Definition 42

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)
$$0 \le f_{xy} \le c_{xy} \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$
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Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

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\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
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$$
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\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} \qquad + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
 & f_{st}: & 1\ell_{st} & \ge & 1 \\
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 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - \ 0 + 1p_y & \geq & 0 \\
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\end{array}
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We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

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LP-Formulation of Maxflow

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\begin{array}{llrcl}
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 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
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 & f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
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with $p_t = 0$ and $p_s = 1$.

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This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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Degeneracy Revisited

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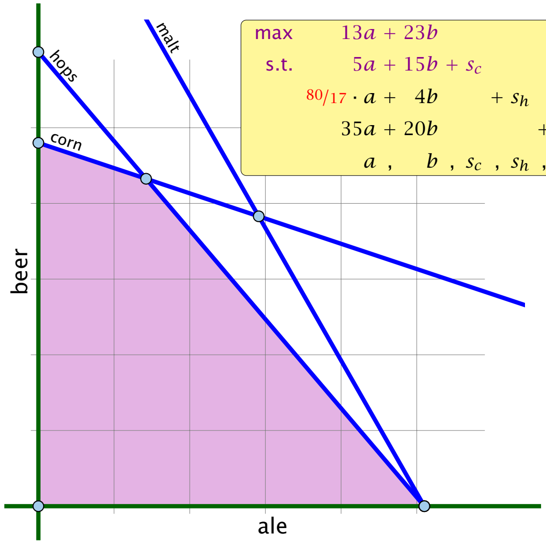
If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Degenerate Example



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\begin{array}{llr}
\text{max} & 13a + 23b & \\
\text{s.t.} & 5a + 15b + s_c & = 480 \\
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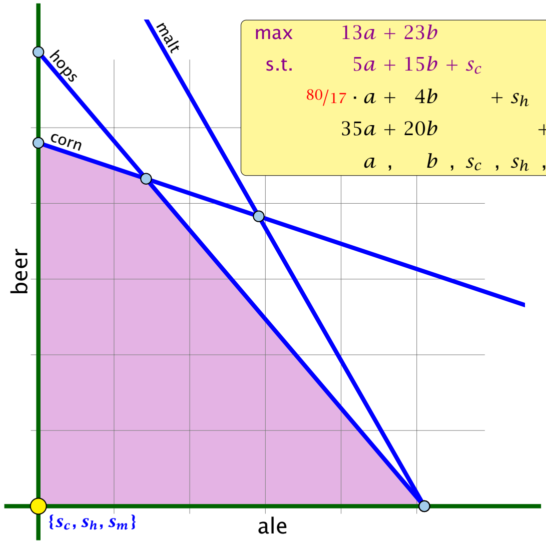

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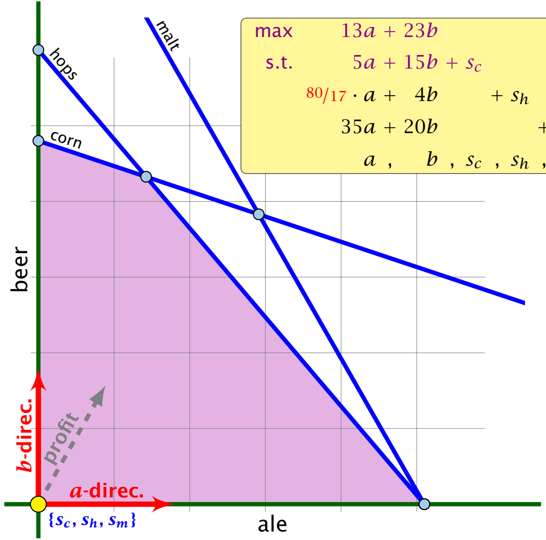

$$\begin{array}{lrcrcrcrcrl} \text{max} & 13a & + & 23b & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & = & 480 \\ & \tfrac{80}{17}\cdot a & + & 4b & & & + & s_h & & = & 160 \\ & 35a & + & 20b & & & & & + s_m & = & 1190 \\ & a & , & b & , & s_c & , & s_h & , s_m & \geq & 0 \end{array}$$

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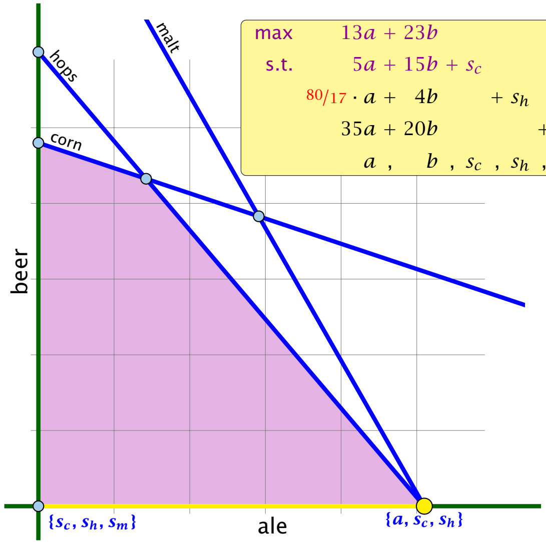

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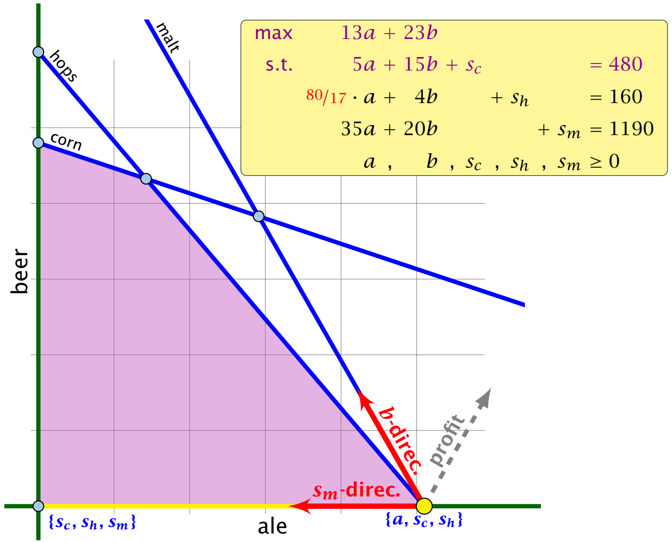

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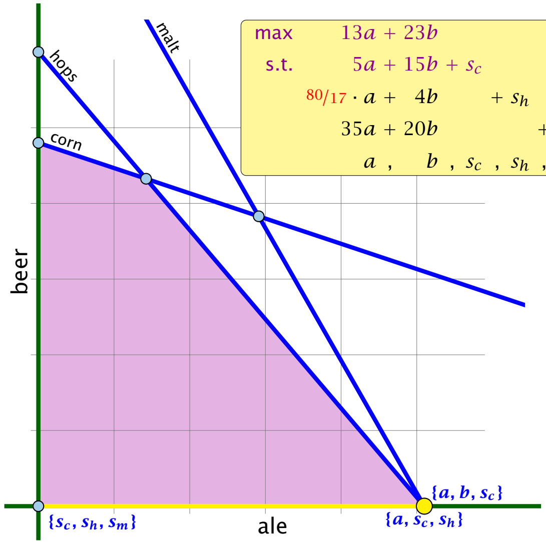

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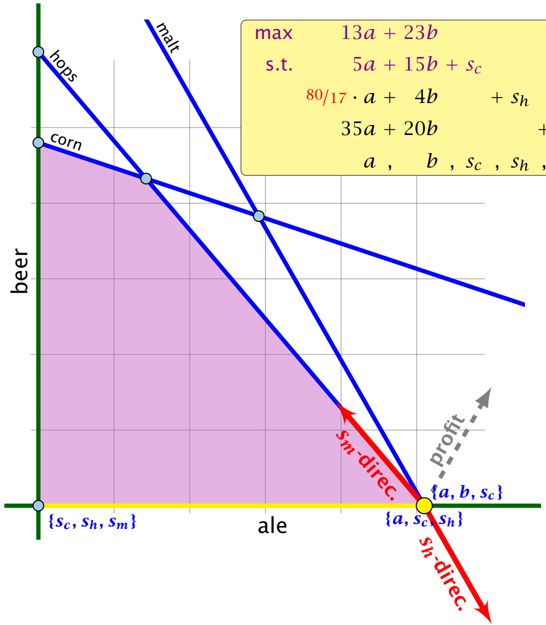
$$\begin{array}{llcl} \text{max} & 13a + 23b & & \\ \text{s.t.} & 5a + 15b + s_c & = & 480 \\ & {\color{red}80/17} \cdot a + 4b \quad + s_h & = & 160 \\ & 35a + 20b \quad\quad + s_m & = & 1190 \\ & a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m & \geq & 0 \end{array}$$



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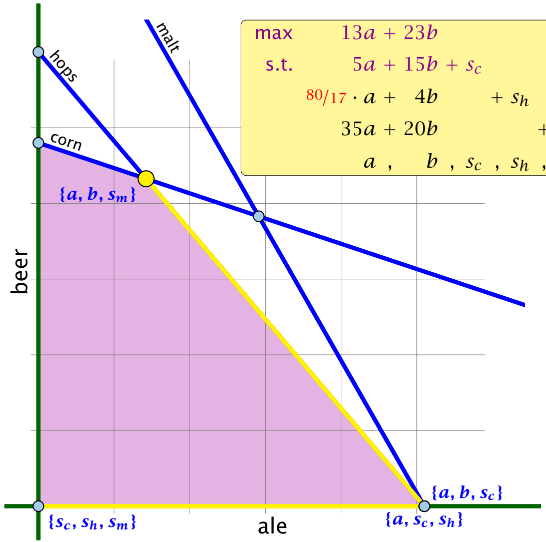

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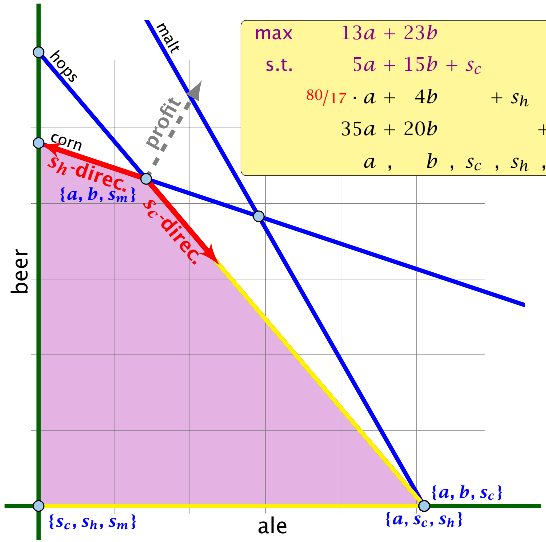

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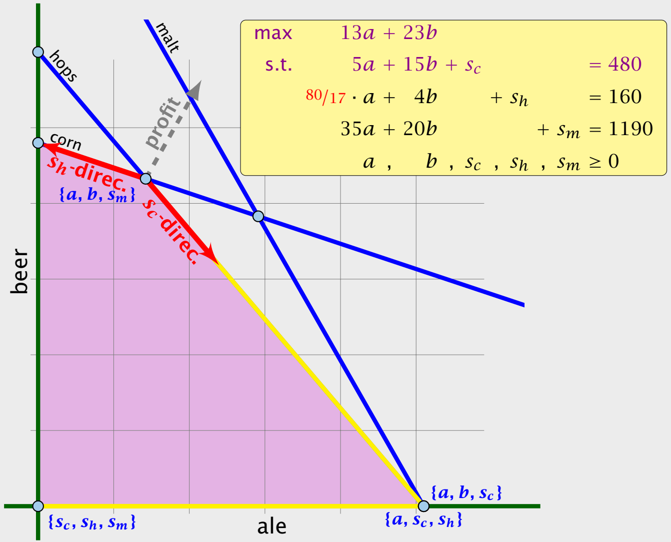

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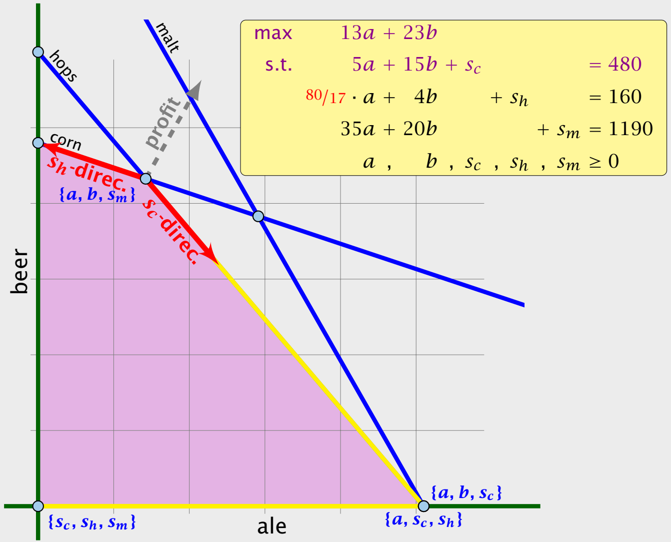

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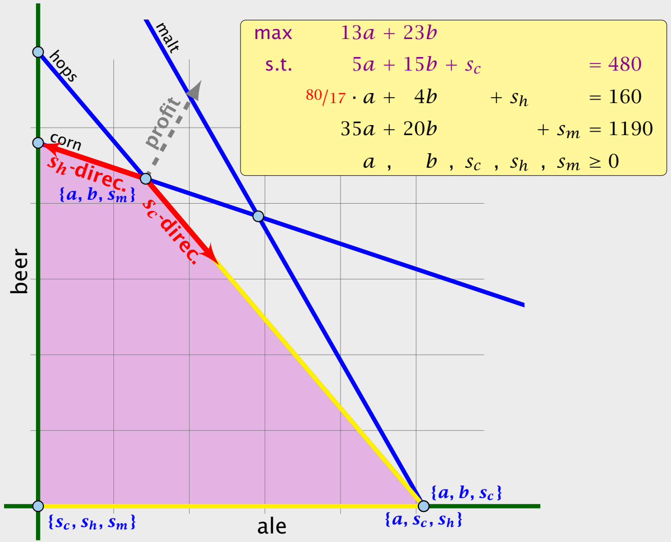

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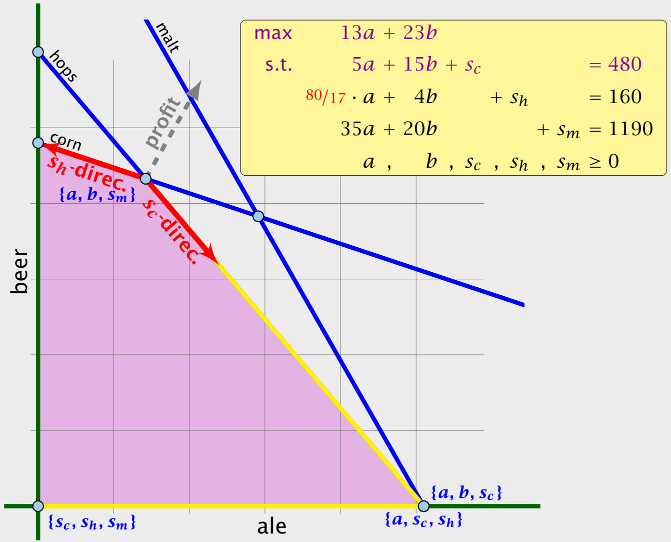

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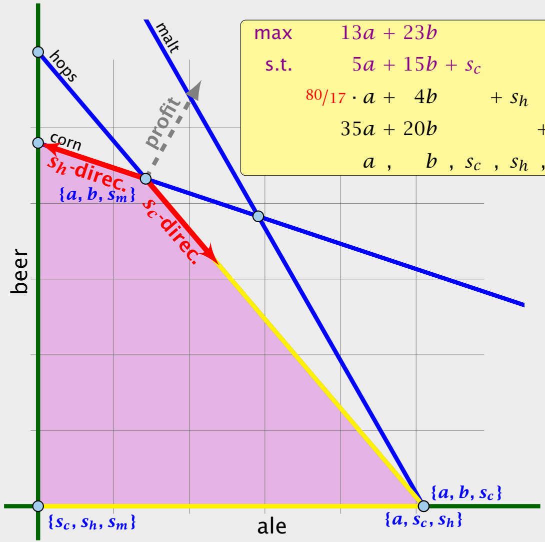

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Perturbation

Let B be index set of some basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 \ .$$

This is the perturbation that we are using.

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Property I

The new LP is feasible because the set B of basis variables provides a feasible basis:

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Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1}b)_i < 0$ for some row i.

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$$\left(A_{\tilde{B}}^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)\right)_i = (A_{\tilde{B}}^{-1}b)_i + \left(A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)_i < 0$$

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Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1}b)_i < 0$ for some row i.

Then for small enough $\epsilon > 0$

$$\left(A_{\tilde{B}}^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)\right)_i = (A_{\tilde{B}}^{-1}b)_i + \left(A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)_i < 0$$

Hence, $\tilde{B}$ is not feasible.

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Idea:
Given feasible $\mathrm{LP} := \max\{c^Tx, Ax = b; x \geq 0\}$. Change it into $\mathrm{LP}' := \max\{c^Tx, Ax = b', x \geq 0\}$ such that

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II. If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \ngeq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).

III. LP' has no degenerate basic solutions

Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x^*_{\tilde{B}} = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable ε of degree at most m.

$A_{\tilde{B}}^{-1}A_B$ has rank m. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

Hence, $\varepsilon > 0$ small enough gives that no component of the above vector is 0. Hence, no degeneracies.

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$$\tilde{c} = (c^T - c_B^T A_B^{-1} A) \le 0$$

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- If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j-th basis direction d, fulfills $d \ge 0$ we know that LP' is unbounded. The basis direction does not depend on b. Hence, we also know that LP is unbounded.

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Lexicographic Pivoting

Doing calculations with perturbed instances may be costly. Also the right choice of ε is difficult.

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Simulate behaviour of LP' without explicitly doing a perturbation.

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Lexicographic Pivoting

We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

Otherwise we have to be careful.

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In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

$$b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

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Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \; .$$

ℓ is the index of a leaving variable within B. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

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Lexicographic Pivoting

Definition 44
$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

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Lexicographic Pivoting

LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

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Lexicographic Pivoting

LP$'$ chooses an index that minimizes

$$\theta_\ell = \frac{\left(A_B^{-1}\left(b + \begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)\right)_\ell}{(A_B^{-1}A_{*e})_\ell} = \frac{\left(A_B^{-1}(b \mid I)\begin{pmatrix}1\\ \varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)_\ell}{(A_B^{-1}A_{*e})_\ell}$$

$$= \frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}\begin{pmatrix}1\\ \varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}$$

Lexicographic Pivoting

LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

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Lexicographic Pivoting

This means you can choose the variable/row ℓ for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

Lexicographic Pivoting

LP' chooses an index that minimizes

$$\theta_\ell = \frac{\left(A_B^{-1}\left(b + \begin{pmatrix}\varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}\right)\right)_\ell}{(A_B^{-1}A_{*e})_\ell} = \frac{\left(A_B^{-1}(b \mid I)\begin{pmatrix}1 \\ \varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}\right)_\ell}{(A_B^{-1}A_{*e})_\ell}$$

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This means you can choose the variable/row ℓ for which the vector

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is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

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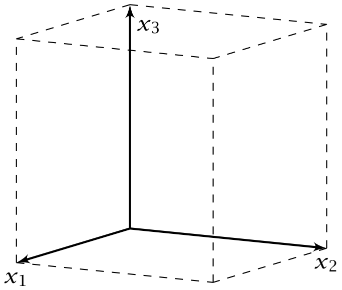

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The feasible region has 2^n vertices.

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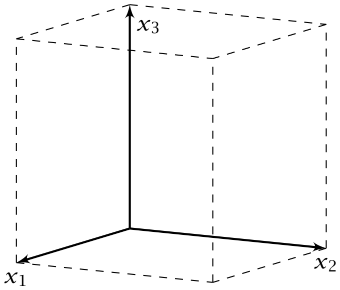

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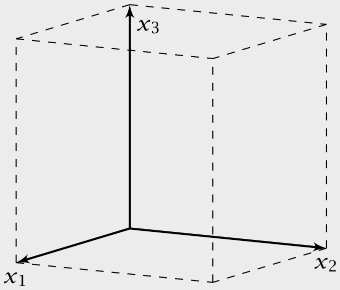

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Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

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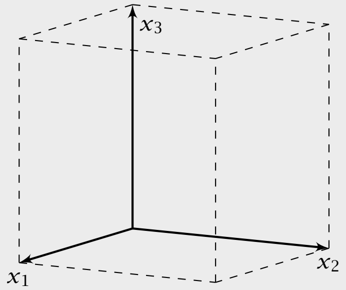

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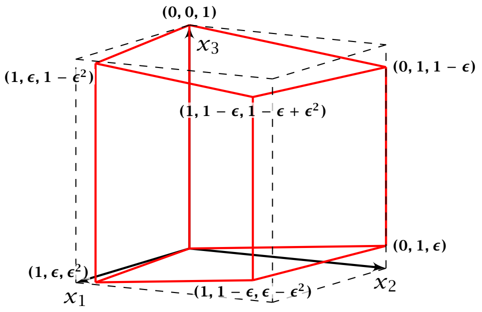

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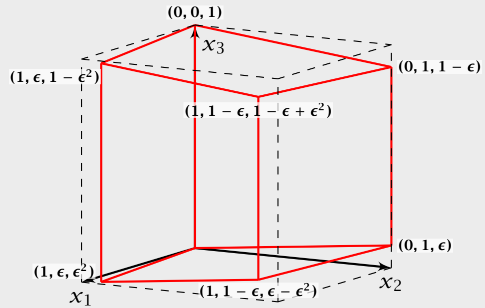

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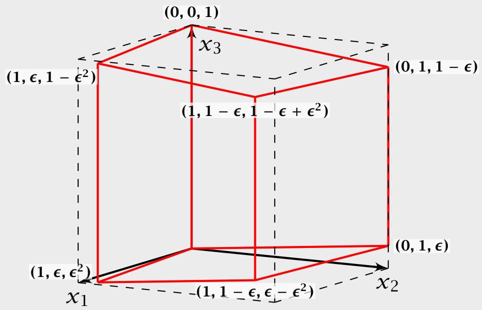

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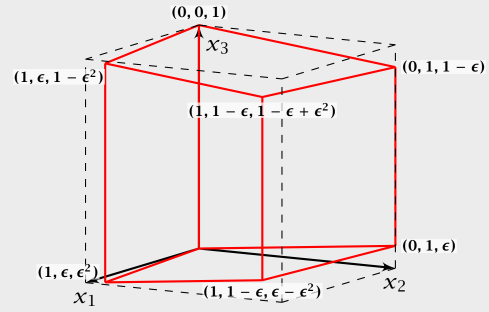

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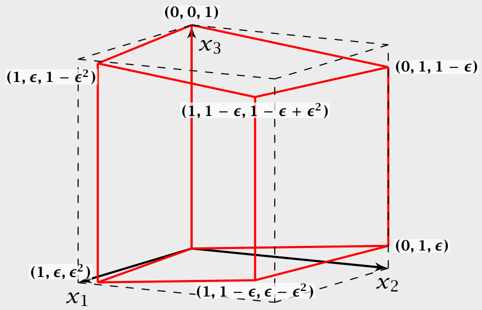

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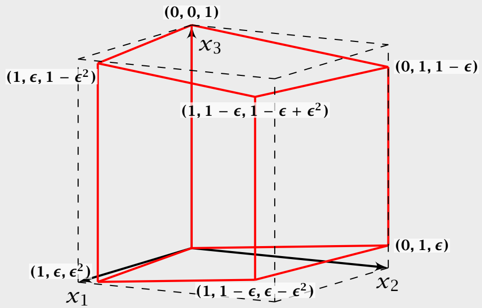

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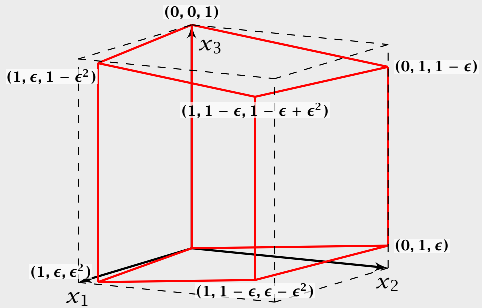

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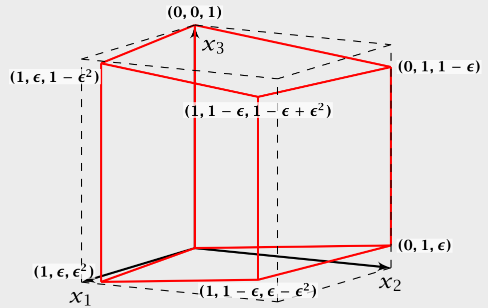

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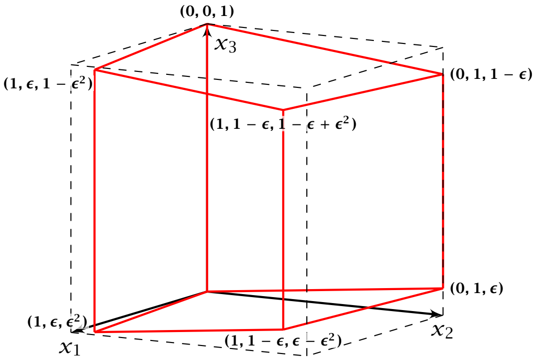

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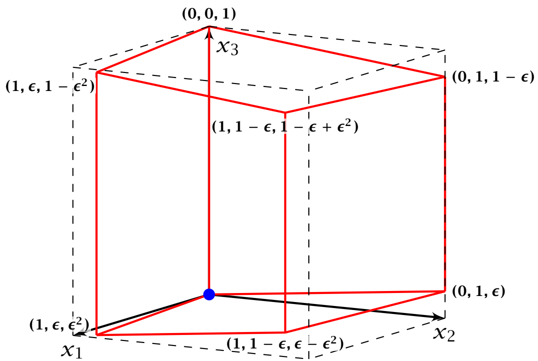

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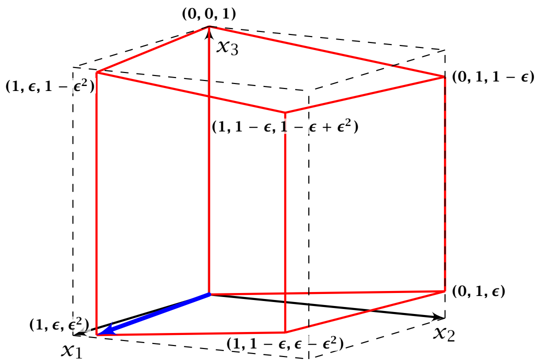

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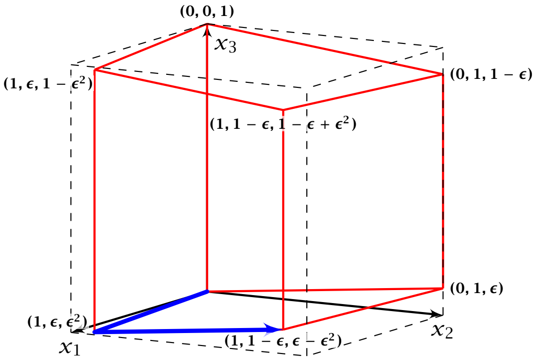

Analysis

- In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
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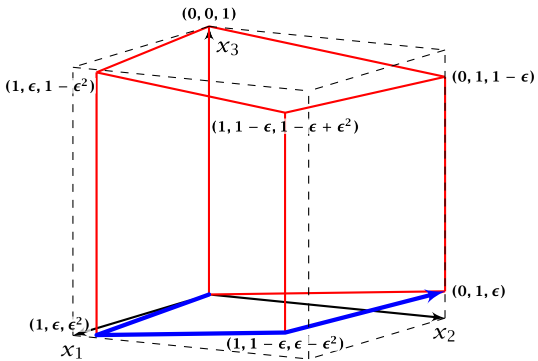

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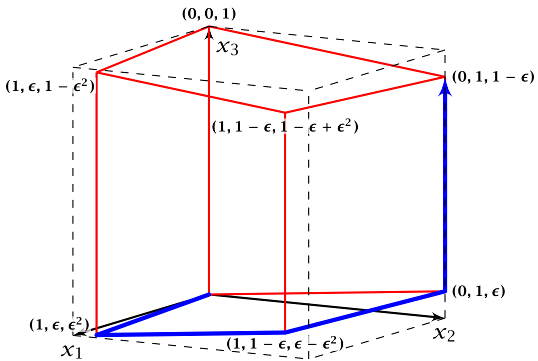

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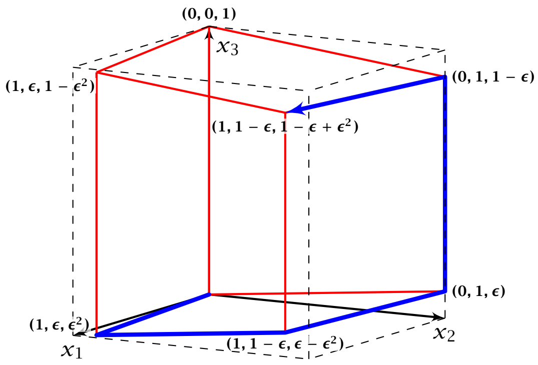

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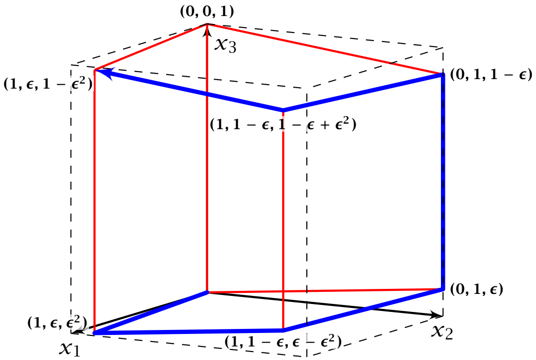

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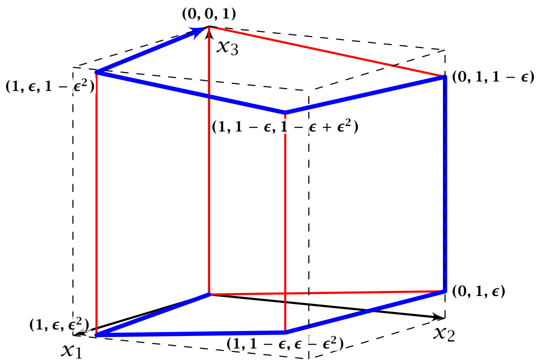

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Analysis

The sequence S_n that visits every node of the hypercube is defined recursively

$$
\begin{array}{c}
(0,\dots,0,0,0) \\
\rightsquigarrow S_{n-1} \\
(0,\dots,0,1,0) \\
\downarrow \\
(0,\dots,0,1,1) \\
\rightsquigarrow S_{n-1}^{\text{rev}} \\
(0,\dots,0,0,1)
\end{array}
\qquad \rightsquigarrow S_n
$$

The non-recursive case is $S_1 = 0 \to 1$

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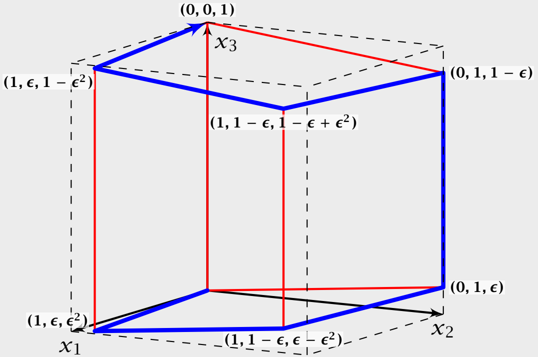

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Lemma 45
The objective value x_n is increasing along path S_n.

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \to 1$, and $1 > 0$.

$n - 1 \to n$

- For the first part the value of $x_n = \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence, also x_n.
- Going from $(0, \ldots, 0, 1, 0)$ to $(0, \ldots, 0, 1, 1)$ increases x_n for small enough ϵ.
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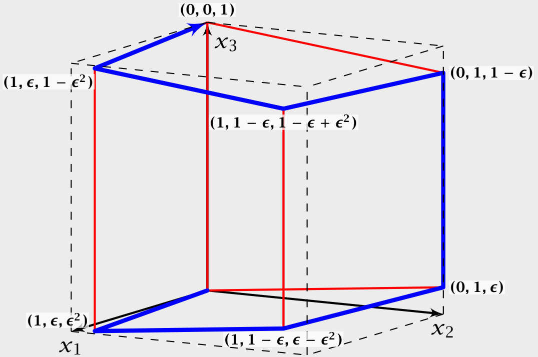

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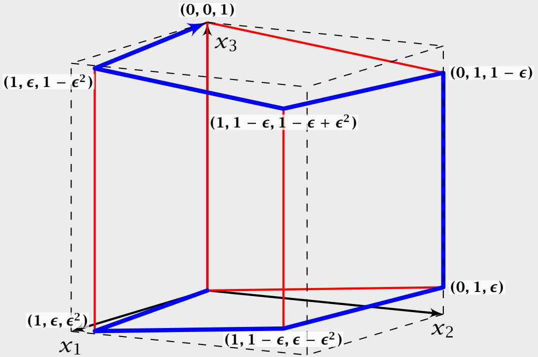

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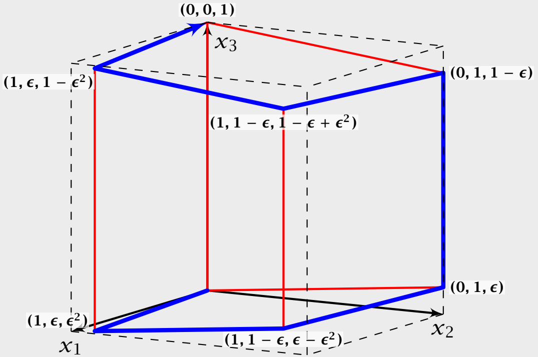

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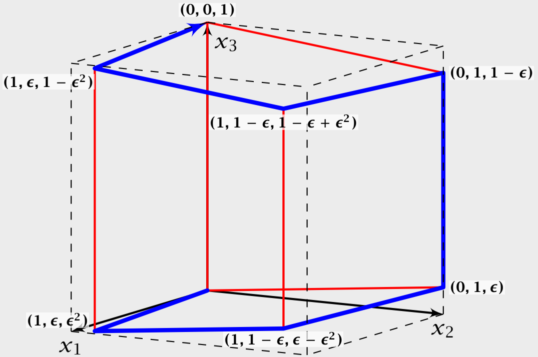

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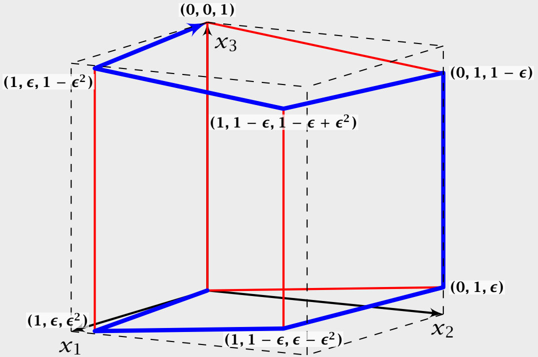

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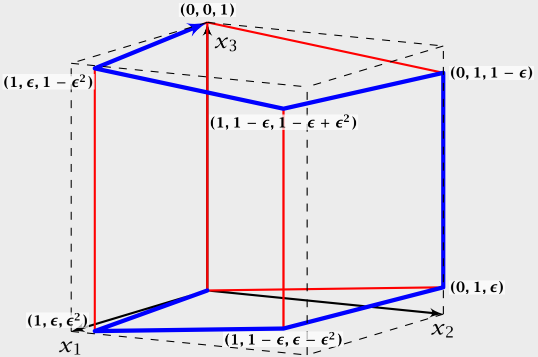

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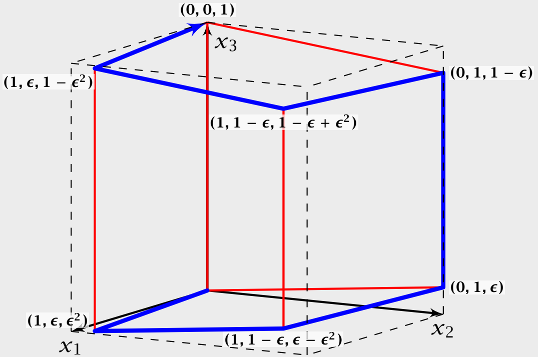

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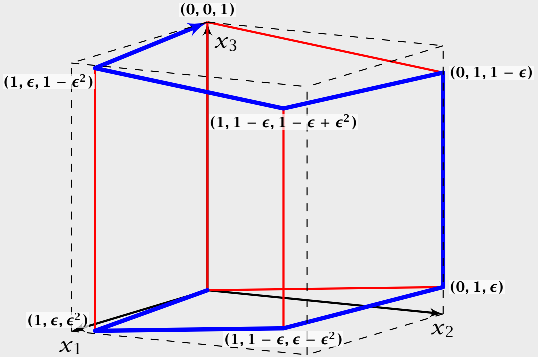

Remarks about Simplex

Observation

The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

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Remarks about Simplex

Conjecture (Hirsch 1957)
The edge-vertex graph of an m-facet polytope in d-dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\text{poly}(m, d))$ is open.

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8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^T x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
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$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

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Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

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Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

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Theorem 46 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_i = \frac{\det(M_j)}{\det(M)} ,$$

where M_i is the matrix obtained from M by replacing the i-th column by the vector b.

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Proof:

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  Note that expanding along the i-th column gives that $\det(X_i) = x_i$.
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Let Z be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or c. Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j-th column with vector $\bar{b}$ (for some j).

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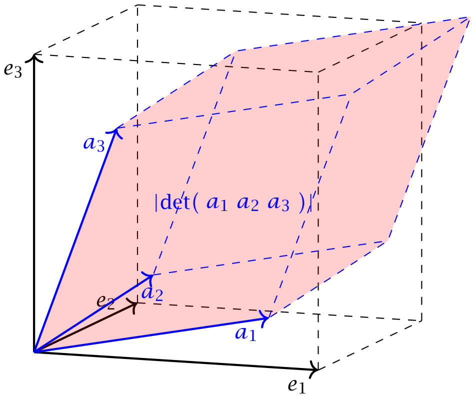

Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

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Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

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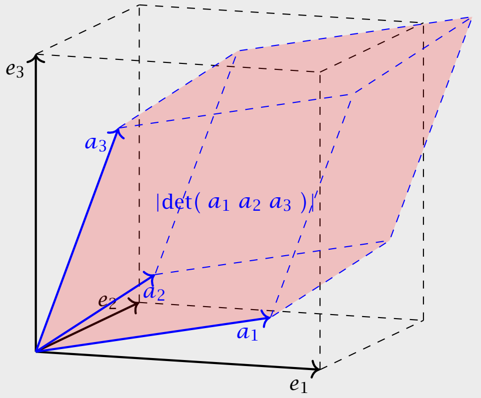

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Compute an optimum basis for the new LP.

- If the cost is $c^Tx = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
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In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^Tx \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes c^Tx over all feasible points.

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if $\mathcal{H} = \emptyset$ **then** return x on implicit constraint hyperplane
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// optimal solution fulfills h with equality, i.e., $a_h^T x = b_h$
solve $a_h^T x = b_h$ for some variable x_ℓ;
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In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^T x \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes $c^T x$ over all feasible points.

In addition it obeys the implicit constraint
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- If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(\max\{m, 1\})$.
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$$T(m,d) = \begin{cases} \mathcal{O}(\max\{1,m\}) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1,d) + & \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1,d-1)) & \text{otw.} \end{cases}$$

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$$T(m,d) = \begin{cases} C\max\{1,m\} & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m-1,d) + & \\ \frac{d}{m}(Cdm + T(m-1,d-1)) & \text{otw.} \end{cases}$$

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Complexity

LP Feasibility Problem (LP feasibility)

Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
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Size of a Basic Feasible Solution

Lemma 47
Let $M \in \mathbb{Z}^{m \times m}$ *be an invertible matrix and let* $b \in \mathbb{Z}^m$*. Further define* $L = \langle M \rangle + \langle b \rangle + n \log_2 n$*. Then a solution to* $Mx = b$ *has rational components* x_j *of the form* $\frac{D_j}{D}$*, where* $|D_j| \le 2^L$ *and* $|D| \le 2^L$*.*

Proof:
Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

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Let $X = A_B$. Then

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Size of a Basic Feasible Solution

Lemma 47

Let $M \in \mathbb{Z}^{m \times m}$ be an invertible matrix and let $b \in \mathbb{Z}^m$. Further define $L = \langle M \rangle + \langle b \rangle + n \log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^L$ and $|D| \le 2^L$.

Proof:

Cramers rules says that we can compute x_j as

$$
x_j = \frac{\det(M_j)}{\det(M)}
$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

Bounding the Determinant

Let $X = A_B$. Then

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\begin{aligned}
|\det(X)| &= \left| \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{1 \le i \le n} X_{i\pi(i)} \right| \\
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Analogously for $\det(M_j)$.

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$$\log_2\left(\frac{2n2^{2L'}}{1/2^{L'}}\right) = \mathcal{O}(L') \ ,$$

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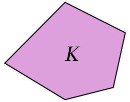

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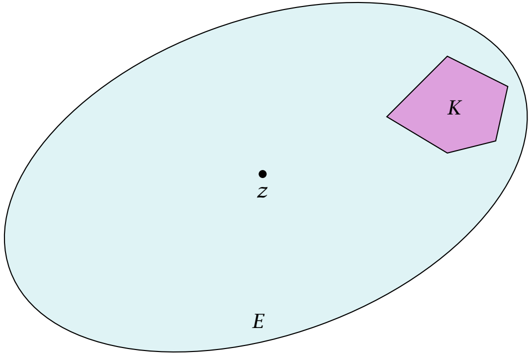

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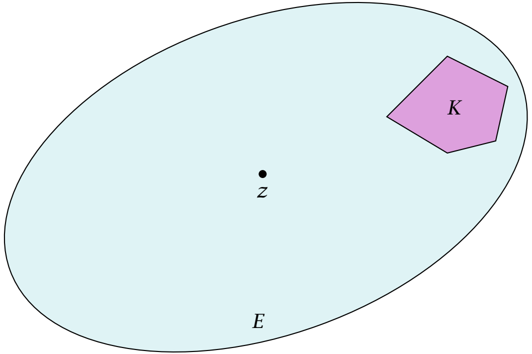

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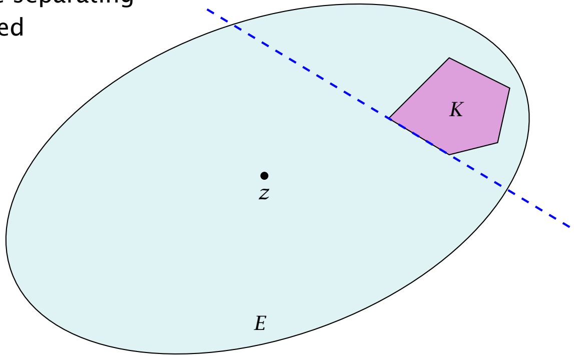

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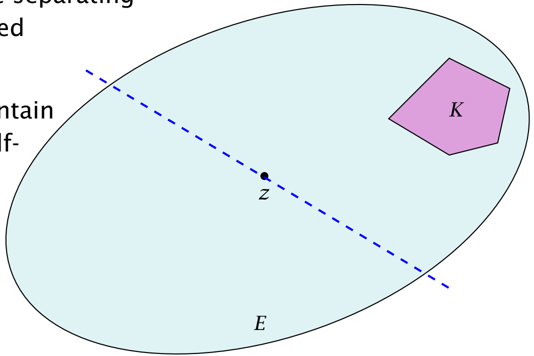

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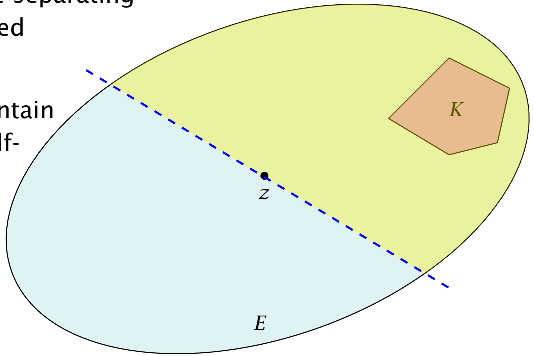

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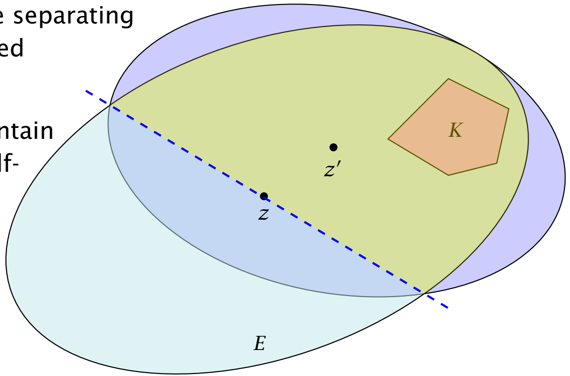

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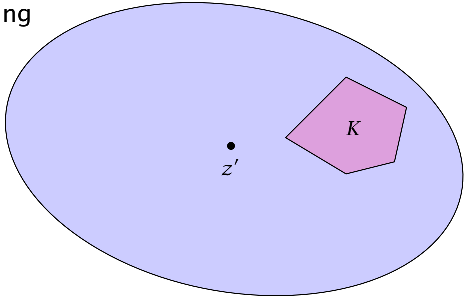

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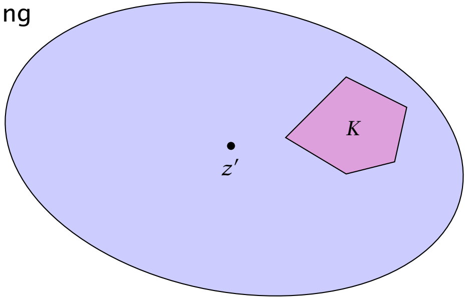

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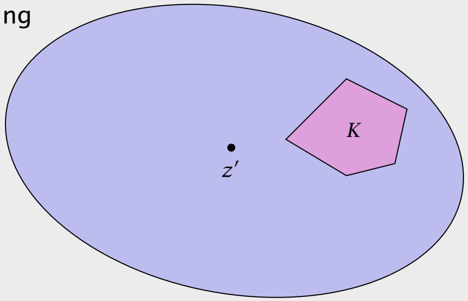

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A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

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$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

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An affine transformation of the unit ball is called an ellipsoid.

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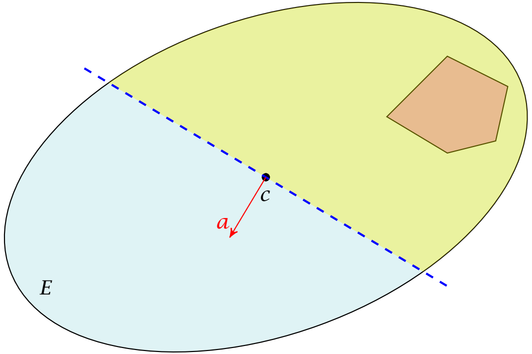
How to Compute the New Ellipsoid



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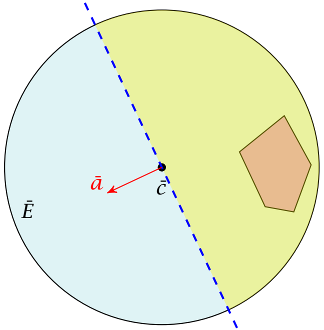

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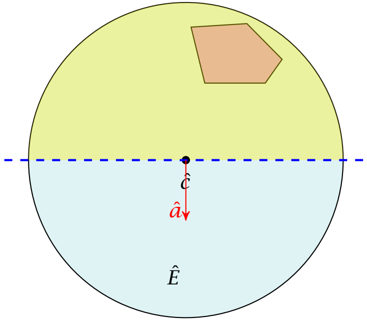

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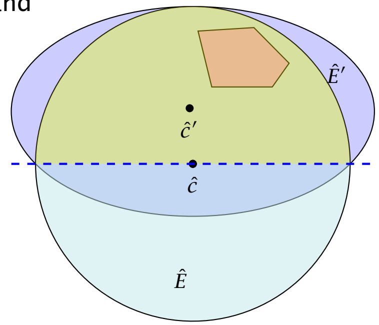

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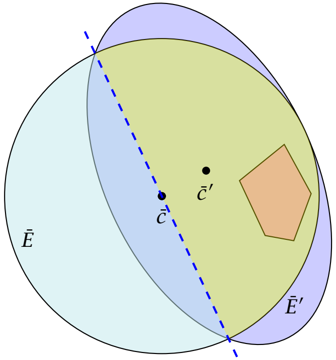

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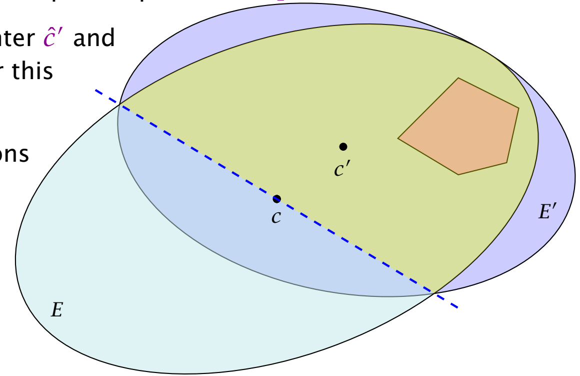

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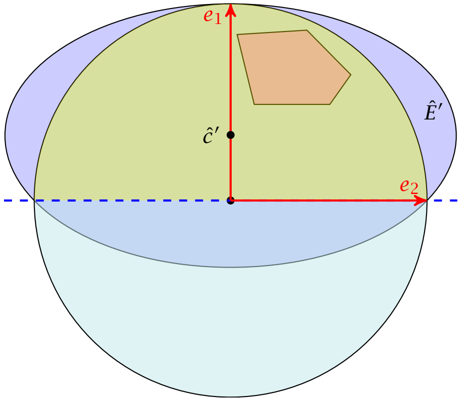

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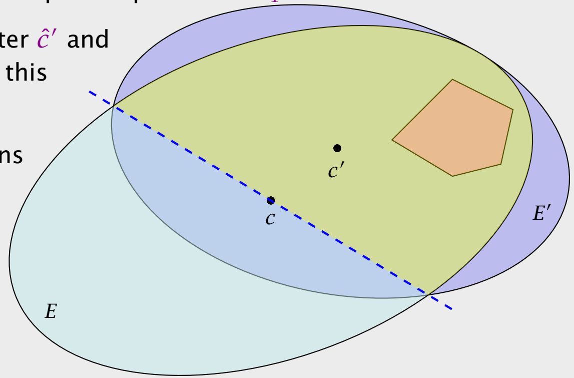

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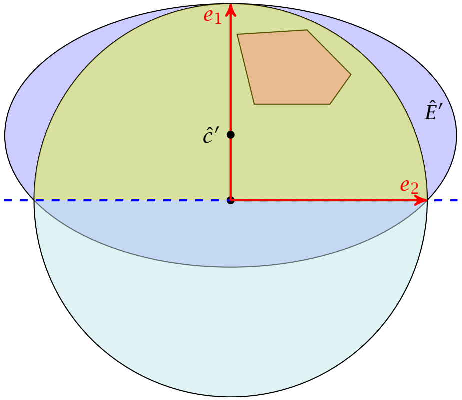

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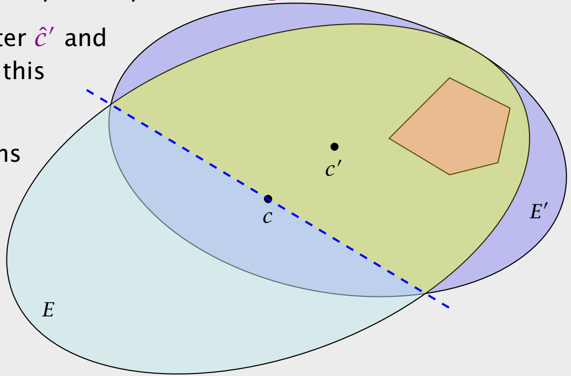

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- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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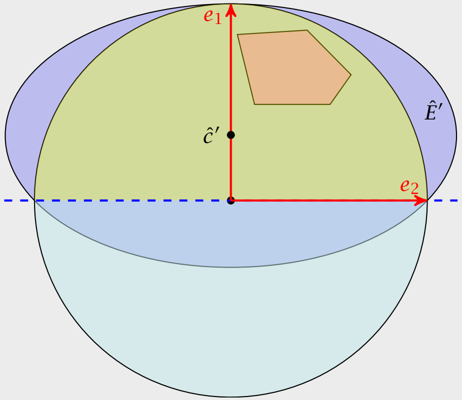

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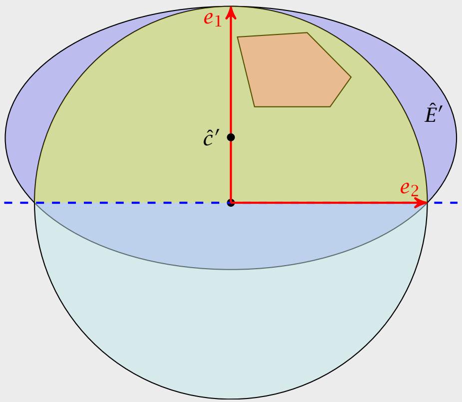

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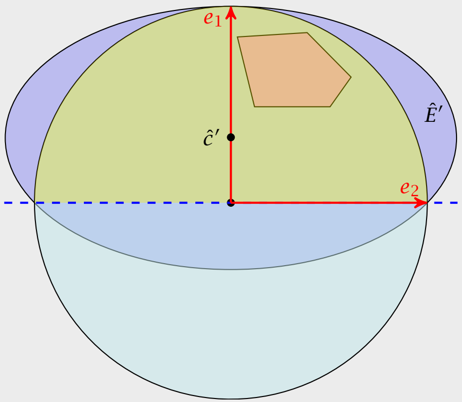

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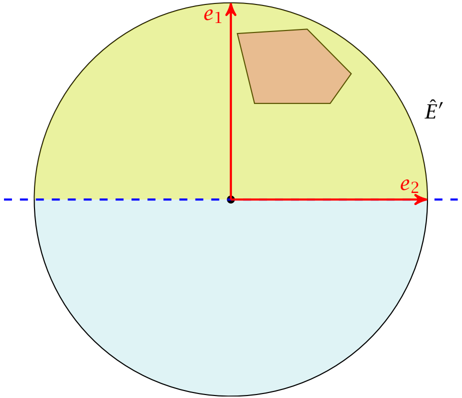

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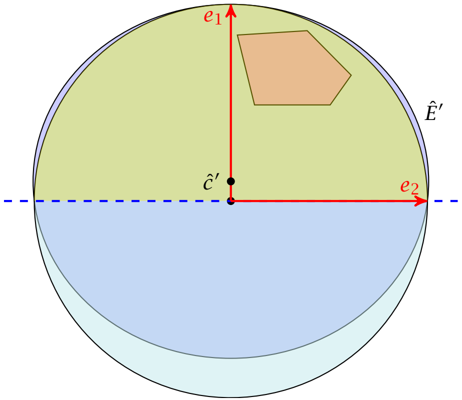

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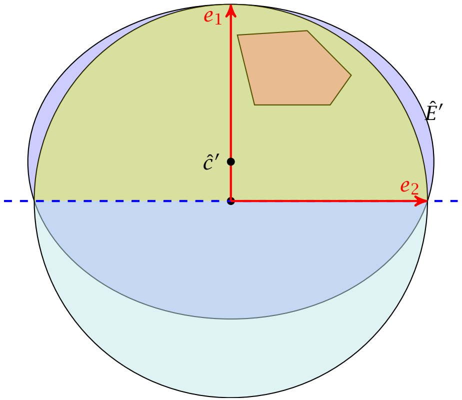

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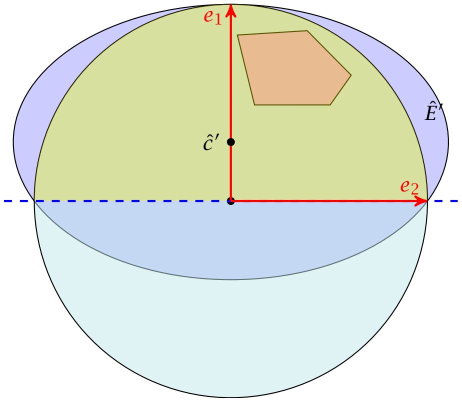

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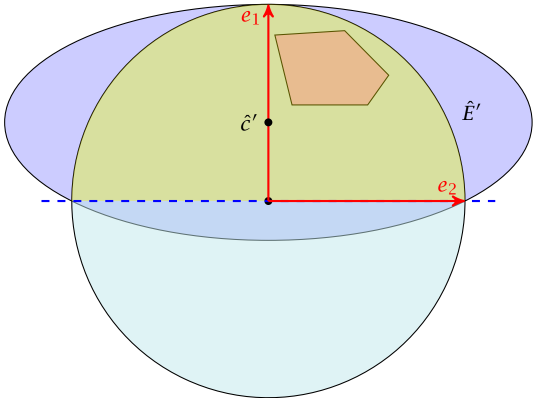

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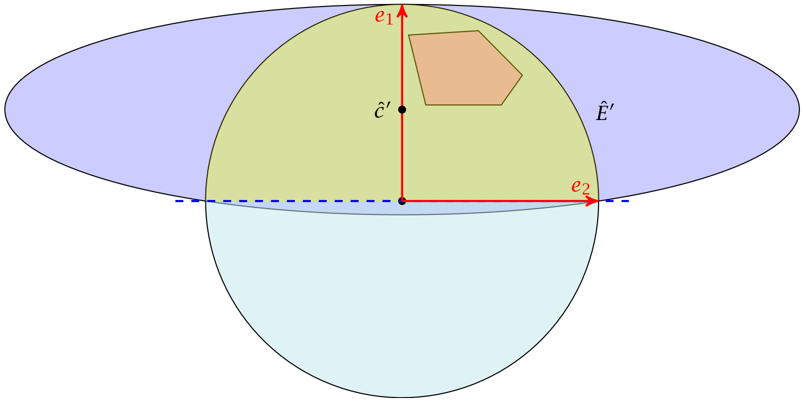

Choose t such that the volume of $\hat{E}'$ is minimal!!!

Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1-t}{\sqrt{1-2t}}$$

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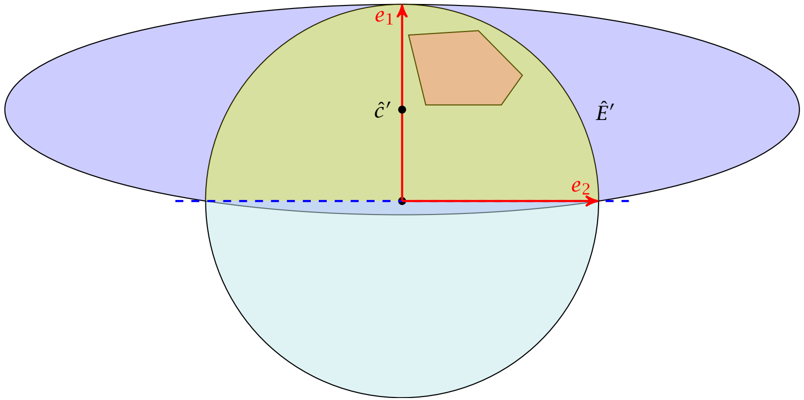

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We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \ .$$

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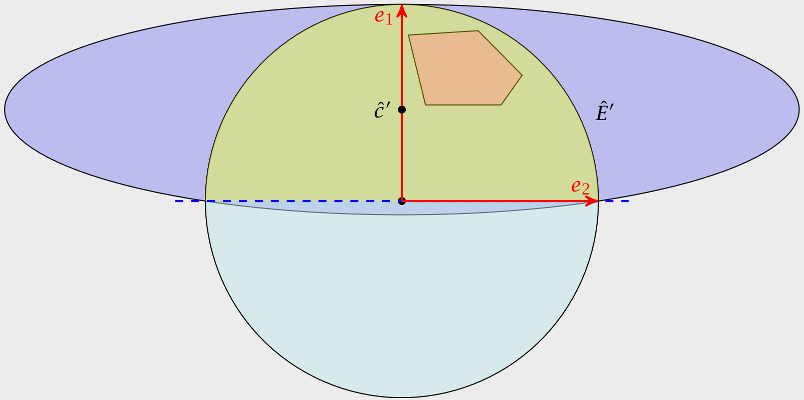

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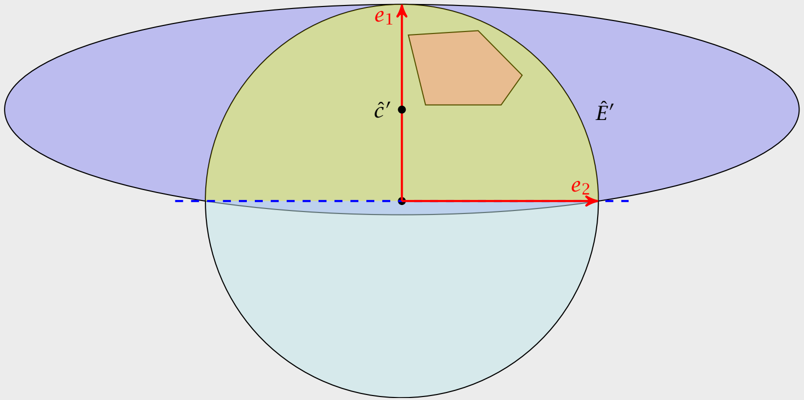

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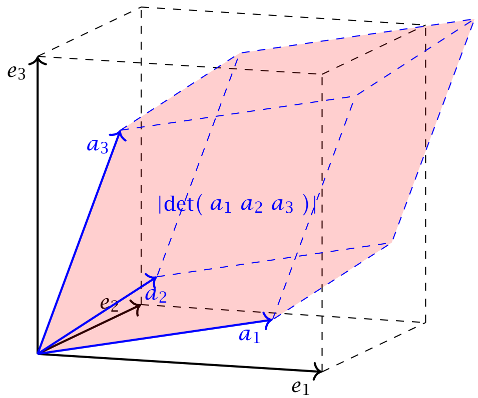
n-dimensional volume



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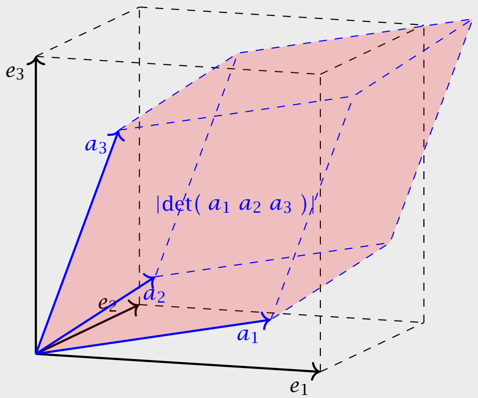

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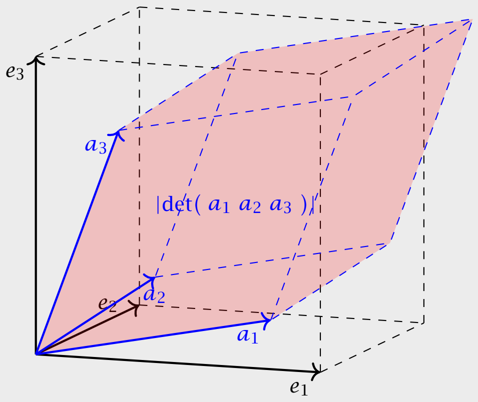

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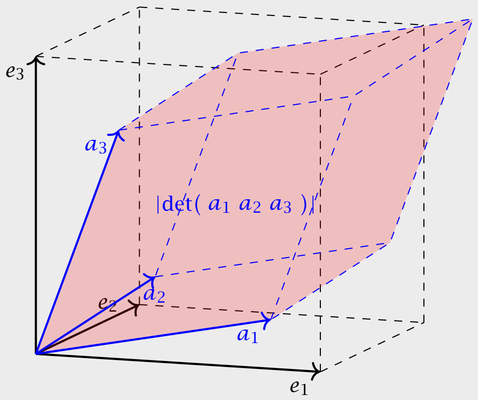

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$$= \frac{\Phi}{N^2}$$

N = denominator

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outer derivative

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&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)
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The Easy Case

$$
\begin{aligned}
\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1))\cdot|\det(\hat{L}')|\\
&= \mathrm{vol}(B(0,1))\cdot ab^{n-1}\\
&= \mathrm{vol}(B(0,1))\cdot(1-t)\cdot\left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1}\\
&= \mathrm{vol}(B(0,1))\cdot\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}
\end{aligned}
$$

We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

The Easy Case

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right) \\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}} \\
&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}
$$

The Easy Case

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\end{aligned}
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We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a$$

The Easy Case

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\
&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^{n}}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t$$

The Easy Case

$$
\begin{aligned}
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&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\
&\quad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}
$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t = \frac{n}{n+1}$$

The Easy Case

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}^{1-2t}\cdot\cancel{(\sqrt{1-2t})^{n-1}}\\
&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\cancel{(1-t)^n}^{1-t}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
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&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overset{1-t}{\cancel{(1-t)^{n}}}\Bigg) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
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&\quad\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overset{1-t}{\cancel{(1-t)^{n}}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
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\begin{aligned}
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&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
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To see the equation for b, observe that

$$b^2$$

The Easy Case

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
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&\quad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overbrace{\cancel{(1-t)^n}}^{1-t}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
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$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}} = \frac{n}{\sqrt{n^2-1}}$$

To see the equation for b, observe that

$$b^2 = \frac{(1-t)^2}{1-2t}$$

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$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
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To see the equation for b, observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}}$$

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&\qquad \cancel{+}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
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\end{aligned}$$

The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$\gamma_n^2$$

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$$= \Big(1 - \frac{1}{n+1}\Big)^2\Big(1 + \frac{1}{(n-1)(n+1)}\Big)^{n-1}$$

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where we used $(1+x)^a \le e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

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This gives $\gamma_n \le e^{-\frac{1}{2(n+1)}}$.

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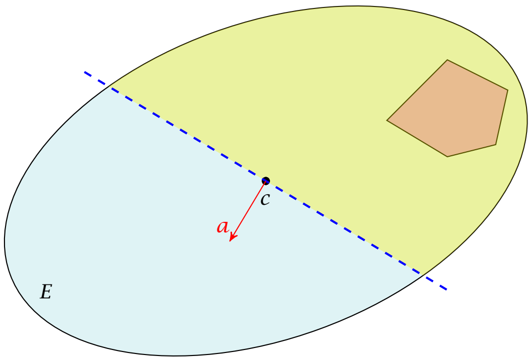
How to Compute the New Ellipsoid



The Easy Case

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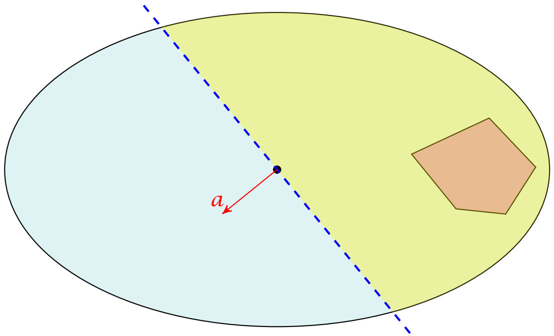
$$\begin{aligned}\gamma_n^2 &= \Big(\frac{n}{n+1}\Big)^2\Big(\frac{n^2}{n^2-1}\Big)^{n-1}\\ &= \Big(1-\frac{1}{n+1}\Big)^2\Big(1+\frac{1}{(n-1)(n+1)}\Big)^{n-1}\\ &\le e^{-2\frac{1}{n+1}}\cdot e^{\frac{1}{n+1}}\\ &= e^{-\frac{1}{n+1}}\end{aligned}$$

where we used $(1+x)^a \le e^{ax}$ for $x\in\mathbb{R}$ and $a>0$.

This gives $\gamma_n \le e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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\begin{aligned}
\gamma_n^2 &= \Big(\frac{n}{n+1}\Big)^2 \Big(\frac{n^2}{n^2-1}\Big)^{n-1} \\
&= \Big(1 - \frac{1}{n+1}\Big)^2 \Big(1 + \frac{1}{(n-1)(n+1)}\Big)^{n-1} \\
&\le e^{-2\frac{1}{n+1}} \cdot e^{\frac{1}{n+1}} \\
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where we used $(1+x)^a \le e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

This gives $\gamma_n \le e^{-\frac{1}{2(n+1)}}$.

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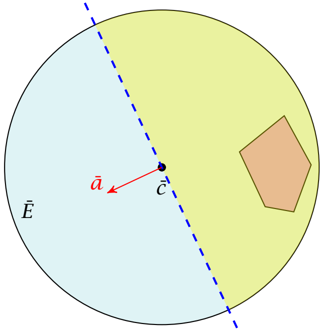

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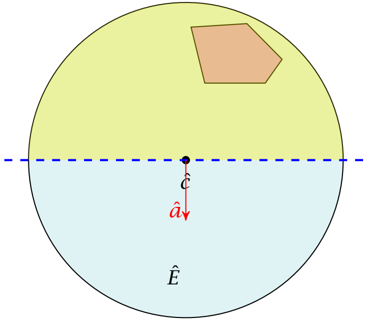

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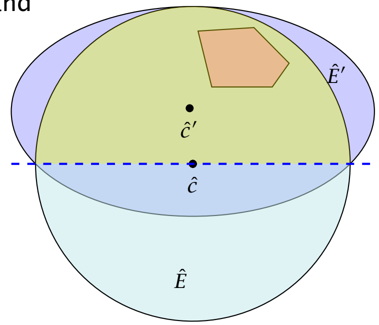

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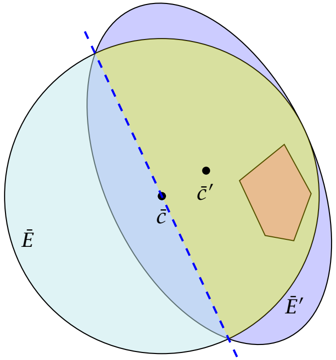

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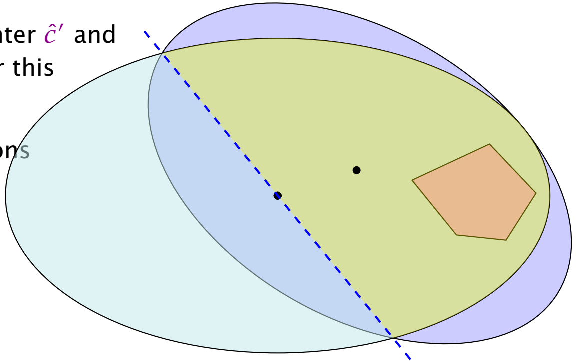

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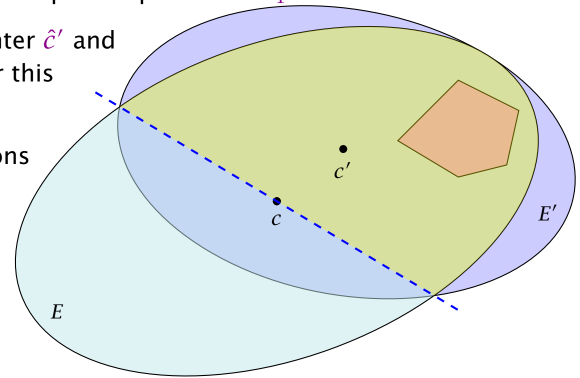

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$$e^{-\frac{1}{2(n+1)}}$$

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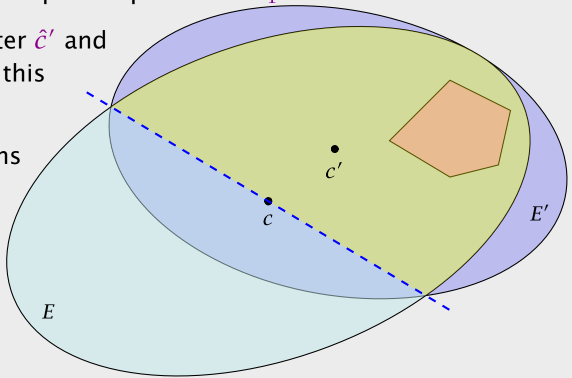

Our progress is the same:

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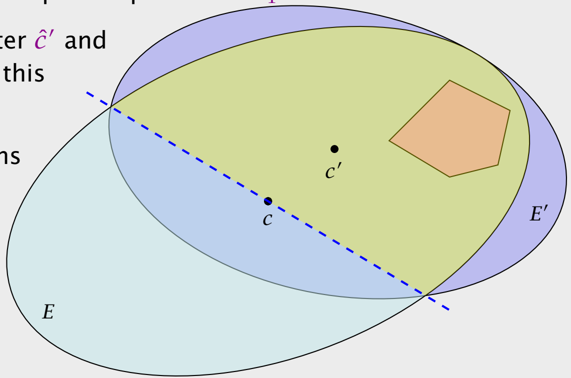

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$$e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})}$$

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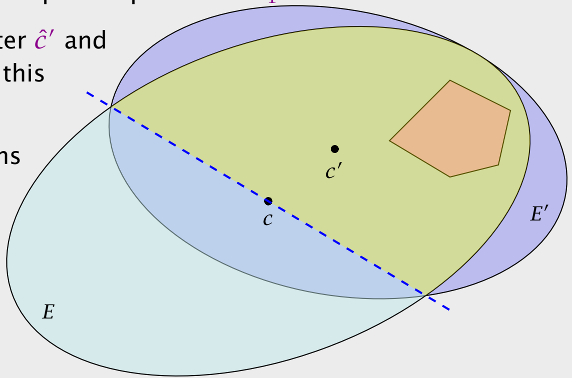

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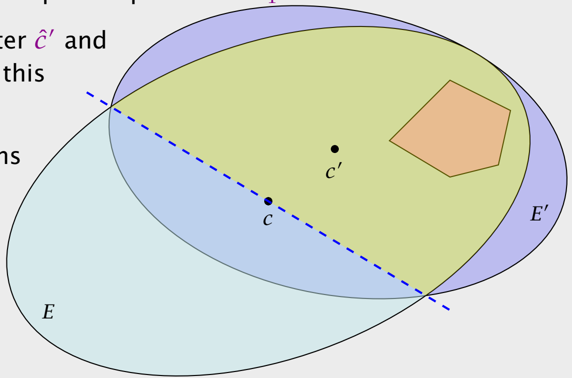

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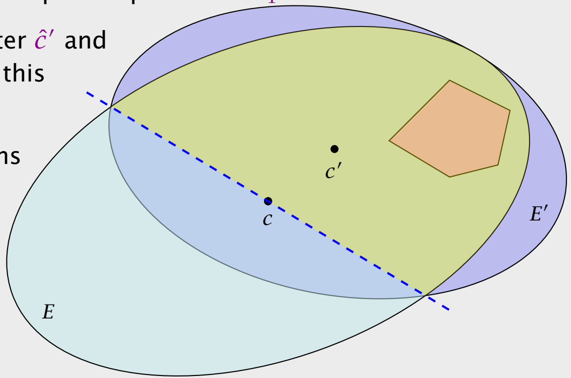

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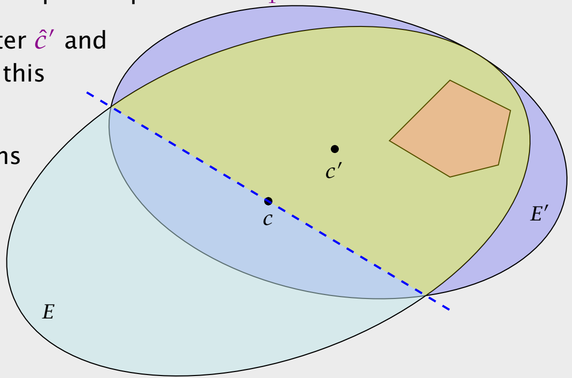

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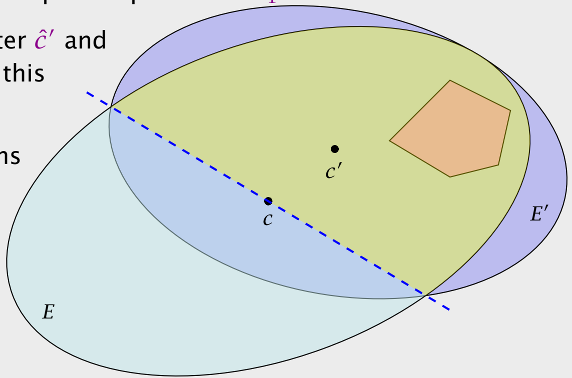

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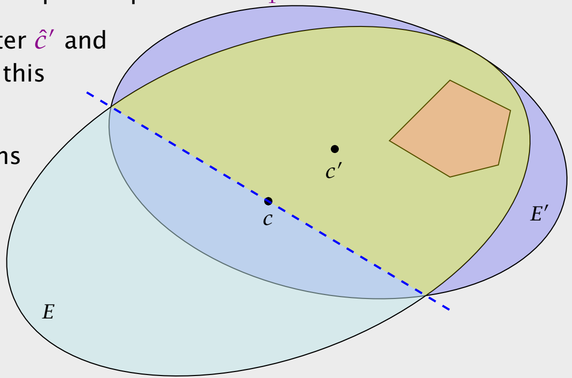

The Ellipsoid Algorithm

How to Compute The New Parameters?

Our progress is the same:

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The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

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Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

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This means $\bar{a} = L^T a$.

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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

The Ellipsoid Algorithm

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The Ellipsoid Algorithm

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Hence,

$$\bar{c}'$$

The Ellipsoid Algorithm

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The Ellipsoid Algorithm

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This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

The Ellipsoid Algorithm

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The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

$$c'$$

The Ellipsoid Algorithm

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The Ellipsoid Algorithm

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$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

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$$c' = f(\bar{c}')$$

The Ellipsoid Algorithm

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$$c' = f(\bar{c}') = L \cdot \bar{c}' + c$$

The Ellipsoid Algorithm

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$$\begin{aligned} c' &= f(\bar{c}') = L\cdot\bar{c}' + c \\ &= -\frac{1}{n+1}L\frac{L^Ta}{\|L^Ta\|} + c \end{aligned}$$

The Ellipsoid Algorithm

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The Ellipsoid Algorithm

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$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\ &= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}} \end{aligned}$$

The Ellipsoid Algorithm

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For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

The Ellipsoid Algorithm

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$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

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$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\ &= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}} \end{aligned}$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

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$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

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$$b^2 - b^2\frac{2}{n+1} = \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2}$$

$$= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

9 The Ellipsoid Algorithm

$\bar{E}'$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$$\bar{E}' = R(\hat{E}')$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}\end{aligned}$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

$$\begin{aligned} b^2 - b^2 \frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\} \\ &= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1} y \le 1\}\end{aligned}$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

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9 The Ellipsoid Algorithm

$$
\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\} \\
&= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1} y \le 1\}
\end{aligned}
$$

Recall that

$$
\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}
$$

This gives

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\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)
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9 The Ellipsoid Algorithm

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Recall that

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9 The Ellipsoid Algorithm

Hence,

$\bar{Q}'$

9 The Ellipsoid Algorithm

$$
\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\} \\
&= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$\bar{Q}' = R\hat{Q}'R^T$$

9 The Ellipsoid Algorithm

$$\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\} \\
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\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^T \\ &= R\cdot\frac{n^2}{n^2-1}\Big(I-\frac{2}{n+1}e_1e_1^T\Big)\cdot R^T\end{aligned}$$

9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

Hence,

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\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}e_1e_1^T\right) \cdot R^T \\
&= \frac{n^2}{n^2-1}\left(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\right)
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$
\begin{aligned}
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&= \{R(x) \mid x^T\hat{Q}'^{-1}x \le 1\} \\
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9 The Ellipsoid Algorithm

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&= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\
&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

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&= \{y \mid y^T(R^T)^{-1}\hat{Q}'^{-1}R^{-1}y \leq 1\} \\
&= \{y \mid y^T(\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1}y \leq 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

E'

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T \\
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\end{aligned}
$$

9 The Ellipsoid Algorithm

$$E' = L(\bar{E}')$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T \\
&= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\
&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)
\end{aligned}$$

9 The Ellipsoid Algorithm

$$
\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\}
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$$

9 The Ellipsoid Algorithm

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&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{\|L^T a\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\ &= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} \bar{Q}' &= R\hat{Q}'R^T \\ &= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big) \cdot R^T \\ &= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\ &= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{\|L^T a\|^2}\Big) \end{aligned}$$

9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

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&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big) \cdot R^T \\
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&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} \frac{L^T a a^T L}{\|L^T a\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$Q'$$

9 The Ellipsoid Algorithm

$$\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$Q' = L\bar{Q}'L^T$$

9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\ &= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\ &= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\ &= \{y \mid y^T \underbrace{(L\bar{Q}'L^T)}_{Q'}{}^{-1} y \le 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
Q' &= L\bar{Q}'L^T \\
&= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{a^T Q a}\Big) \cdot L^T
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$
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E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
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&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} Q' &= L\bar{Q}'L^T \\ &= L\cdot\frac{n^2}{n^2-1}\Big(I-\frac{2}{n+1}\,\frac{L^Taa^TL}{a^TQa}\Big)\cdot L^T \\ &= \frac{n^2}{n^2-1}\Big(Q-\frac{2}{n+1}\,\frac{Qaa^TQ}{a^TQa}\Big) \end{aligned}$$

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Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "K is empty"
$Q \leftarrow$???
repeat
  if $c \in K$ **then return** c
  else
    choose a violated hyperplane a
    $$c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$$
    $$Q \leftarrow \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big)$$
  endif
until ???
return "K is empty"

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}
Q' &= L\bar{Q}'L^T \\
&= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{a^TQa}\Big) \cdot L^T \\
&= \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big)
\end{aligned}$$

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a bounded polyhedron. Let $\langle a_{\max}\rangle$ be the maximum encoding length of an entry in A, b. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \le b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

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  endif
until ???
return "K is empty"

Repeat: Size of basic solutions

Proof:
Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, b, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by b) can become at most

$$\begin{aligned}\det(\bar{A}_B), \det(\bar{M}_j) &\le \|\vec{\ell}_{\max}\|^{2n} \\ &\le (\sqrt{2n} \cdot 2^{\langle a_{\max}\rangle})^{2n} \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} \ ,\end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

Repeat: Size of basic solutions

Lemma 52
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How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \operatorname{vol}(B(0,1)) \leq (n\delta)^n \operatorname{vol}(B(0,1))$.

Repeat: Size of basic solutions

Proof:

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, b, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by b) can become at most

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$$\begin{aligned} \det(\bar{A}_B), \det(\bar{M}_j) &\leq \|\vec{\ell}_{\max}\|^{2n} \\ &\leq (\sqrt{2n} \cdot 2^{\langle a_{\max}\rangle})^{2n} \leq 2^{2n\langle a_{\max}\rangle + 2n \log_2 n} , \end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \le \delta$.

Hence, P is contained in the cube $-\delta \le x_i \le \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

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Repeat: Size of basic solutions

Proof:

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When can we terminate?

Let $P := \{x \mid Ax \le b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \le b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\} ,$$

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(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1} b)_i < 0 \le (\bar{A}_B^{-1} b)_i + \frac{1}{\lambda} (\bar{A}_B^{-1} \vec{1})_i$$

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Lemma 54

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}}\mathrm{vol}(B(0,1))$.

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By Cramers rule we get

$$(\bar{A}_B^{-1}b)_i < 0 \quad \Longrightarrow \quad (\bar{A}_B^{-1}b)_i \le -\frac{1}{\det(\bar{A}_B)}$$

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$$(\bar{A}_B^{-1}\vec{1})_i \le \det(\bar{M}_j) \enspace ,$$

where $\bar{M}_j$ is obtained by replacing the j-th column of $\bar{A}_B$ by $\vec{1}$.

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If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}}\mathrm{vol}(B(0,1))$.

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Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \mathrm{diag}(R^2, \ldots, R^2)$ // i.e., $L = \mathrm{diag}(R, \ldots, R)$
repeat
  if $c \in K$ **then return** c
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    $c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$
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Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
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We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
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We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
- an initial ball $B(c, R)$ with radius R that contains K,
- a separation oracle for K.

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
    with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \text{diag}(R^2, \ldots, R^2)$ // i.e., $L = \text{diag}(R, \ldots, R)$
repeat
    if $c \in K$ **then return** c
    else
        choose a violated hyperplane a

$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$

$$Q \leftarrow \frac{n^2}{n^2 - 1} \Big(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \Big)$$

    endif
until $\det(Q) \leq r^{2n}$ // i.e., $\det(L) \leq r^n$
return "K is empty"

Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
- or finds a hyperplane separating x from K.

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10 Karmarkars Algorithm

- inequalities $Ax \leq b$; $m \times n$ matrix A with rows a_i^T
- $P = \{x \mid Ax \leq b\}$; $P^\circ := \{x \mid Ax < b\}$
- interior point algorithm: $x \in P^\circ$ throughout the algorithm
- for $x \in P^\circ$ define
$$s_i(x) := b_i - a_i^T x$$
as the slack of the i-th constraint

logarithmic barrier function:

$$\phi(x) = -\sum_{i=1}^{m} \log(s_i(x))$$

Penalty for point x; points close to the boundary have a very large penalty.

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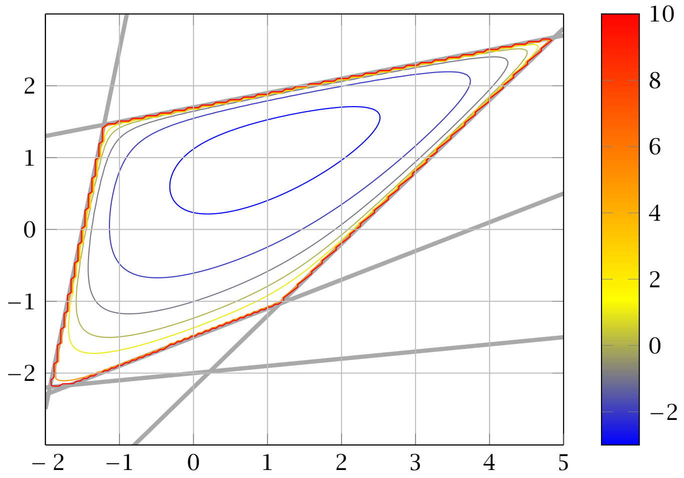

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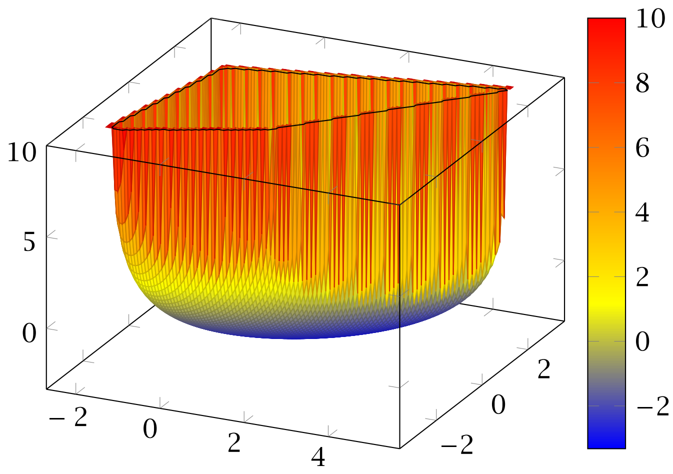

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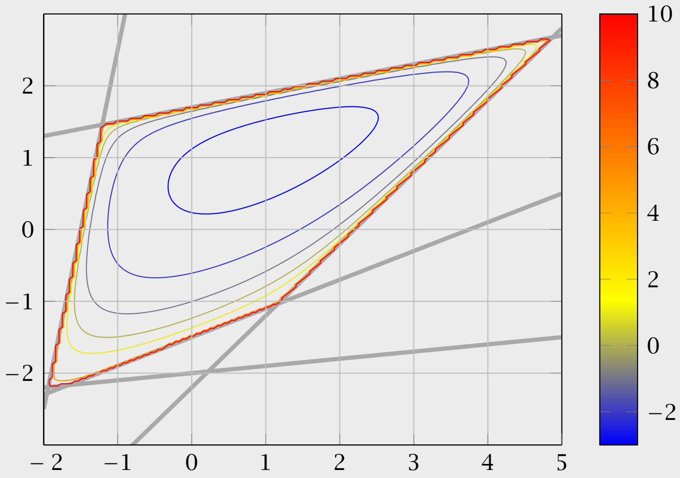

Gradient and Hessian

Taylor approximation:

$$\phi(x+\epsilon) \approx \phi(x) + \nabla\phi(x)^T\epsilon + \frac{1}{2}\epsilon^T\nabla^2\phi(x)\epsilon$$

Gradient:

$$\nabla\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)} \cdot a_i = A^T d_x$$

where $d_x^T = (1/s_1(x), \dots, 1/s_m(x))$. ($d_x$ vector of inverse slacks)

Hessian:

$$H_x := \nabla^2\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)^2} a_i a_i^T = A^T D_x^2 A$$

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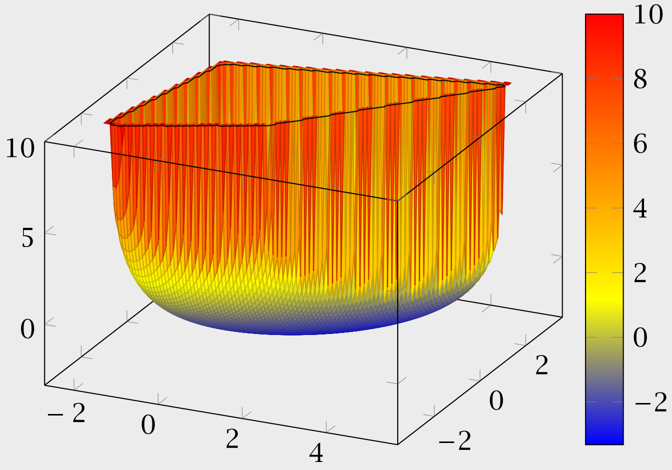

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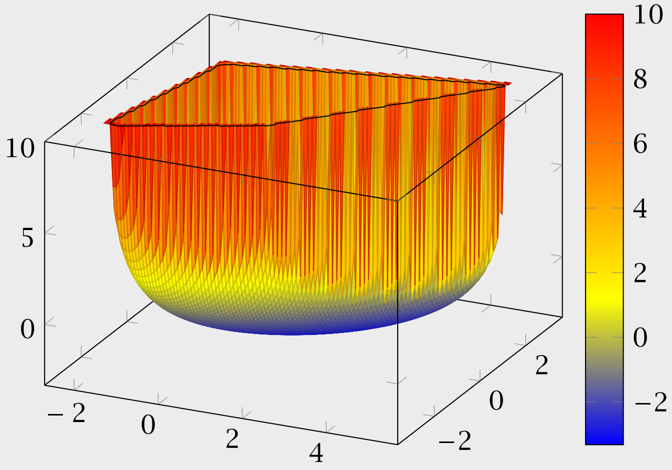

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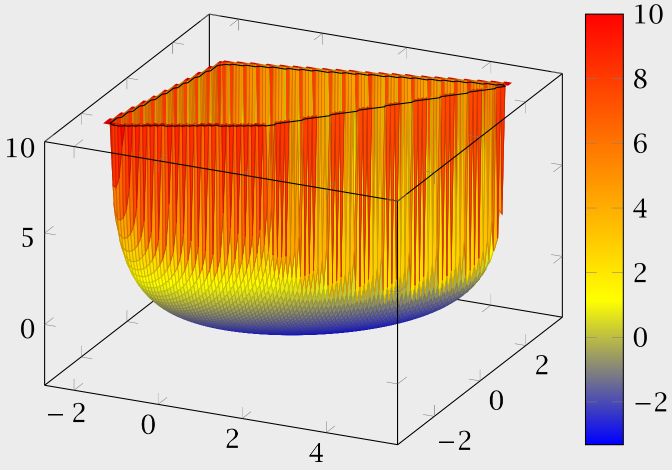

Proof for Gradient

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x_i} &= \frac{\partial}{\partial x_i}\Big(-\sum_r \ln(s_r(x))\Big) \\ &= -\sum_r \frac{\partial}{\partial x_i}\Big(\ln(s_r(x))\Big) = -\sum_r \frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(s_r(x)\Big) \\ &= -\sum_r \frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(b_r - a_r^T x\Big) = \sum_r \frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(a_r^T x\Big) \\ &= \sum_r \frac{1}{s_r(x)} A_{ri}\end{aligned}$$

The i-th entry of the gradient vector is $\sum_r 1/s_r(x) \cdot A_{ri}$. This gives that the gradient is

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Proof for Hessian

$$\frac{\partial}{\partial x_j}\left(\sum_r \frac{1}{s_r(x)}A_{ri}\right) = \sum_r A_{ri}\left(-\frac{1}{s_r(x)^2}\right)\cdot\frac{\partial}{\partial x_j}\left(s_r(x)\right)$$
$$= \sum_r A_{ri}\frac{1}{s_r(x)^2}A_{rj}$$

Note that $\sum_r A_{ri}A_{rj} = (A^TA)_{ij}$. Adding the additional factors $1/s_r(x)^2$ can be done with a diagonal matrix.

Hence the Hessian is

$$H_x = A^TD^2A$$

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Properties of the Hessian

H_x is positive semi-definite for $x \in P^\circ$

$$u^T H_x u = u^T A^T D_x^2 A u = \|D_x A u\|_2^2 \geq 0$$

This gives that $\phi(x)$ is convex.

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$$E_x = \{y \mid (y - x)^T H_x (y - x) \le 1\} = \{y \mid \|y - x\|_{H_x} \le 1\}$$

Points in E_x are feasible!!!

In order to become infeasible when going from x to y one of the terms in the sum would need to be larger than 1.

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&= \sum_{i=1}^{m} \frac{(a_i^T(y-x))^2}{s_i(x)^2} \\
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&\le 1
\end{aligned}$$

In order to become infeasible when going from x to y one of the terms in the sum would need to be larger than 1.

Properties of the Hessian

H_x is positive semi-definite for $x \in P^\circ$

$$u^T H_x u = u^T A^T D_x^2 A u = \|D_x A u\|_2^2 \ge 0$$

This gives that $\phi(x)$ is convex.

If $\mathrm{rank}(A) = n$, H_x is positive definite for $x \in P^\circ$

$$u^T H_x u = \|D_x A u\|_2^2 > 0 \text{ for } u \neq 0$$

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$\|u\|_{H_x} := \sqrt{u^T H_x u}$ is a (semi-)norm; the unit ball w.r.t. this norm is an ellipsoid.

Dikin Ellipsoid

$$E_x = \{y \mid (y-x)^T H_x (y-x) \le 1\} = \{y \mid \|y-x\|_{H_x} \le 1\}$$

Points in E_x are feasible!!!

$$\begin{aligned}(y-x)^T H_x (y-x) &= (y-x)^T A^T D_x^2 A (y-x)\\ &= \sum_{i=1}^{m} \frac{(a_i^T(y-x))^2}{s_i(x)^2}\\ &= \sum_{i=1}^{m} \frac{(\text{change of distance to } i\text{-th constraint going from } x \text{ to } y)^2}{(\text{distance of } x \text{ to } i\text{-th constraint})^2}\\ &\le 1\end{aligned}$$

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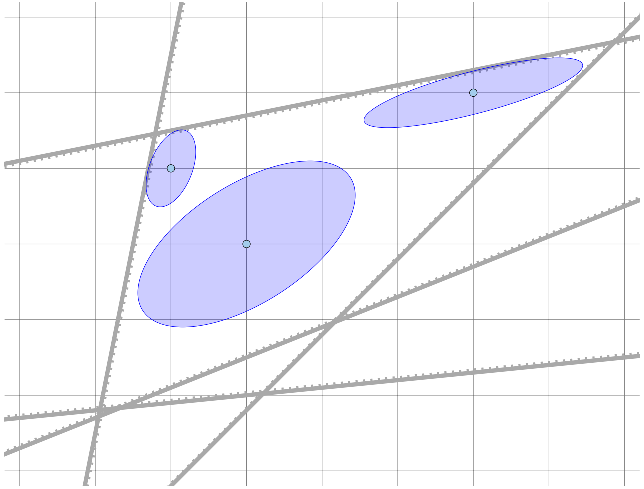

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Analytic Center

$$x_{\text{ac}} := \arg\min_{x \in P^\circ} \phi(x)$$

- x_{ac} is solution to

$$\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)} a_i = 0$$

- depends on the description of the polytope
- x_{ac} exists and is unique iff P° is nonempty and bounded

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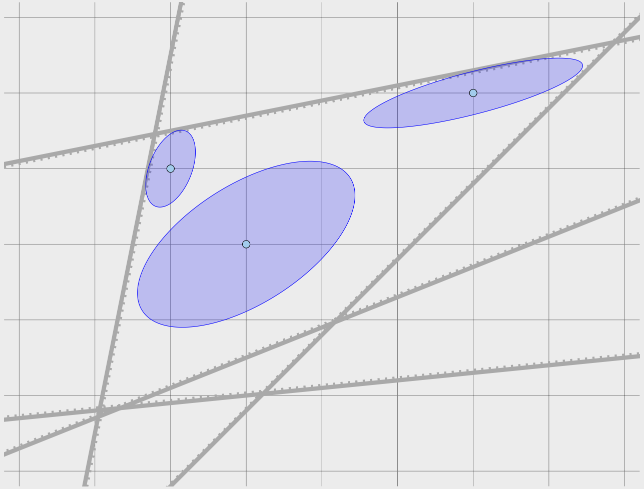

Central Path

In the following we assume that the LP and its dual are strictly feasible and that $\operatorname{rank}(A) = n$.

Central Path:
Set of points $\{x^*(t) \mid t > 0\}$ with

$$x^*(t) = \operatorname{argmin}_x \{tc^T x + \phi(x)\}$$

- $t = 0$: analytic center
- $t = \infty$: optimum solution

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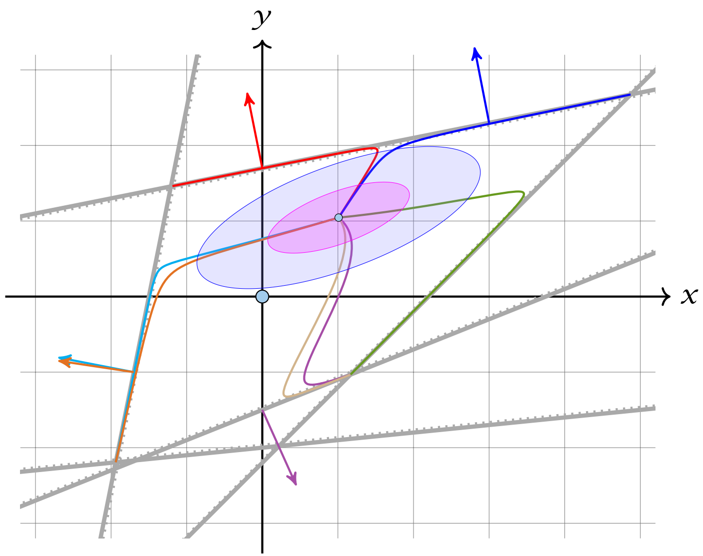
Different Central Paths



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Central Path

Intuitive Idea:
Find point on central path for large value of t. Should be close to optimum solution.

Questions:

- Is this really true? How large a t do we need?
- How do we find corresponding point $x^*(t)$ on central path?

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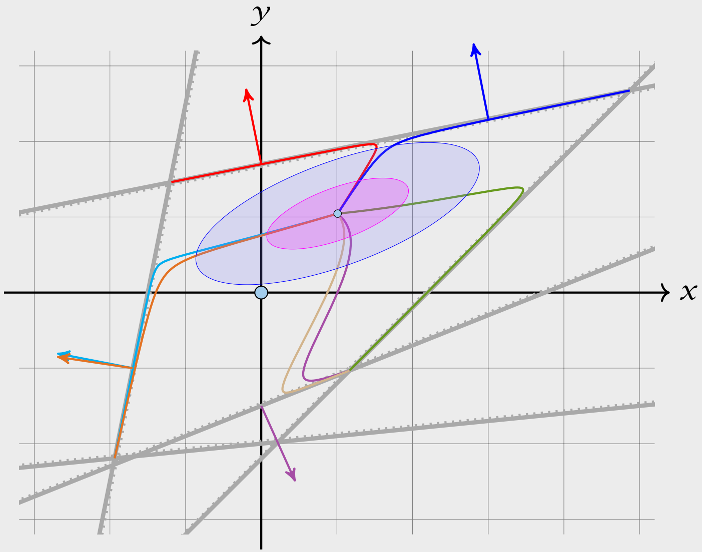

The Dual

primal-dual pair:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \le b \end{aligned}$$

$$\begin{aligned} \max\ & -b^T z \\ \text{s.t.}\ & A^T z + c = 0 \\ & z \ge 0 \end{aligned}$$

Assumptions

- primal and dual problems are strictly feasible;
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Find point on central path for large value of t. Should be close to optimum solution.

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Force Field Interpretation

Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$

- We can view each constraint as generating a repelling force. The combination of these forces is represented by $\nabla\phi(x)$.
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Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$.

This means

$$tc + \sum_{i=1}^{m} \frac{1}{s_i(x^*(t))} a_i = 0$$

or

$$c + \sum_{i=1}^{m} z_i^*(t) a_i = 0 \quad \text{with} \quad z_i^*(t) = \frac{1}{t s_i(x^*(t))}$$

- $z^*(t)$ is strictly dual feasible: $(A^T z^* - c = 0; z^* > 0)$
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$$c^T x + b^T z = (b - Ax)^T z = \frac{m}{t}$$

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- start somewhere in the polytope
- use iterative method (Newtons method) to minimize $f_t(x) := tc^T x + \phi(x)$

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Newton Method

Quadratic approximation of f_t

$$f_t(x+\epsilon) \approx f_t(x) + \nabla f_t(x)^T\epsilon + \frac{1}{2}\epsilon^T H_{f_t}(x)\,\epsilon$$

Suppose this were exact:

$$f_t(x+\epsilon) = f_t(x) + \nabla f_t(x)^T\epsilon + \frac{1}{2}\epsilon^T H_{f_t}(x)\,\epsilon$$

Then gradient is given by:

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Newton Step at $x \in P^\circ$

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$$x := x + \Delta x_{\text{nt}}$$

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Measuring Progress of Newton Step

Newton decrement:

$$\begin{aligned}\lambda_t(x) &= \|D_x A \Delta x_{\mathrm{nt}}\| \\ &= \|\Delta x_{\mathrm{nt}}\|_{H_x}\end{aligned}$$

Square of Newton decrement is linear estimate of reduction if we do a Newton step:

$$-\lambda_t(x)^2 = \nabla f_t(x)^T \Delta x_{\mathrm{nt}}$$

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Convergence of Newtons Method

Theorem 55
If $\lambda_t(x) < 1$ *then*

- $x_+ := x + \Delta x_{nt} \in P^\circ$ *(new point feasible)*
- $\lambda_t(x_+) \leq \lambda_t(x)^2$

This means we have quadratic convergence. Very fast.

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feasibility:

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Convergence of Newtons Method

bound on $\lambda_t(x^+)$:

we use $D := D_x = \mathrm{diag}(d_x)$ and $D_+ := D_{x^+} = \mathrm{diag}(d_{x^+})$

To see the last equality we use Pythagoras

$$\|a\|^2 + \|a+b\|^2 = \|b\|^2$$

if $a^T(a+b) = 0$.

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&\le \|D_+ A\Delta x^+_{\mathrm{nt}}\|^2 + \|D_+ A\Delta x^+_{\mathrm{nt}} + (I - D_+^{-1}D)DA\Delta x_{\mathrm{nt}}\|^2 \\
&= \|(I - D_+^{-1}D)DA\Delta x_{\mathrm{nt}}\|^2
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To see the last equality we use Pythagoras

$$\|a\|^2 + \|a+b\|^2 = \|b\|^2$$

if $a^T(a+b) = 0$.

Convergence of Newtons Method

feasibility:

- $\lambda_t(x) = \|\Delta x_{\mathrm{nt}}\|_{H_x} < 1$; hence x_+ lies in the Dikin ellipsoid around x.

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&= \|(I - D_+^{-1}D)^2\vec{1}\|^2 \\
&\le \|(I - D_+^{-1}D)\vec{1}\|^4 \\
&= \|DA\Delta x_{\mathrm{nt}}\|^4 \\
&= \lambda_t(x)^4
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The second inequality follows from $\sum_i y_i^4 \le (\sum_i y_i^2)^2$

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The second inequality follows from $\sum_i y_i^4 \le (\sum_i y_i^2)^2$

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Try to avoid this case!!!

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Path-following Methods

Try to slowly travel along the central path.

Algorithm 1 PathFollowing

1: start at analytic center
2: **while** solution not good enough **do**
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Short Step Barrier Method

simplifying assumptions:

- a first central point $x^*(t_0)$ is given
- $x^*(t)$ is computed exactly in each iteration

ϵ is approximation we are aiming for

start at $t = t_0$, repeat until $m/t \leq \epsilon$

- compute $x^*(\mu t)$ using Newton starting from $x^*(t)$
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where $\mu = 1 + 1/(2\sqrt{m})$

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gradient of f_{t^+} at ($x = x^*(t)$)

$$\begin{aligned}\nabla f_{t^+}(x) &= \nabla f_t(x) + (\mu - 1)tc \\ &= -(\mu - 1)A^T D_x \vec{1}\end{aligned}$$

This holds because $0 = \nabla f_t(x) = tc + A^T D_x \vec{1}$.

The Newton decrement is

$$\begin{aligned}\lambda_{t^+}(x)^2 &= \nabla f_{t^+}(x)^T H^{-1} \nabla f_{t^+}(x) \\ &= (\mu - 1)^2 \vec{1}^T B(B^T B)^{-1} B^T \vec{1} \qquad B = D_x^T A \\ &\le (\mu - 1)^2 m \\ &= 1/4\end{aligned}$$

This means we are in the range of quadratic convergence!!!

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Number of Iterations

the number of Newton iterations per outer iteration is very small; in practise only 1 or 2

Number of outer iterations:
We need $t_k = \mu^k t_0 \geq m/\epsilon$. This holds when

$$k \geq \frac{\log(m/(\epsilon t_0))}{\log(\mu)}$$

We get a bound of

$$\mathcal{O}\left(\sqrt{m}\log\frac{m}{\epsilon t_0}\right)$$

We show how to get a starting point with $t_0 = 1/2^L$. Together with $\epsilon \approx 2^{-L}$ we get $\mathcal{O}(L\sqrt{m})$ iterations.

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Damped Newton Method

For $x \in P^\circ$ and direction $v \neq 0$ define

$$\sigma_x(v) := \max_i \frac{a_i^T v}{s_i(x)}$$

Observation:

$$x + \alpha v \in P \qquad \text{for } \alpha \in \{0, 1/\sigma_x(v)\}$$

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Suppose that we move from x to $x + \alpha v$. The linear estimate says that $f_t(x)$ should change by $\nabla f_t(x)^T \alpha v$.

The following argument shows that f_t is well behaved. For small α the reduction of $f_t(x)$ is close to linear estimate.

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&\le -\sum_{w_i>0} \frac{w_i^2}{\sigma^2}\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big) + \frac{(\alpha\sigma)^2}{2} \sum_{w_i\le 0} \frac{w_i^2}{\sigma^2}
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Suppose that we move from x to $x + \alpha v$. The linear estimate says that $f_t(x)$ should change by $\nabla f_t(x)^T \alpha v$.

The following argument shows that f_t is well behaved. For small α the reduction of $f_t(x)$ is close to linear estimate.

$$f_t(x+\alpha v) - f_t(x) = tc^T \alpha v + \phi(x+\alpha v) - \phi(x)$$

$$
\begin{aligned}
\phi(x+\alpha v) - \phi(x) &= -\sum_i \log(s_i(x+\alpha v)) + \sum_i \log(s_i(x)) \\
&= -\sum_i \log(s_i(x+\alpha v)/s_i(x)) \\
&= -\sum_i \log(1 - a_i^T \alpha v / s_i(x))
\end{aligned}
$$

Damped Newton Method

Define $w_i = a_i^T v / s_i(x)$ and $\sigma = \max_i w_i$. Then

$$
\begin{aligned}
f_t(x+\alpha v) - f_t(x) - \nabla f_t(x)^T \alpha v
&= -\sum\nolimits_i (\alpha w_i + \log(1-\alpha w_i)) \\
&\le -\sum_{w_i>0} (\alpha w_i + \log(1-\alpha w_i)) + \sum_{w_i\le 0} \frac{\alpha^2 w_i^2}{2} \\
&\le -\sum_{w_i>0} \frac{w_i^2}{\sigma^2}\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big) + \frac{(\alpha\sigma)^2}{2} \sum_{w_i\le 0} \frac{w_i^2}{\sigma^2}
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Suppose that we move from x to $x + \alpha v$. The linear estimate says that $f_t(x)$ should change by $\nabla f_t(x)^T \alpha v$.

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Damped Newton Method

$$\leq -\sum_i \frac{w_i^2}{\sigma^2}\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big)$$

$$= -\frac{1}{\sigma^2}\|v\|_{H_x}^2\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big)$$

Damped Newton Iteration:

In a damped Newton step we choose

$$x_+ = x + \frac{1}{1+\sigma_x(\Delta x_{\text{nt}})}\Delta x_{\text{nt}}$$

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$$\begin{aligned}
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Damped Newton Method

Theorem:
In a damped Newton step the cost decreases by at least

$$\lambda_t(x) - \log(1 + \lambda_t(x))$$

Proof: The decrease in cost is

$$-\alpha\nabla f_t(x)^T v + \frac{1}{\sigma^2}\|v\|_{H_x}(\alpha\sigma + \log(1-\alpha\sigma))$$

Choosing $\alpha = \frac{1}{1+\sigma}$ and $v = \Delta x_{\text{nt}}$ gives

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&\geq \lambda_t(x) - \log(1 + \lambda_t(x)) \\
&\geq 0.09
\end{aligned}
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for $\lambda_t(x) \geq 0.5$

Centering Algorithm:

Input: precision δ; starting point x

1. compute Δx_{nt} and $\lambda_t(x)$
2. if $\lambda_t(x) \leq \delta$ return x
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$$
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Centering

Lemma 56
The centering algorithm starting at x_0 reaches a point with $\lambda_t(x) \leq \delta$ after

$$\frac{f_t(x_0) - \min_y f_t(y)}{0.09} + \mathcal{O}(\log\log(1/\delta))$$

iterations.

This can be very, very slow...

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How to get close to analytic center?

Let $P = \{Ax \le b\}$ be our (feasible) polyhedron, and x_0 a feasible point.

We change $b \to b + \frac{1}{\lambda} \cdot \vec{1}$, where $L = \langle A \rangle + \langle b \rangle + \langle c \rangle$ (encoding length) and $\lambda = 2^{2L}$. Recall that a basis is feasible in the old LP iff it is feasible in the new LP.

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Lemma [without proof]
The inverse of a matrix M can be represented with rational numbers that have denominators $z_{ij} = \det(M)$.

For two basis solutions x_B, $x_{\bar{B}}$, the cost-difference $c^T x_B - c^T x_{\bar{B}}$ can be represented by a rational number that has denominator $z = \det(A_B) \cdot \det(A_{\bar{B}}) \cdot \lambda$.

This means that in the perturbed LP it is sufficient to decrease the duality gap to $1/2^{4L}$ (i.e., $t \approx 2^{4L}$). This means the previous analysis essentially also works for the perturbed LP.

For a point x from the polytope (not necessarily BFS) the objective value $\bar{c}^T x$ is at most $n2^M 2^L$, where $M \le L$ is the encoding length of the largest entry in $\bar{c}$.

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How to get close to analytic center?

Start at x_0.

Choose $\hat{c} := -\nabla\phi(x)$.

$x_0 = x^*(1)$ is point on central path for $\hat{c}$ and $t = 1$.

You can travel the central path in both directions. Go towards 0 until $t \approx 1/2^{\Omega(L)}$. This requires $O(\sqrt{m}L)$ outer iterations.

Let $x_{\hat{c}}$ denote this point.

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Lemma [without proof]
The inverse of a matrix M can be represented with rational numbers that have denominators $z_{ij} = \det(M)$.

For two basis solutions $x_B, x_{\bar{B}}$, the cost-difference $c^T x_B - c^T x_{\bar{B}}$ can be represented by a rational number that has denominator $z = \det(A_B) \cdot \det(A_{\bar{B}}) \cdot \lambda$.

This means that in the perturbed LP it is sufficient to decrease the duality gap to $1/2^{4L}$ (i.e., $t \approx 2^{4L}$). This means the previous analysis essentially also works for the perturbed LP.

For a point x from the polytope (not necessarily BFS) the objective value $\bar{c}^T x$ is at most $n2^M 2^L$, where $M \le L$ is the encoding length of the largest entry in $\bar{c}$.

How to get close to analytic center?

Clearly,

$$t \cdot \hat{c}^T x_{\hat{c}} + \phi(x_{\hat{c}}) \le t \cdot \hat{c}^T x_c + \phi(x_c)$$

The different between $f_t(x_{\hat{c}})$ and $f_t(x_c)$ is

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For $t = 1/2^{\Omega(L)}$) the last term becomes constant. Hence, using damped Newton we can move from $x_{\hat{c}}$ to x_c quickly.

In total for this analysis we require $\mathcal{O}(\sqrt{m}L)$ outer iterations for the whole algorithm.

One iteration can be implemented in $\tilde{\mathcal{O}}(m^3)$ time.

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