Technique 1: Round the LP solution.

We first solve the LP-relaxation and then we round the fractional values so that we obtain an integral solution.

Set Cover relaxation:

$$\begin{array}{lrrcl} \min & & \sum_{i=1}^{k} w_i x_i & & \\ \text{s.t.} & \forall u \in U & \sum_{i: u \in S_i} x_i & \geq & 1 \\ & \forall i \in \{1,\dots,k\} & x_i & \in & [0,1] \end{array}$$

Let f_u be the number of sets that the element u is contained in (the frequency of u). Let $f = \max_u \{f_u\}$ be the maximum frequency.

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The rounding algorithm gives an f-approximation.

Proof: Every $u \in U$ is covered.

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Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lll} \min & & \sum_{i\in I} w_i x_i \\ \text{s.t. } \forall u & & \sum_{i:u\in S_i} x_i \ge 1 \\ & & x_i \ge 0 \end{array}$$

Dual:

$$\begin{array}{lll} \max & & \sum_{u\in U} y_u \\ \text{s.t. } \forall i & & \sum_{u:u\in S_i} y_u \le w_i \\ & & y_u \ge 0 \end{array}$$

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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

Of course, we also need that I is a cover.

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  where x^* is an optimum solution to the primal LP.
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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

1. The solution is dual feasible and, hence,

$$\sum_u y_u \leq \text{cost}(x^*) \leq \text{OPT}$$

   where x^* is an optimum solution to the primal LP.

2. The set I contains only sets for which the dual inequality is tight.

Of course, we also need that I is a cover.

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$$I \subseteq I' .$$

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while exists $u \notin \bigcup_{i \in I} S_i$ **do**
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In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

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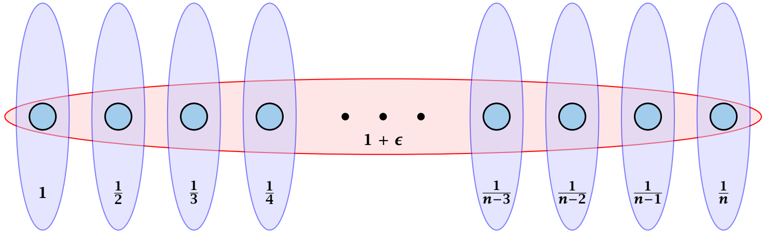

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$$
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One round of randomized rounding:
Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

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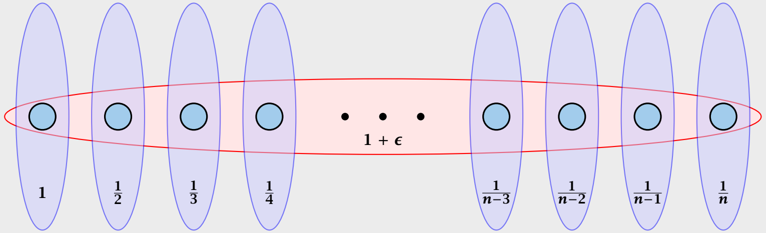

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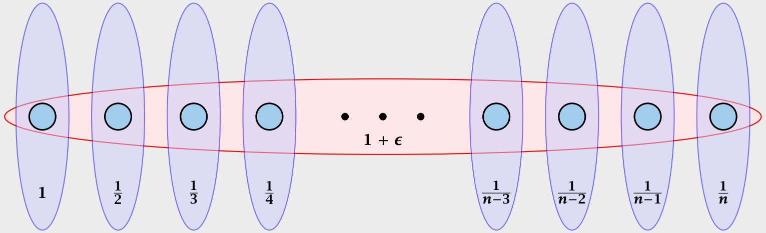

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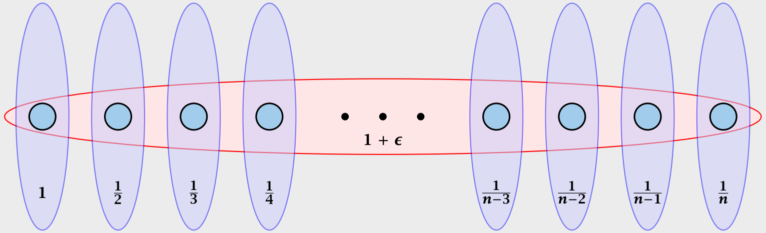

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Integrality Gap

The integrality gap of the SetCover LP is $\Omega(\log n)$.

- $n = 2^k - 1$
- Elements are all vectors $\vec{x}$ over $GF[2]$ of length k (excluding zero vector).
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- each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
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Techniques:

- Deterministic Rounding
- Rounding of the Dual
- Primal Dual
- Greedy
- Randomized Rounding
- Local Search
- Rounding Data + Dynamic Programming

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