There are many practically important optimization problems that are NP-hard.

## What can we do?

- Heuristics.
- Exploit special structure of instances occurring in practise.
- Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

## **Definition 2**

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**Definition 3** 

• goal  $\in$  {min, max}

m(x, y) at most/at least z is in NP.

An  $\alpha$ -approximation for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of  $\alpha$  of the value of an optimal solution.

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An optimization problem  $P = (\mathcal{I}, \text{sol}, m, \text{goal})$  is in **NPO** if

•  $x \in \mathcal{I}$  can be decided in polynomial time

 $\blacktriangleright$  *m* can be computed in polynomial time

•  $\gamma \in \text{sol}(\mathcal{I})$  can be verified in polynomial time

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260

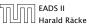
## Why approximation algorithms?

- > We need algorithms for hard problems.
- It gives a rigorous mathematical base for studying heuristics.
- It provides a metric to compare the difficulty of various optimization problems.
- Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

#### Why not?

 Sometimes the results are very pessimistic due to the fact that an algorithm has to provide a close-to-optimum solution on every instance.

262



In other words: the decision problem is there a solution  $\gamma$  with

- $\blacktriangleright x$  is problem instance
- y is candidate solution
- $m^*(x)$  cost/profit of an optimal solution

## **Definition 4 (Performance Ratio)**

$$R(x, y) := \max\left\{\frac{m(x, y)}{m^*(x)}, \frac{m^*(x)}{m(x, y)}\right\}$$

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### **Definition 6 (PTAS)**

A PTAS for a problem *P* from NPO is an algorithm that takes as input  $x \in I$  and  $\epsilon > 0$  and produces a solution  $\gamma$  for x with

# $R(x, y) \leq 1 + \epsilon$ .

The running time is polynomial in |x|.

approximation with arbitrary good factor... fast?

Definition 5 ( <i>r</i> -approximation)
An algorithm $A$ is an $r$ -approximation algorithm iff
$\forall x \in \mathcal{I} : R(x, A(x)) \leq r$ ,

and A runs in polynomial time.

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Problems that have a PTAS

**Scheduling.** Given m jobs with known processing times; schedule the jobs on n machines such that the MAKESPAN is minimized.



#### **Definition 7 (FPTAS)**

An FPTAS for a problem *P* from NPO is an algorithm that takes as input  $x \in \mathcal{I}$  and  $\epsilon > 0$  and produces a solution  $\mathcal{Y}$  for x with

# $R(x,y) \le 1 + \epsilon \ .$

The running time is polynomial in |x| and  $1/\epsilon$ .

## approximation with arbitrary good factor... fast!

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## **Definition 8 (APX – approximable)**

A problem *P* from NPO is in APX if there exist a constant  $r \ge 1$ and an *r*-approximation algorithm for *P*.

## constant factor approximation...

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## Problems that have an FPTAS

**KNAPSACK**. Given a set of items with profits and weights choose a subset of total weight at most W s.t. the profit is maximized.

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#### Problems that are in APX

**MAXCUT.** Given a graph G = (V, E); partition V into two disjoint pieces A and B s.t. the number of edges between both pieces is maximized.

**MAX-3SAT.** Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.

Problems with polylogarithmic approximation guarantees

- Set Cover
- Minimum Multicut
- Sparsest Cut
- Minimum Bisection

There is an r-approximation with  $r \leq O(\log^{c}(|x|))$  for some constant c.

Note that only for some of the above problem a matching lower bound is known.

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There are weird problems!

Asymmetric *k*-Center admits an  $O(\log^* n)$ -approximation.

There is no  $o(\log^* n)$ -approximation to Asymmetric *k*-Center unless  $NP \subseteq DTIME(n^{\log \log \log n})$ .

## There are really difficult problems!

## **Theorem 9**

For any constant  $\epsilon > 0$  there does not exist an  $\Omega(n^{1-\epsilon})$ -approximation algorithm for the maximum clique problem on a given graph *G* with *n* nodes unless P = NP.

Note that an *n*-approximation is trivial.

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Class APX not important in practise. Instead of saying problem *P* is in APX one says problem *P* admits a 4-approximation. One only says that a problem is APX-hard.