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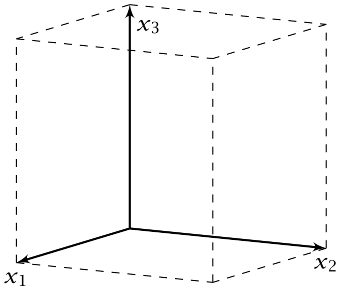

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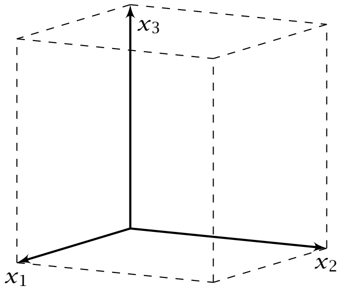

However, Simplex may still run quickly as it usually does not visit all feasible bases.

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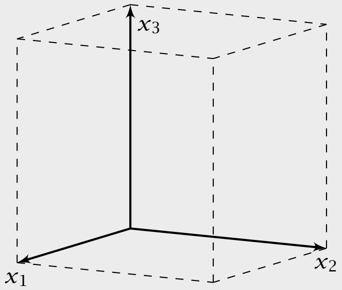

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Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

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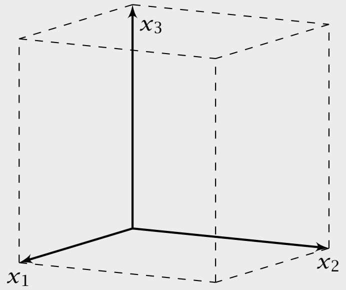

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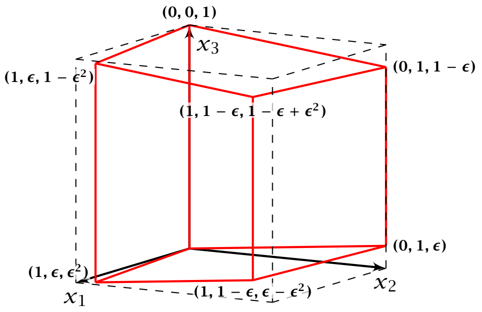

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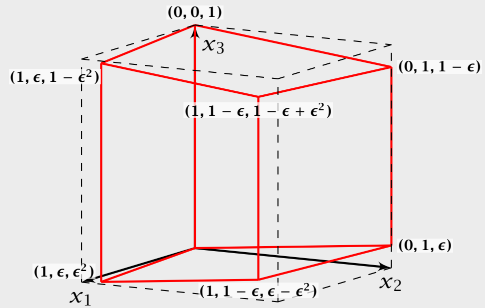

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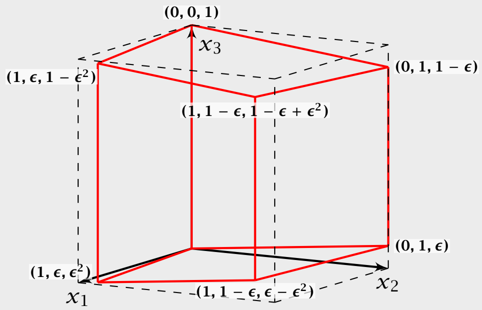

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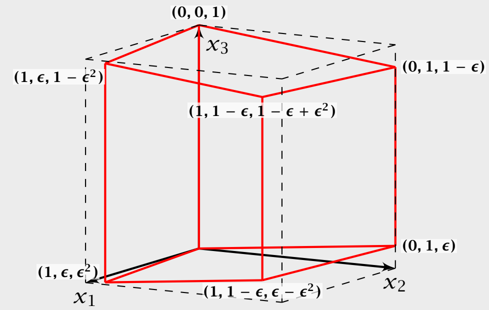

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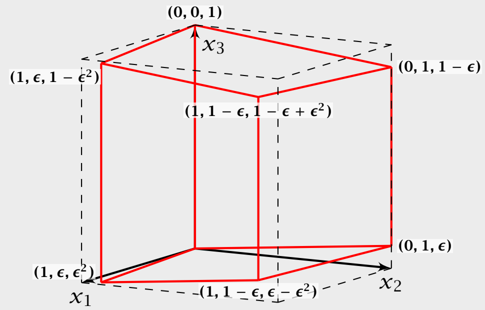

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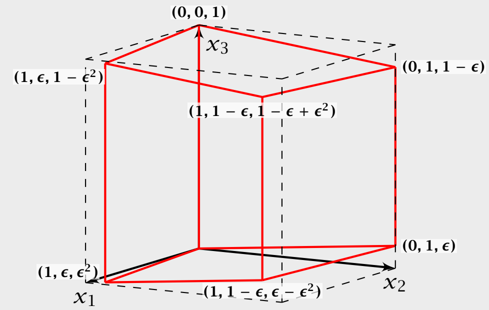

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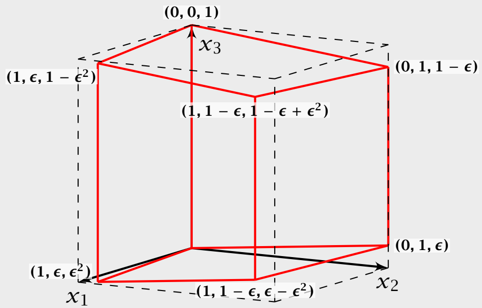

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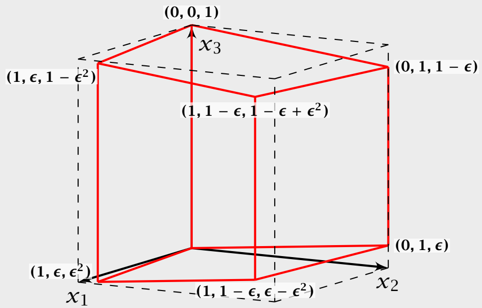

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- In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- The basis $(0, \dots, 0, 1)$ is the unique optimal basis.
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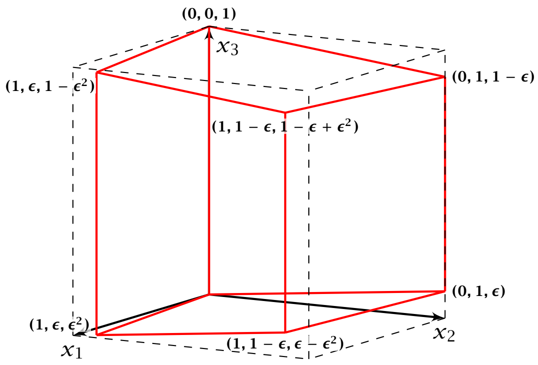

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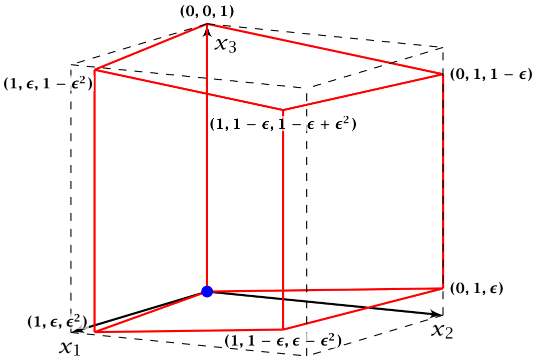

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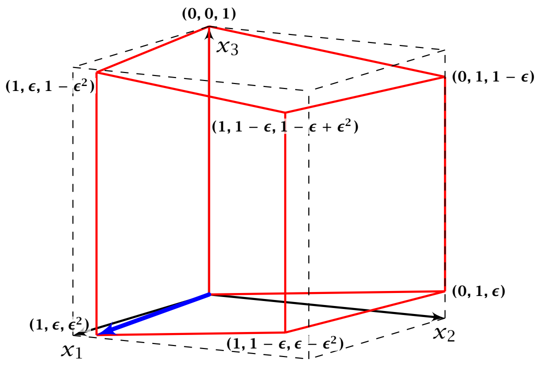

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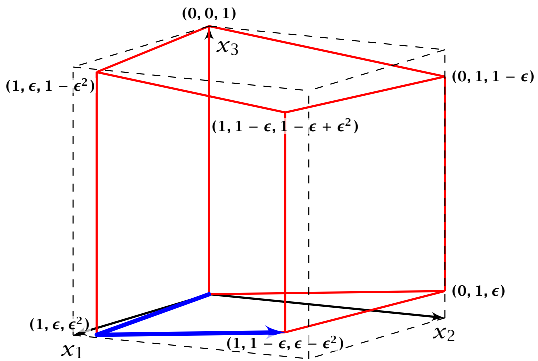

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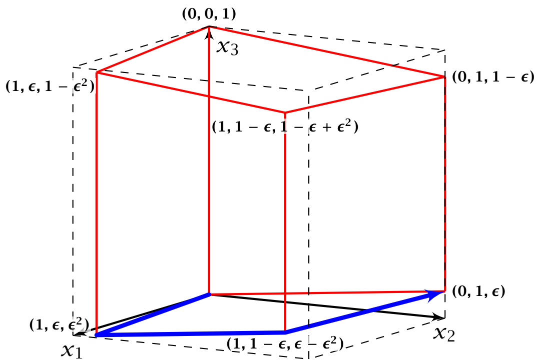

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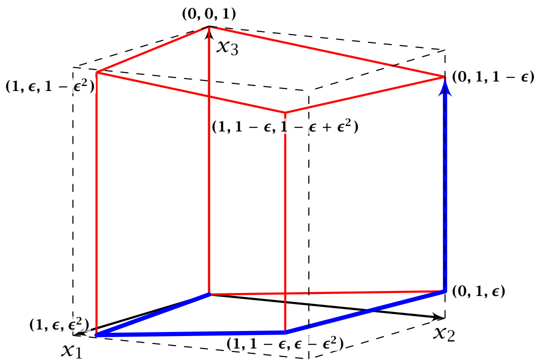

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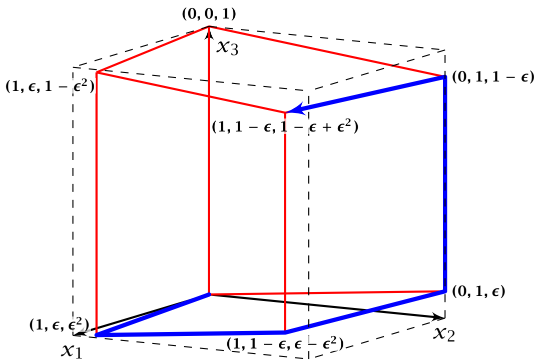

Analysis

- In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- The basis $(0,\dots,0,1)$ is the unique optimal basis.
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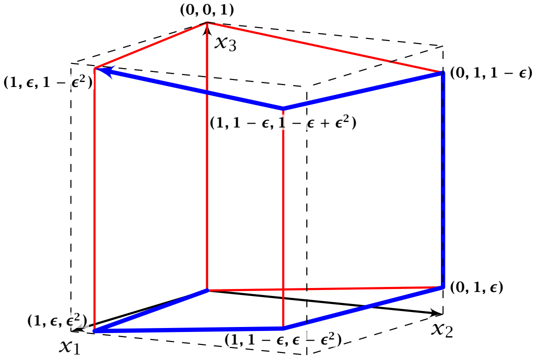

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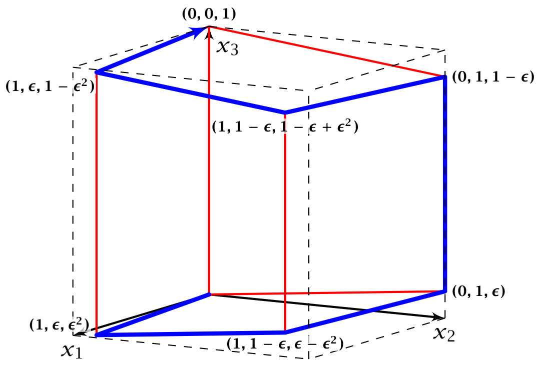

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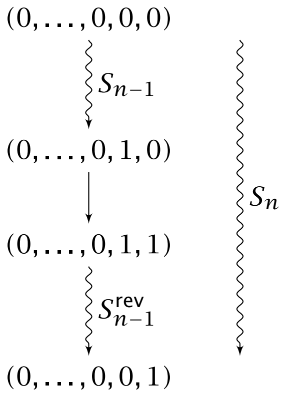
The sequence S_n that visits every node of the hypercube is defined recursively



The non-recursive case is $S_1 = 0 \to 1$

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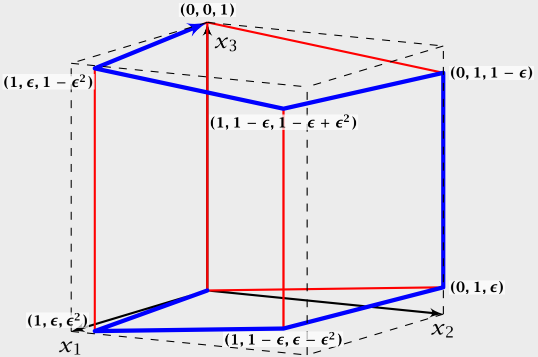

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Lemma 2

The objective value x_n is increasing along path S_n.

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \to 1$, and $1 > 0$.

$n - 1 \to n$

- For the first part the value of $x_n = \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence, also x_n.
- Going from $(0, \ldots, 0, 1, 0)$ to $(0, \ldots, 0, 1, 1)$ increases x_n for small enough ϵ.
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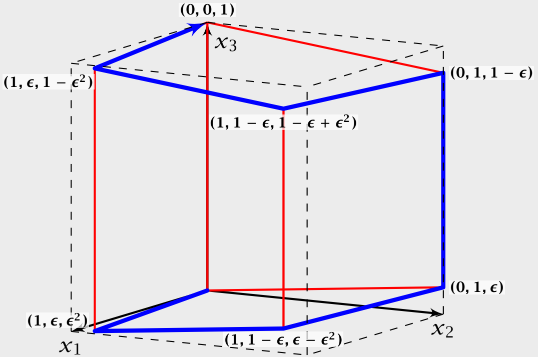

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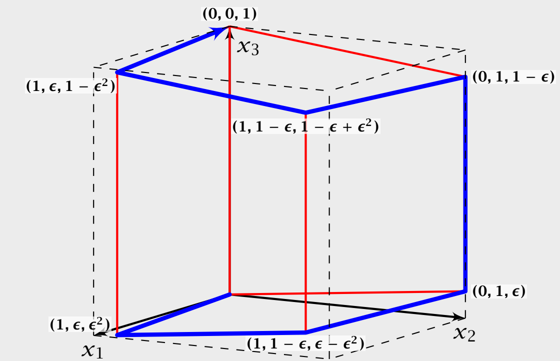

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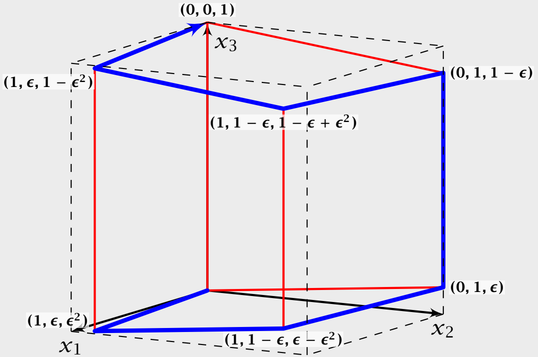

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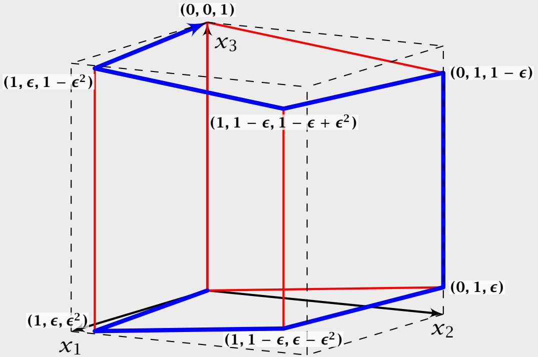

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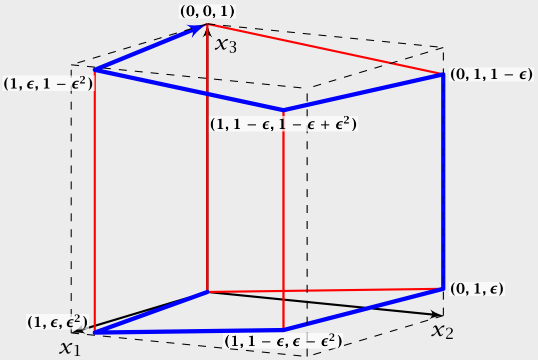

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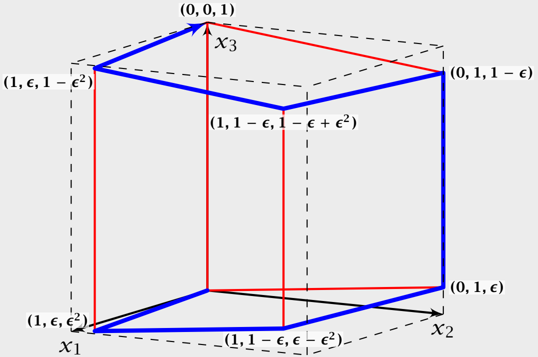

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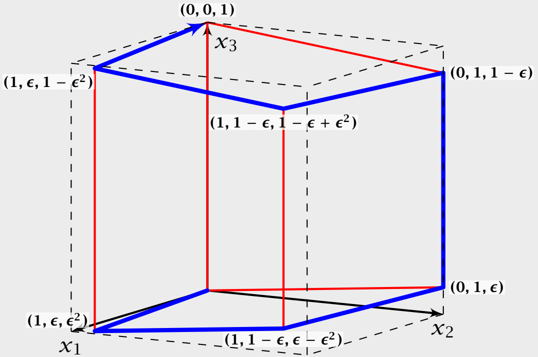

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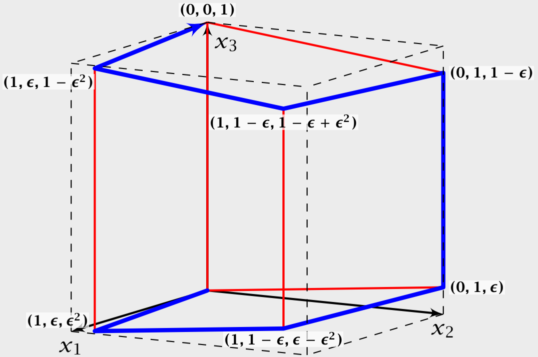

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The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

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Remarks about Simplex

Conjecture (Hirsch 1957)
The edge-vertex graph of an m-facet polytope in d-dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\text{poly}(m, d))$ is open.

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