Lemma 2 (Chernoff Bounds)

Let $X_1, ..., X_n$ be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^n X_i$ and $\mu = E[X], L \le \mu \le U$, and $\delta > 0$

$$\Pr[X \ge (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U$$
,

and

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Lemma 3

For $0 \le \delta \le 1$ we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

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Markovs Inequality:

Let \boldsymbol{X} be random variable taking non-negative values. Then

$$\Pr[X \ge a] \le \mathrm{E}[X]/a$$

Trivial!

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That's awfully weak :(

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Cool Trick:

$$\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

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This may be a lot better (!?)

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$$\mathbb{E}\left[e^{tX}
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17.1 Chernoff Bounds

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Cool Trick:

Proof of Chernoff Bounds

Now, we apply Markov:

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

 $\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \ .$$

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EADS II

Harald Räcke

$$\mathrm{E}\left[e^{tX}\right] = \mathrm{E}\left[e^{t\sum_{i}X_{i}}\right]$$

Proof of Chernoff Bounds

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{\mathbb{E}[e^{tX}]}{\rho t(1+\delta)U}$$
.

 $\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right]$$

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Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

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Proof of Chernoff Bounds

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)U}}$$
.

$$e^{i(1+\theta)\theta}$$

EADS II

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17.1 Chernoff Bounds

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17.1 Chernoff Bounds

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$$E\left[e^{tX_i}\right] = (1 - p_i) + p_i e^t$$

17.1 Chernoff Bounds

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Now, we apply Markov:

 $\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)U}} \ .$$

$$X \ge e^{t(1+\delta)U} \Big] \le \frac{-t^2}{e^{t(1+\delta)U}} .$$

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Cool Trick:

Proof of Chernoff Bounds

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

17.1 Chernoff Bounds

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}E\left[e^{tX_{i}}\right]$$

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1)$$

Now, we apply Markov:
$$\Pr[e^{tX}]$$

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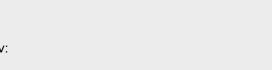
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Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

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17.1 Chernoff Bounds

Cool Trick:



$$tX$$
]

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \ .$$

EADS II

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}E\left[e^{tX_{i}}\right]$$

$$\mathbb{E}\left[atX_i\right] = (1 - n_i) + n_i a^t - 1 + n_i (a^t - 1) < a^{p_i(e^t - 1)}$$

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i(e^t - 1) \le e^{p_i(e^t - 1)}$$

17.1 Chernoff Bounds

$$e^t = 1 + p_i(e^t - 1) \le e^{p_i(e^t - 1)}$$

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Cool Trick:

$$\Pr[e^{t\lambda}]$$

This may be a lot better (!?)

Proof of Chernoff Bounds

Now, we apply Markov:
$$\Pr[e^{tX} \geq e^{t(1+\delta)U}] \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \ .$$

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

17.1 Chernoff Bounds

$$\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

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$$\Box = \begin{bmatrix} -tX_i \end{bmatrix}$$

$$\prod_i \operatorname{E}\left[e^{tX_i}\right]$$

17.1 Chernoff Bounds

$$\mathbb{E}\left[e^{tX_i}
ight]$$



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Cool Trick:

Now, we apply Markov:
$$\Pr[e^{tX} \geq e^{t(1+\delta)U}] \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \ .$$

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17.1 Chernoff Bounds

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EADS II

EADS II

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}E\left[e^{tX_{i}}\right]$$

$$E[e^{tX_i}] = E[e^{tX_i}] - E[II_i e^{t}] - II_i E[e^{t}]$$

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

$$\prod_{i=1}^{n} \mathbb{E}\left[\rho^{tX_{i}}\right] < \prod_{i=1}^{n} \rho^{p_{i}(e^{t}-1)}$$

$$\prod_{i} \mathbb{E}\left[e^{tX_{i}}\right] \leq \prod_{i} e^{p_{i}(e^{t}-1)}$$

$$\prod_{i} \mathbf{E}\left[e^{tX_{i}}\right] \leq \prod_{i} e^{p_{i}(e^{t}-1)}$$

17.1 Chernoff Bounds

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Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

Proof of Chernoff Bounds

17.1 Chernoff Bounds

 $\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$

Markov:
$$\Pr[e^{tX} \geq e^{t(1+\delta)U}] \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \; .$$

$$X \ge e^{t(1+\delta)U}$$
] $\le \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}}$.

$$X \ge e^{t(1+\delta)U}] \le \frac{1}{e^{t(1+\delta)U}} .$$

Cool Trick:

$$\mathbf{E}\left[e^{tX}\right] = \mathbf{E}\left[e^{t\sum_{i}X_{i}}\right] = \mathbf{E}\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}\mathbf{E}\left[e^{tX_{i}}\right]$$

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$$\mathbf{E}\left[e^{tX_i}\right] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

$$\left[e^{tX_i}\right] \leq \prod e^{p_i(e^t-1)} = e^{\sum p_i(e^t-1)}$$

 $\prod_{i} \mathbb{E}\left[e^{tX_{i}}\right] \leq \prod_{i} e^{p_{i}(e^{t}-1)} = e^{\sum p_{i}(e^{t}-1)}$

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Cool Trick:

Proof of Chernoff Bounds

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

This may be a lot better (!?)

 $\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$

17.1 Chernoff Bounds

Markov:
$$\Pr[e^{tX} \geq e^{t(1+\delta)U}] \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \ .$$

$$\frac{1}{1}$$
 .

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}E\left[e^{tX_{i}}\right]$$

$$\mathbb{E}\left[\prod_{i}e^{-\tau}\right] - \prod_{i}\mathbb{E}\left[e^{-\tau}\right]$$

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

$$\prod_{i} \mathbb{E}\left[e^{tX_{i}}\right] \leq \prod_{i} e^{p_{i}(e^{t}-1)} = e^{\sum p_{i}(e^{t}-1)} = e^{(e^{t}-1)U}$$

$$\prod_i \mathbf{E}\left[e^{tX_i}\right] \leq \prod_i e^{p_i(e^t-1)} = e^{\sum p_i(e^t-1)} = e^{(e^t-1)U}$$

17.1 Chernoff Bounds

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Cool Trick:

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)U}} .$$

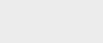
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Proof of Chernoff Bounds

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

17.1 Chernoff Bounds

$$\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$



Now, we apply Markov:

Pr
$$[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

$$\le \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}}$$

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}E\left[e^{tX_{i}}\right]$$

Proof of Chernoff Bounds

$$p_i e^t = 1 +$$

 $\prod_{i} \mathbb{E}\left[e^{tX_{i}}\right] \leq \prod_{i} e^{p_{i}(e^{t}-1)} = e^{\sum p_{i}(e^{t}-1)} = e^{(e^{t}-1)U}$

17.1 Chernoff Bounds

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

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EADS II

Now, we apply Markov:
$$\Pr[X \geq (1+\delta)U] = \Pr[e^{tX} \geq e^{t(1+\delta)U}]$$

$$\leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \leq \frac{e^{(e^t-1)U}}{e^{t(1+\delta)U}}$$

17.1 Chernoff Bounds

$$\mathbf{E}\left[e^{tX}\right] = \mathbf{E}\left[e^{t\sum_{i}X_{i}}\right] = \mathbf{E}\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}\mathbf{E}\left[e^{tX_{i}}\right]$$

$$\mathbf{E}\left[e^{tX_{i}}\right] = (1 - p_{i}) + p_{i}e^{t} = 1 + p_{i}(e^{t} - 1) \le e^{p_{i}(e^{t} - 1)}$$

Proof of Chernoff Bounds

$$\prod_{i} \mathbb{E} \left[e^{tX_i} \right] \le \prod_{i} e^{p_i(e^t - 1)} = e^{\sum p_i(e^t - 1)} = e^{(e^t - 1)U}$$

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17.1 Chernoff Bounds

Now, we apply Markov:

$$\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

$$E[e^{tX}] \qquad e^{(e^t-1)U}$$

17.1 Chernoff Bounds

 $\leq \frac{\mathbb{E}[e^{tX}]}{\rho t(1+\delta)U} \leq \frac{e^{(e^t-1)U}}{\rho t(1+\delta)U}$

We choose $t = \ln(1 + \delta)$.

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$$\prod_i \mathsf{E}$$

Proof of Chernoff Bounds

$$E\left[e^{tX}\right] = E\left[e^{t\sum_{i}X_{i}}\right] = E\left[\prod_{i}e^{tX_{i}}\right] = \prod_{i}E\left[e^{tX_{i}}\right]$$

$$E\left[e^{tX_{i}}\right] = (1 - p_{i}) + p_{i}e^{t} = 1 + p_{i}(e^{t} - 1) \le e^{p_{i}(e^{t} - 1)}$$

17.1 Chernoff Bounds

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

$$(e^{t}-1) = a(e^{t}-1)U$$

$$\prod_{i} \mathbb{E}\left[e^{tX_{i}}\right] \leq \prod_{i} e^{p_{i}(e^{t}-1)} = e^{\sum p_{i}(e^{t}-1)} = e^{(e^{t}-1)U}$$

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EADS II

Now, we apply Markov: $\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$

$$J = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

$$\le \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)U}} \le \frac{e^{(e^t-1)U}}{e^{t(1+\delta)U}} \le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U}$$

We choose $t = \ln(1 + \delta)$.

17.1 Chernoff Bounds

$$\mathbb{P}[tX:]$$

Proof of Chernoff Bounds

$$E[e^{tX_i}] = (1 - p_i) + p_i e^t = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

$$\prod_i \mathbb{E}\left[e^{tX_i}\right] \leq \prod_i e^{p_i(e^t - 1)} = e^{\sum p_i(e^t - 1)} = e^{(e^t - 1)U}$$

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 $E[e^{tX}] = E[e^{t\sum_i X_i}] = E[\prod_i e^{tX_i}] = \prod_i E[e^{tX_i}]$

Lemma 4

For $0 < \delta < 1$ we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U \le e^{-U\delta^2/3}$$

and

nd
$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \leq e^{-L\delta^2/2}$$

Now, we apply Markov:

$$\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

$$\le \frac{E[e^{tX}]}{e^{t(1+\delta)U}} \le \frac{e^{(e^t-1)U}}{e^{t(1+\delta)U}} \le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U$$

We choose $t = \ln(1 + \delta)$.

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^{2}/3}$$

Lemma 4

For
$$0 \le \delta \le 1$$
 we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

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17.1 Chernoff Bounds 402/575

17.1 Chernoff Bounds

 $\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \le e^{-L\delta^2/2}$

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^{2}/3}$$

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta^2/3$$

$$\delta \ln(1+\delta) \le -U\delta^2/3$$

Lemma 4

and

For
$$0 \le \delta \le 1$$
 we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

$$(1+\delta)^{1+\delta}$$

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \le e^{-L\delta^2/2}$$

17.1 Chernoff Bounds

17.1 Chernoff Bounds

True for $\delta = 0$.

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Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^{2}/3}$$

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta^2/3$$

17.1 Chernoff Bounds

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Lemma 4

For $0 < \delta < 1$ we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

17.1 Chernoff Bounds

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \le e^{-L\delta^2/2}$$

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U \leq e^{-U\delta^2/3}$$
 Take logarithms:
$$U(\delta-(1+\delta)\ln(1+\delta)) \leq -U\delta^2/3$$
 True for $\delta=0$. Divide by U and take derivatives:
$$-\ln(1+\delta) \leq -2\delta/3$$

For $0 \le \delta \le 1$ we have that $\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U \le e^{-U\delta^2/3}$ $\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \le e^{-L\delta^2/2}$

17.1 Chernoff Bounds

Reason:

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Lemma 4

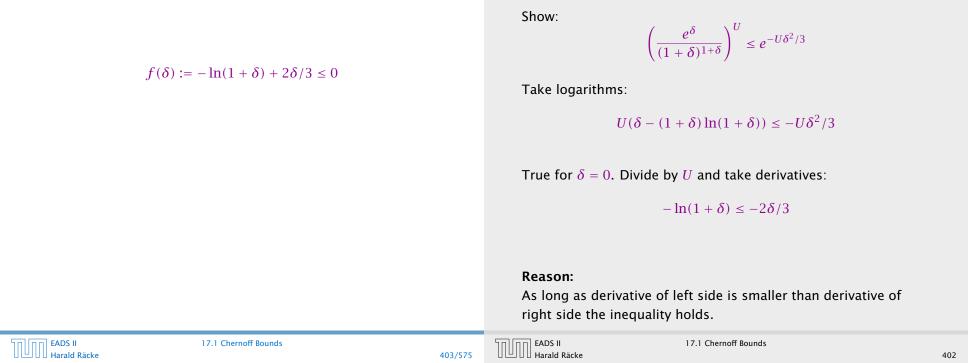
and

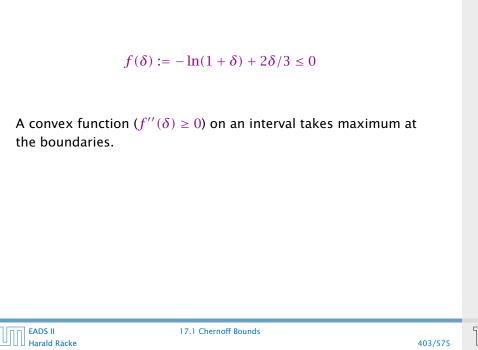
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Show:

17.1 Chernoff Bounds





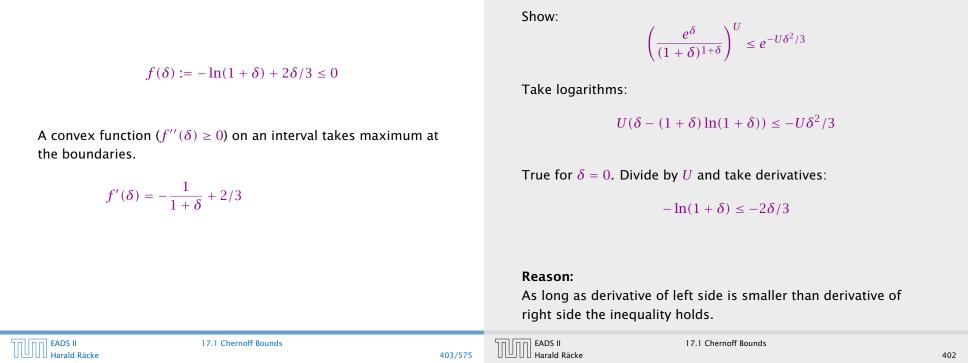
Take logarithms: $U(\delta - (1 + \delta) \ln(1 + \delta)) \le -U\delta^2/3$ True for $\delta = 0$. Divide by U and take derivatives: $-\ln(1+\delta) < -2\delta/3$

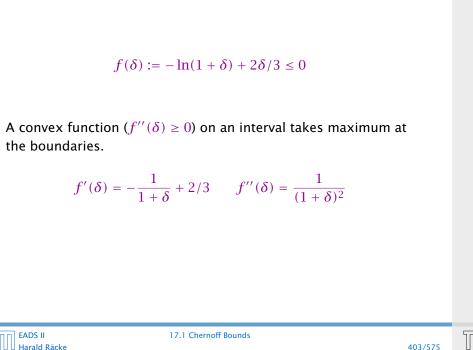
 $\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$

17.1 Chernoff Bounds

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Show:





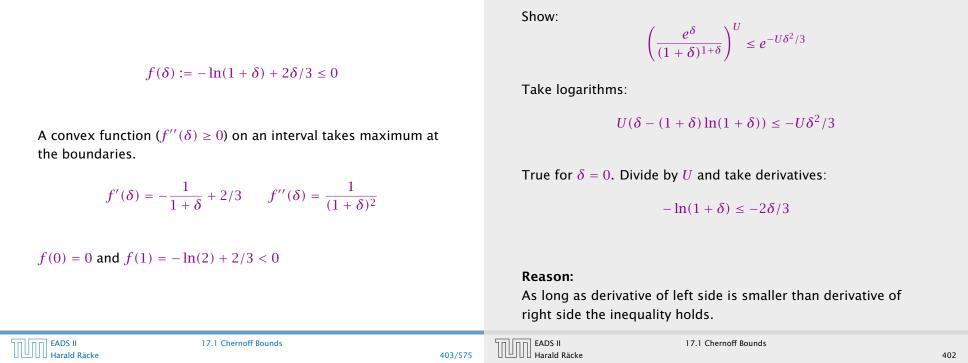
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17.1 Chernoff Bounds

 $\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$

Show:

Take logarithms:



For $\delta \geq 1$ we show

the boundaries.
$$f'(\delta)$$

$$f(0) = 0 \text{ and } f(0)$$

A convex function (
$$f''(\delta) \ge 0$$
) on an interval takes maximum at the boundaries.
$$f'(\delta) = -\frac{1}{1+\delta} + 2/3 \qquad f''(\delta) = \frac{1}{(1+\delta)^2}$$

$$f(0) = 0 \text{ and } f(1) = -\ln(2) + 2/3 < 0$$

 $\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta/3}$

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 $f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$

For $\delta > 1$ we show

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta/3}$$

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta/3$$

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta$$

$$(1+0) \ln(1+0)) \le -0.0/3$$

$$f(0)=0 \text{ and } f(1)=-\ln(2)+2/3<0$$

the boundaries.

$$f'(\delta)$$
 = and $f(1)$

daries.
$$f'(\delta) = -\frac{1}{1+\delta} + 2/3 \qquad f''(\delta) = \frac{1}{(1+\delta)^2}$$

 $f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$

A convex function (
$$f^{\prime\prime}(\delta)\geq 0$$
) on an interval takes maximum at the boundaries.

$$\frac{1}{(1+\delta)^2}$$

$$=\frac{1}{(1+\delta)^2}$$

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17.1 Chernoff Bounds

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- - 17.1 Chernoff Bounds

For $\delta > 1$ we show

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$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta/3}$$

Take logarithms:

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True for $\delta = 0$.

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$$f'(\delta) = -\frac{1}{1+\delta} + 2/3$$
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17.1 Chernoff Bounds

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$$\frac{1}{(\delta)^2}$$

$$(1+\delta)^2$$

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17.1 Chernoff Bounds

Take logarithms:

For $\delta > 1$ we show

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$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta/3$$

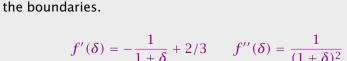
True for
$$\delta=0$$
. Divide by U and take derivatives:

$$-\ln(1+\delta) \le -1/3 \iff \ln(1+\delta) \ge 1/3$$
 (true)

17 1 Chernoff Bounds

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$$f'(\delta$$



f(0) = 0 and $f(1) = -\ln(2) + 2/3 < 0$

 $f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$

17.1 Chernoff Bounds

A convex function (
$$f''(\delta) \ge 0$$
) on an interval takes maximum at the boundaries.

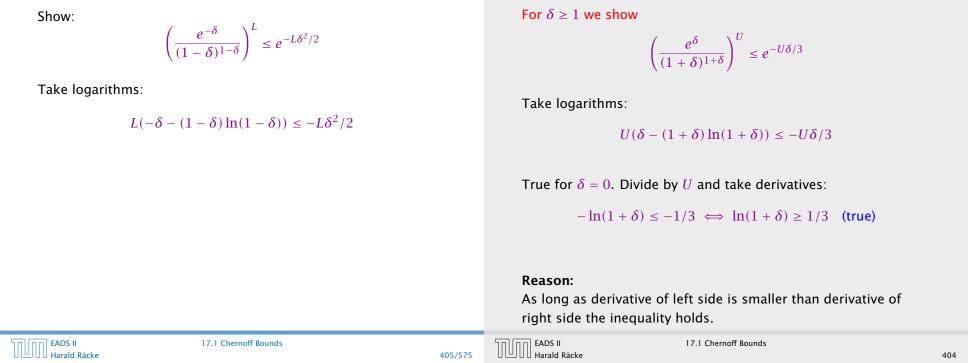
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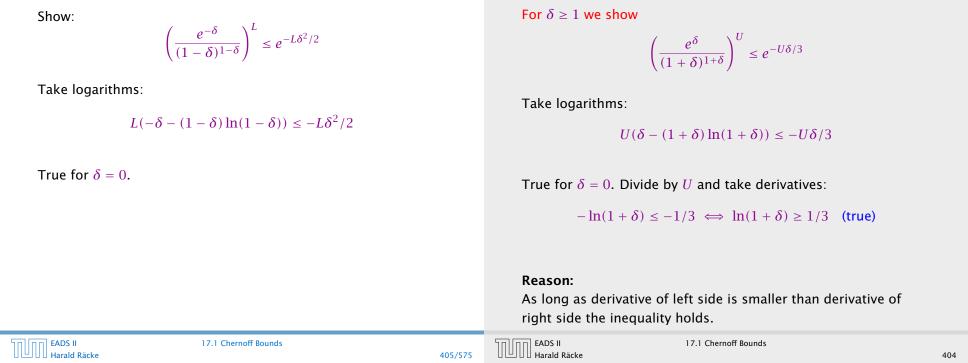
Show:
$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \leq e^{-L\delta^2/2}$$

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U \leq e^{-U\delta/3}$$
 Take logarithms:
$$U(\delta-(1+\delta)\ln(1+\delta)) \leq -U\delta/3$$
 True for $\delta=0$. Divide by U and take derivatives:
$$-\ln(1+\delta) \leq -1/3 \iff \ln(1+\delta) \geq 1/3 \text{ (true)}$$
 Reason: As long as derivative of left side is smaller than derivative of right side the inequality holds.

Reason: As long as derivative of left side is smaller than derivative of right side the inequality holds.

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$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \leq e^{-L\delta^2/2}$$
 Take logarithms:
$$L(-\delta-(1-\delta)\ln(1-\delta)) \leq -L\delta^2/2$$
 True for $\delta=0$. Divide by L and take derivatives:
$$\ln(1-\delta) \leq -\delta$$

 $U(\delta - (1 + \delta) \ln(1 + \delta)) \le -U\delta/3$ True for $\delta = 0$. Divide by *U* and take derivatives: $-\ln(1+\delta) \le -1/3 \iff \ln(1+\delta) \ge 1/3$ (true) Reason:

As long as derivative of left side is smaller than derivative of

17 1 Chernoff Bounds

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 $\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta/3}$

For $\delta > 1$ we show

Take logarithms:

right side the inequality holds.

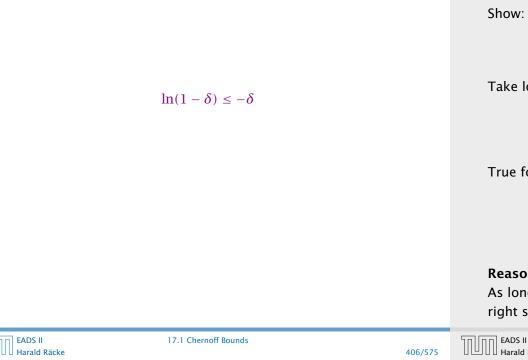
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Show:

As long as derivative of left side is smaller than derivative of

17.1 Chernoff Bounds



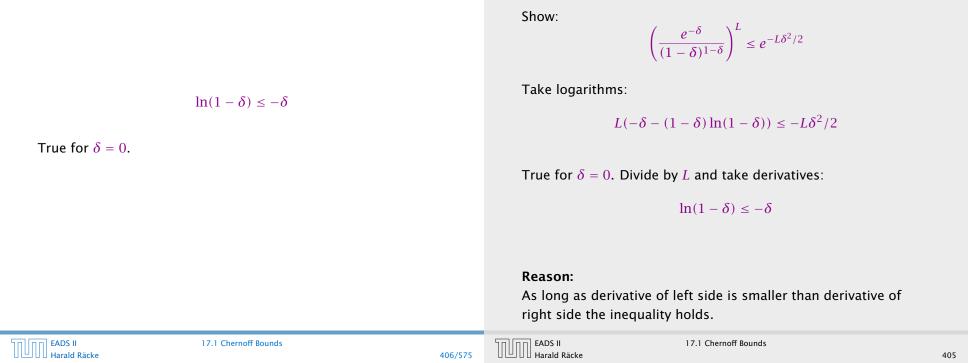
Take logarithms: $L(-\delta - (1 - \delta) \ln(1 - \delta)) \le -L\delta^2/2$ True for $\delta = 0$. Divide by L and take derivatives: $ln(1-\delta) \leq -\delta$ Reason:

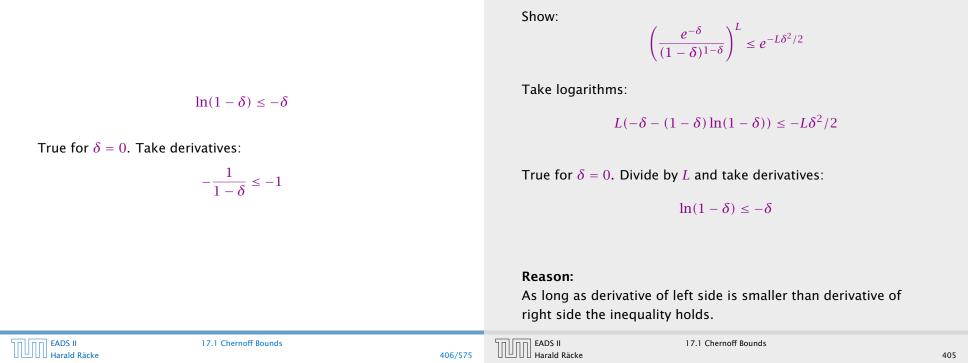
17.1 Chernoff Bounds

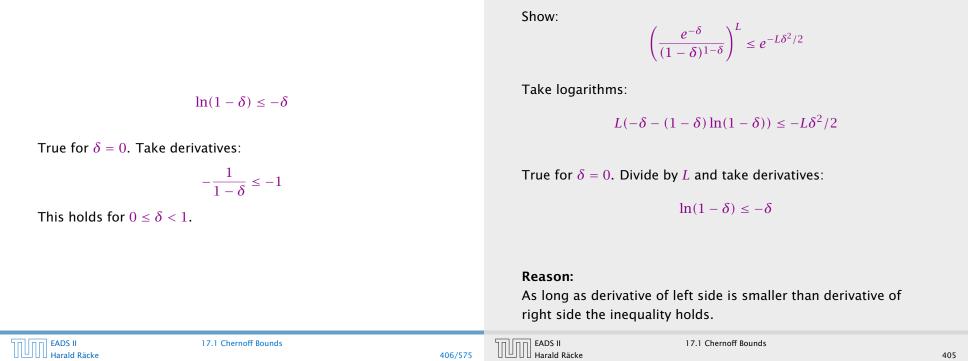
 $\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \le e^{-L\delta^2/2}$

As long as derivative of left side is smaller than derivative of

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- Given s_i - t_i pairs in a graph.
- Connect each pair by a path such that not too many path use any given edge.

$$\ln(1-\delta) \le -\delta$$

True for $\delta = 0$. Take derivatives:

$$-\frac{1}{1-\delta} \le -1$$

This holds for $0 \le \delta < 1$.

Randomized Rounding:

For each i choose one path from the set \mathcal{P}_i at random according to the probability distribution given by the Linear Programming solution.

Integer Multicommodity Flows

- Given s_i - t_i pairs in a graph.
- ► Connect each pair by a path such that not too many path use any given edge.

Theorem 5

If $W^* \ge c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

Theorem 6

With probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + c \ln n$.

Integer Multicommodity Flows

Randomized Rounding:

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For each i choose one path from the set \mathcal{P}_i at random according to the probability distribution given by the Linear Programming solution.

Let X_e^i be a random variable that indicates whether the path for s_i - t_i uses edge e.

Then the number of paths using edge e is $Y_e = \sum_i X_e^i$.

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$$E[Y_e] = \sum_{\substack{i \text{ } n \in P: o \in n}} x_p^* = \sum_{\substack{n \text{ } o \in P}} x_p^* \le W^*$$

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17.2 Integer Multicommodity Flows

Theorem 5

If $W^* \ge c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

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Theorem 6

Choose
$$\delta = \sqrt{(c \ln n)/W^*}$$
.

$$\Pr[Y_e \ge (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{m^c}$$

Integer Multicommodity Flows

Let X_o^i be a random variable that indicates whether the path for s_i - t_i uses edge e.

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Problem definition:

- n Boolean variables

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

Integer Multicommodity Flows

 $\Pr[Y_e \ge (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$

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Then

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Problem definition:

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- ightharpoonup m clauses C_1, \ldots, C_m . For example

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Terminology:

- ▶ A variable x_i and its negation \bar{x}_i are called literals.
- ▶ Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \lor x_i \lor \bar{x}_i$ is **not** a clause).
- ▶ We assume a clause does not contain x_i and \bar{x}_i for any i.
- \triangleright x_i is called a positive literal while the negation \bar{x}_i is called a negative literal.
- ▶ For a given clause C_j the number of its literals is called its length or size and denoted with ℓ_i .
- Clauses of length one are called unit clauses

17.3 MAXSAT

Problem definition:

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173 MAXSAT

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MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

17.3 MAXSAT

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- ► Clauses of length one are called unit clauses.

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

$$W = \sum_{i} w_{j} X_{i}$$

MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

MAXSAT: Flipping Coins

Define random variable X_i with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_{j} w_{j} X_{j}$$

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_{j} w_{j} X_{j}$$

$$E[W] = \sum_{j} w_{j} E[X_{j}]$$

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$$E[W] = \sum_{j} w_{j} E[X_{j}]$$

$$= \sum_{j} w_{j} Pr[C_{j} \text{ is satisified}]$$

$$= \sum_{j} w_{j} \left(1 - \left(\frac{1}{2}\right)^{\ell_{j}}\right)$$

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$$W = \sum_{j} w_{j} X_{j}$$

$$E[W] = \sum_{j} w_{j} E[X_{j}]$$

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$$= \sum_{j} w_{j} \left(1 - \left(\frac{1}{2}\right)^{\ell_{j}}\right)$$

$$\geq \frac{1}{2} \sum_{j} w_{j}$$

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

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$$\geq \frac{1}{2} \text{OPT}$$

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$$W = \sum_{j} w_{j} X_{j}$$

MAXSAT: LP formulation

Let for a clause C_j , P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{j \in P_j} x_i \vee \bigvee_{j \in N_j} \bar{x}_i$$

$$E[W] = \sum_{j} w_{j} E[X_{j}]$$

$$= \sum_{j} w_{j} \Pr[C_{j} \text{ is satisified}]$$

$$= \sum_{j} w_{j} \left(1 - \left(\frac{1}{2}\right)^{\ell_{j}}\right)$$

$$\geq \frac{1}{2} \sum_{j} w_{j}$$

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MAXSAT: LP formulation

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17.3 MAXSAT

MAXSAT: Randomized Rounding

Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

MAXSAT: LP formulation

► Let for a clause C_i , P_i be the set of positive literals and N_i

the set of negative literals.

$$C_j = \bigvee_{j \in P_j} x_i \vee \bigvee_{j \in N_j} \bar{x}_i$$

$$\begin{bmatrix} \max & \sum_{j} w_{j} z_{j} \\ \text{s.t.} & \forall j & \sum_{i \in P_{j}} y_{i} + \sum_{i \in N_{j}} (1 - y_{i}) \geq z_{j} \\ & \forall i & y_{i} \in \{0, 1\} \\ & \forall j & z_{j} \leq 1 \end{bmatrix}$$

MAXSAT: Randomized Rounding

Lemma 7 (Geometric Mean ≤ Arithmetic Mean)

For any nonnegative a_1, \ldots, a_k

$$\left(\prod_{i=1}^k a_i\right)^{1/k} \le \frac{1}{k} \sum_{i=1}^k a_i$$

Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1 - \lambda)r) \ge \lambda f(s) + (1 - \lambda)f(r)$$

Lemma 9

Let f be a concave function on the interval [0,1], with f(0)=a and f(1)=a+b. Then

$$f(\lambda)$$

for $\lambda \in [0,1]$.

Lemm For ar

Lemma 7 (Geometric Mean ≤ Arithmetic Mean)

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Let f be a concave function on the interval [0,1], with f(0)=a and f(1)=a+b. Then

$$f(\lambda) = f((1 - \lambda)0 + \lambda 1)$$

$$\geq (1 - \lambda) f(0) + \lambda f(1)$$

$$= a + \lambda b$$

for
$$\lambda \in [0,1]$$
.

EADS II

Lemma 7 (Geometric Mean \leq **Arithmetic Mean)** *For any nonnegative* a_1, \ldots, a_k

$$\left(\prod_{i=1}^k a_i\right)^{1/k} \le \frac{1}{k} \sum_{i=1}^k a_i$$

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for $\lambda \in [0,1]$.

Lemma 7 (Geometric Mean ≤ Arithmetic Mean) For any nonnegative a_1, \ldots, a_k

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17.3 MAXSAT

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Lemma 7 (Geometric Mean ≤ Arithmetic Mean) For any nonnegative a_1, \ldots, a_k

$$\left(\prod_{i=1}^k a_i\right)^{1/k} \le \frac{1}{k} \sum_{i=1}^k a_i$$

 $Pr[C_i \text{ not satisfied}]$

A function f on an interval I is concave if for any two points sand r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1 - \lambda)r) \ge \lambda f(s) + (1 - \lambda)f(r)$$

Lemma 9

Definition 8

Let
$$f$$
 be a concave function on the interval $[0,1]$, with $f(0)=a$ and $f(1)=a+b$. Then

and
$$f(1) = a + b$$
. Then
$$f(\lambda)$$

$$f(\lambda) = f((1 - \lambda)0 + \lambda 1)$$

$$\geq (1 - \lambda)f(0) + \lambda f(1)$$
$$= a + \lambda b$$

for
$$\lambda \in [0,1]$$
.

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$Pr[C_j \text{ not satisfied}] = \prod (1 - y_i) \prod y_i$

Definition 8 A function f on an interval I is concave if for any two points s

and r from I and any $\lambda \in [0,1]$ we have $f(\lambda s + (1 - \lambda)r) \ge \lambda f(s) + (1 - \lambda) f(r)$

Let
$$f$$
 be a concave function on the interval $[0,1]$, with $f(0)=a$

and
$$f(1)=a+b$$
 . Then
$$f(\lambda)=f((1-\lambda)0+\lambda 1)$$

$$\geq (1 - \lambda)f(0) + \lambda f(1)$$

$$= a + \lambda b$$

for
$$\lambda \in [0, 1]$$

$$\begin{aligned} \Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i \\ &\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j} \end{aligned}$$

A function f on an interval I is concave if for any two points sand r from I and any $\lambda \in [0,1]$ we have

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$$= a + \lambda b$$

for
$$\lambda \in [0,1]$$
.

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17.3 MAXSAT

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$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i$$

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$$= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j}$$

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$$f(\lambda) = f((1 - \lambda)0 + \lambda 1)$$

$$\geq (1 - \lambda)f(0) + \lambda f(1)$$

$$= a + \lambda b$$

for $\lambda \in [0,1]$.

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$$\begin{aligned} \Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i \\ &\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j} \\ &= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j} \\ &\leq \left(1 - \frac{z_j}{\ell_j} \right)^{\ell_j} \end{aligned}$$

17.3 MAXSAT

Definition 8

A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1 - \lambda)r) \ge \lambda f(s) + (1 - \lambda)f(r)$$

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Let f be a concave function on the interval [0,1], with f(0) = aand f(1) = a + b. Then

$$f(\lambda) = f((1 - \lambda)0 + \lambda 1)$$

$$\geq (1 - \lambda)f(0) + \lambda f(1)$$

$$= a + \lambda h$$

$$= a + \lambda b$$

for
$$\lambda \in [0, 1]$$

for $\lambda \in [0,1]$.

EADS II

The function
$$f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$$
 is concave. Hence,

$$Pr[C_j \text{ satisfied}]$$

$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i$$

$$\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j}$$

$$= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j}$$

$$\leq \left(1 - \frac{z_j}{\ell_j} \right)^{\ell_j}.$$

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$$\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j}$$

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$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i$$

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$$\leq \left(1 - \frac{z_j}{\ell_j} \right)^{\ell_j}.$$

$$\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j}$$

$$\ge \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j .$$

$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i$$

$$\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j}$$

$$= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j}$$

$$\leq \left(1 - \frac{z_j}{\ell_i} \right)^{\ell_j}.$$



$$\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j}$$

$$\ge \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j.$$

$$f''(z)=-rac{\ell-1}{\ell}\Big[1-rac{z}{\ell}\Big]^{\ell-2}\leq 0$$
 for $z\in[0,1].$ Therefore, f is concave.

 $Pr[C_i \text{ not satisfied}] = [(1 - y_i) | y_i]$ $\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_i} (1 - y_i) + \sum_{i \in N_j} y_i\right)\right]^{\ell_j}$ $= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_i} y_i + \sum_{i \in N_i} (1 - y_i)\right)\right]^{\ell_j}$ $\leq \left(1 - \frac{z_j}{\varrho}\right)^{\ell_j}$.

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 $\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j$.

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

 $\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_i}\right)^{\ell_j}$

concave.

$$f''(z)=-rac{\ell-1}{\ell}\Big[1-rac{z}{\ell}\Big]^{\ell-2}\leq 0$$
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$$E[W] = \sum_{j} w_{j} \Pr[C_{j} \text{ is satisfied}]$$

 $\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_i}\right)^{\ell_j}$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j .$$

$$f''(z)=-rac{\ell-1}{\ell}\Big[1-rac{z}{\ell}\Big]^{\ell-2}\leq 0 ext{ for } z\in[0,1].$$
 Therefore, f is

concave.

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$$E[W] = \sum_{j} w_{j} \Pr[C_{j} \text{ is satisfied}]$$

$$\geq \sum_{j} w_{j} z_{j} \left[1 - \left(1 - \frac{1}{\ell_{j}} \right)^{\ell_{j}} \right]$$

 $\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_i}\right)^{\ell_j}$

$$\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j .$$

 $f''(z) = -\frac{\ell-1}{\ell} \left[1 - \frac{z}{\ell}\right]^{\ell-2} \le 0$ for $z \in [0,1]$. Therefore, f is concave.

EADS II

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$$E[W] = \sum_{j} w_{j} \Pr[C_{j} \text{ is satisfied}]$$

$$\geq \sum_{j} w_{j} z_{j} \left[1 - \left(1 - \frac{1}{\ell_{j}} \right)^{\ell_{j}} \right]$$

$$\geq \left(1 - \frac{1}{\rho} \right) \text{ OPT }.$$

•

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\Pr[C_j \text{ satisfied}] \ge 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j}$$

$$\ge \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j .$$

 $f''(z)=-rac{\ell-1}{\ell}\Big[1-rac{z}{\ell}\Big]^{\ell-2}\leq 0$ for $z\in[0,1].$ Therefore, f is concave.

MAXSAT: The better of two

Theorem 10

Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$ -approximation.

$$\begin{split} E[W] &= \sum_{j} w_{j} \text{Pr}[C_{j} \text{ is satisfied}] \\ &\geq \sum_{j} w_{j} z_{j} \left[1 - \left(1 - \frac{1}{\ell_{j}} \right)^{\ell_{j}} \right] \\ &\geq \left(1 - \frac{1}{e} \right) \text{OPT }. \end{split}$$

obtained by coin flipping. $E[\max\{W_1, W_2\}]$

Let W_1 be the value of randomized rounding and W_2 the value

MAXSAT: The better of two

Theorem 10 Choosing the better of the two solutions given by randomized

rounding and coin flipping yields a $\frac{3}{4}$ -approximation.

17.3 MAXSAT

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

$$\geq E[\frac{1}{2}W_1 + \frac{1}{2}W_2]$$

MAXSAT: The better of two

Theorem 10

Choosing the better of the two solutions given by randomized

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EADS II

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_{1}, W_{2}\}]$$

$$\geq E[\frac{1}{2}W_{1} + \frac{1}{2}W_{2}]$$

$$\geq \frac{1}{2} \sum_{j} w_{j} z_{j} \left[1 - \left(1 - \frac{1}{\ell_{j}}\right)^{\ell_{j}}\right] + \frac{1}{2} \sum_{j} w_{j} \left(1 - \left(\frac{1}{2}\right)^{\ell_{j}}\right)$$

MAXSAT: The better of two

Theorem 10 Choosing the better of the two solutions given by randomized

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17.3 MAXSAT





Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

$$\geq E[\frac{1}{2}W_1 + \frac{1}{2}W_2]$$

$$\geq \frac{1}{2}\sum_{j}w_jz_j\left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_{j}w_j\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)$$

$$\geq \sum_{j}w_jz_j\left[\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)\right]$$

$$\geq \frac{3}{4} \text{ for all integers}$$

MAXSAT: The better of two

Theorem 10

Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$ -approximation.

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

$$\geq E\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right]$$

$$\geq \frac{1}{2}\sum_{j}w_{j}z_{j}\left[1 - \left(1 - \frac{1}{\ell_{j}}\right)^{\ell_{j}}\right] + \frac{1}{2}\sum_{j}w_{j}\left(1 - \left(\frac{1}{2}\right)^{\ell_{j}}\right)$$

$$\geq \sum_{j}w_{j}z_{j}\left[\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_{j}}\right)^{\ell_{j}}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_{j}}\right)\right]$$

$$\geq \frac{3}{4}\text{for all integers}$$

$$\geq \frac{3}{4}\text{OPT}$$

17.3 MAXSAT

MAXSAT: The better of two

Theorem 10

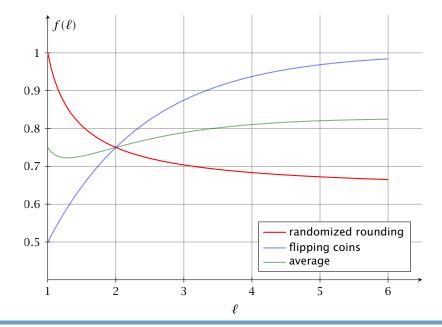
Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$ -approximation.

EADS II

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Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

$$\geq E\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right]$$

$$\geq \frac{1}{2}\sum_{j}w_jz_j\left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_{j}w_j\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)$$

$$\geq \sum_{j}w_jz_j\left[\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)\right]$$

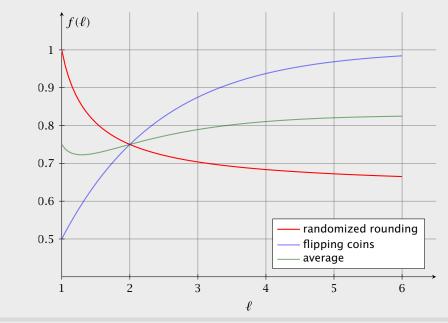
$$\geq \frac{3}{4} \text{for all integers}$$

 $\geq \frac{3}{4}$ OPT

MAXSAT: Nonlinear Randomized Rounding

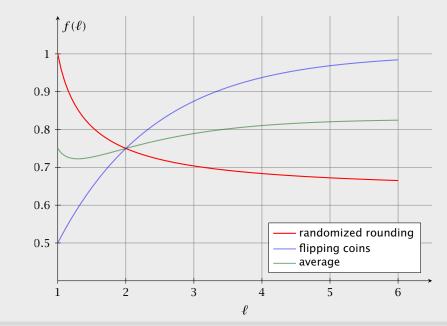
So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f:[0,1] \to [0,1]$ and set x_i to true with probability $f(y_i)$.



So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f:[0,1] \to [0,1]$ and set x_i to true with probability $f(\gamma_i)$.



Let $f:[0,1] \rightarrow [0,1]$ be a function with

$$1 - 4^{-x} \le f(x) \le 4^{x - 1}$$

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the

corresponding variable in the linear program.

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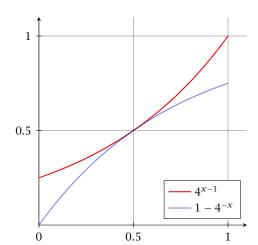
$$1 - 4^{-x} \le f(x) \le 4^{x - 1}$$

Theorem 11 Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$ -approximation.

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

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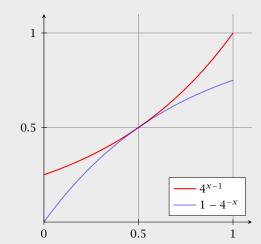
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Theorem 11

EADS II

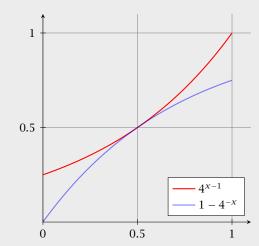
Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$ -approximation.

 $Pr[C_j \text{ not satisfied}]$

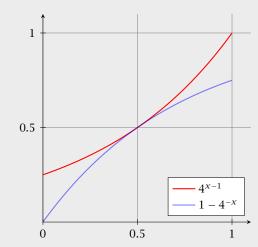




$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - f(y_i)) \prod_{i \in N_j} f(y_i)$$

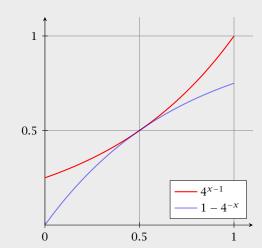


$$\begin{split} \Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - f(y_i)) \prod_{i \in N_j} f(y_i) \\ &\leq \prod_{i \in P_j} 4^{-y_i} \prod_{i \in N_j} 4^{y_i - 1} \end{split}$$



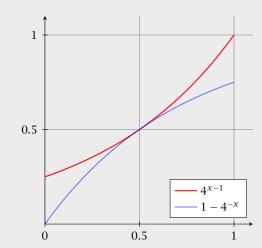


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The function $g(z) = 1 - 4^{-z}$ is concave on [0,1]. Hence,

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17.3 MAXSAT

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17.3 MAXSAT

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17.3 MAXSAT

The function $g(z) = 1 - 4^{-z}$ is concave on [0, 1]. Hence,

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430

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Not if we compare ourselves to the value of an optimum LP-solution.

Definition 12 (Integrality Gap

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

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Lemma 13

Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

Can we do better?

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Consider: $(x_1 \lor x_2) \land (\bar{x}_1 \lor x_2) \land (x_1 \lor \bar{x}_2) \land (\bar{x}_1 \lor \bar{x}_2)$

- any solution can satisfy at most 3 clauses
- we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
- hence, the LP has value 4.

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MaxCut

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Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

MaxCut

Given a weighted graph $G = (V, E, w), w(v) \ge 0$, partition the vertices into two parts. Maximize the weight of edges between the parts.

Trivial 2-approximation

$$\max \qquad \sum_{j} w_{j} z_{j}$$
s.t. $\forall j \quad \sum_{i \in P_{j}} y_{i} + \sum_{i \in N_{j}} (1 - y_{i}) \geq z_{j}$

$$\forall i \quad y_{i} \in \{0, 1\}$$

$$\forall j \quad z_{j} \leq 1$$

- Consider: $(x_1 \lor x_2) \land (\bar{x}_1 \lor x_2) \land (x_1 \lor \bar{x}_2) \land (\bar{x}_1 \lor \bar{x}_2)$
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EADS II 17.4 MAXCUT Harald Räcke 434/575

Semidefinite Programming

- linear objective, linear contraints
- we can constrain a square matrix of variables to be symmetric positive definite

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Vector Programming

- variables are vectors in n-dimensional space
- objective functions and contraints are linear in inner products of the vectors

This is equivalent!

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Fact [without proof]

We (essentially) can solve Semidefinite Programs in polynomial time...

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Quadratic Programs

Quadratic Program for MaxCut:

$$\max \frac{\frac{1}{2} \sum_{i,j} w_{ij} (1 - y_i y_j)}{\forall i} \quad \forall i \quad y_i \in \{-1, 1\}$$

This is exactly MaxCut!

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We (essentially) can solve Semidefinite Programs in polynomial time...

Semidefinite Relaxation

$$\begin{array}{cccc}
\max & \frac{1}{2} \sum_{i,j} w_{ij} (1 - v_i^t v_j) \\
\forall i & v_i^t v_i &= 1 \\
\forall i & v_i \in \mathbb{R}^n
\end{array}$$

- this is clearly a relaxation the solution will be vectors on the unit sphere

Quadratic Programs

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$$\max_{\substack{\frac{1}{2}\sum_{i,j}w_{ij}(1-y_iy_j)\\\forall i}} \frac{\frac{1}{2}\sum_{i,j}w_{ij}(1-y_iy_j)}{v_i \in \{-1,1\}}$$

This is exactly MaxCut!



- ▶ Choose a random vector r such that $r/\|r\|$ is uniformly distributed on the unit sphere.
- If $r^t v_i > 0$ set $y_i = 1$ else set $y_i = -1$

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- ► the solution will be vectors on the unit sphere

Choose the *i*-th coordinate r_i as a Gaussian with mean 0 and variance 1, i.e., $r_i \sim \mathcal{N}(0,1)$.

Density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}$$

Then

$$\Pr[r = (x_1, ..., x_n)]$$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{x_1^2/2} \cdot e^{x_2^2/2} \cdot ... \cdot e^{x_n^2/2} dx_1 \cdot ... \cdot dx_n$$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{\frac{1}{2}(x_1^2 + ... + x_n^2)} dx_1 \cdot ... \cdot dx_n$$

Hence the probability for a point only depends on its distance to the origin.

Rounding the SDP-Solution

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Fact

The projection of r onto two unit vectors e_1 and e_2 are independent and are normally distributed with mean 0 and variance 1 iff e_1 and e_2 are orthogonal.

Note that this is clear if e_1 and e_2 are standard basis vectors.

Rounding the SDP-Solution

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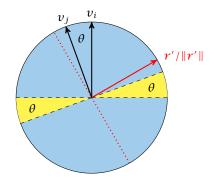
Corollary If we project r onto a hyperplane its normalized projection (r'/||r'||) is uniformly distributed on the unit circle within the hyperplane.

Rounding the SDP-Solution

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- if the normalized projection falls into the shaded region, v_i and v_i are rounded to different values
- this happens with probability θ/π

Rounding the SDP-Solution

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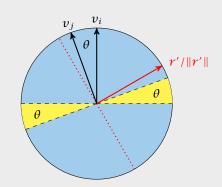
• contribution of edge (i, j) to the SDP-relaxation:

$$\frac{1}{2}w_{ij}\big(1-v_i^tv_j\big)$$

- (expected) contribution of edge (i, j) to the rounded instance $w_{i,i} \arccos(v_i^t v_i)/\pi$
- ratio is at most

$$\min_{x \in [-1,1]} \frac{2\arccos(x)}{\pi(1-x)} \ge 0.876$$

Rounding the SDP-Solution



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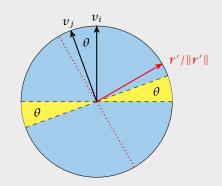
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Rounding the SDP-Solution



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17.4 MAXCUT

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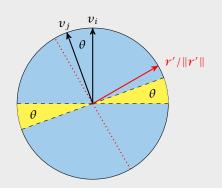
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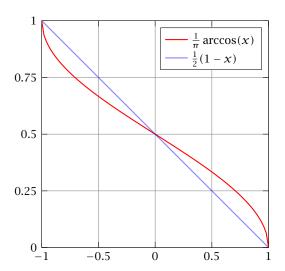
Rounding the SDP-Solution



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17.4 MAXCUT

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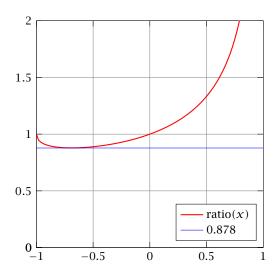
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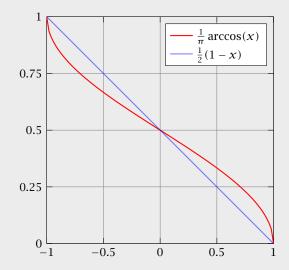
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- ► ratio is at most

$$\min_{x \in [-1,1]} \frac{2\arccos(x)}{\pi(1-x)} \ge 0.878$$



Rounding the SDP-Solution





Theorem 14

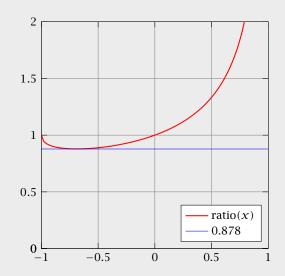
Given the unique games conjecture, there is no α -approximation for the maximum cut problem with constant

$$\alpha > \min_{x \in [-1,1]} \frac{2\arccos(x)}{\pi(1-x)}$$

17.4 MAXCUT

unless P = NP.

Rounding the SDP-Solution



17.4 MAXCUT