Complexity

LP Feasibility Problem (LP feasibility)

Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

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- In the following we sometimes refer to $L := \langle A\rangle + \langle b\rangle$ as the input size (even though the real input size is something in $\Theta(\langle A\rangle + \langle b\rangle)$).
- In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

Suppose that $Ax = b; x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

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Size of a Basic Feasible Solution

Lemma 2

Let $M \in \mathbb{Z}^{m\times m}$ be an invertible matrix and let $b \in \mathbb{Z}^m$. Further define $L = \langle M\rangle + \langle b\rangle + n\log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^L$ and $|D| \le 2^L$.

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

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Analogously for $\det(M_j)$.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^Tx \mid Ax = b; x \geq 0\}$ do a binary search for the optimum solution

(Add constraint $c^Tx - \delta = M$; $\delta \geq 0$ or ($c^Tx \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2\left(\frac{2n2^{2L'}}{1/2^{L'}}\right) = \mathcal{O}(L') \ ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

Here we use $L' = \langle A\rangle + \langle b\rangle + \langle c\rangle + n\log_2 n$ (it also includes the encoding size of c).

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Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

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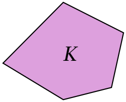

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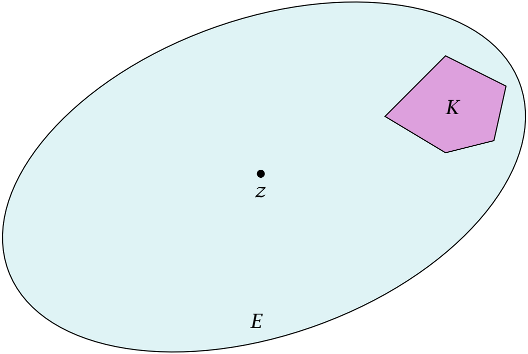

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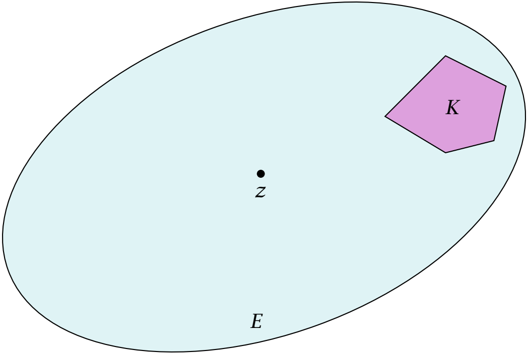

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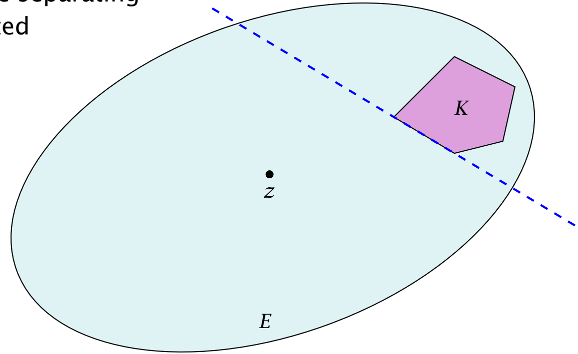

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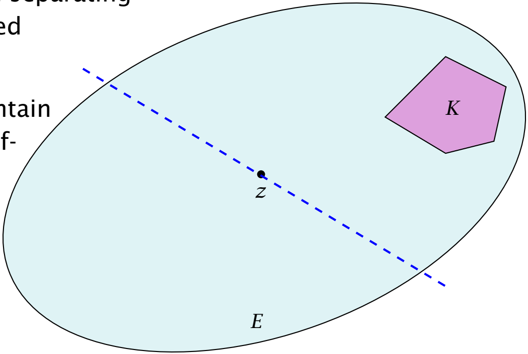

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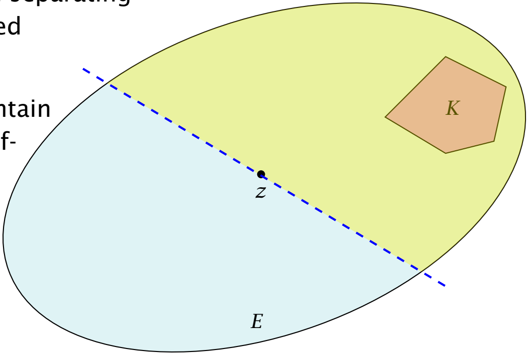

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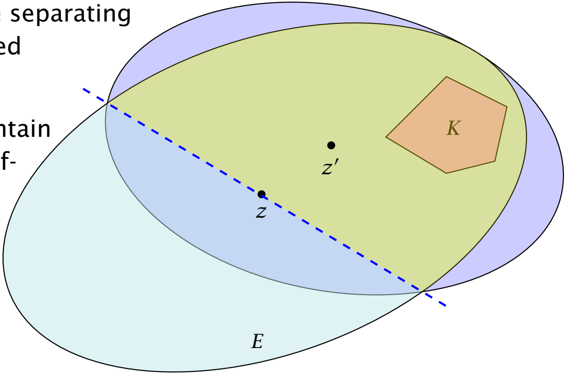
- Compute (smallest) ellipsoid E' that contains $E \cap H$.



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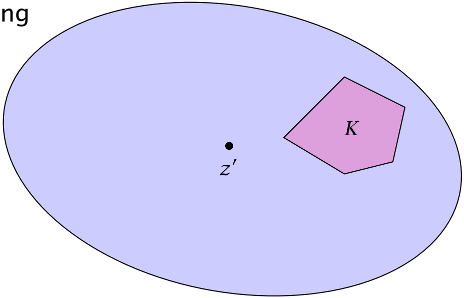

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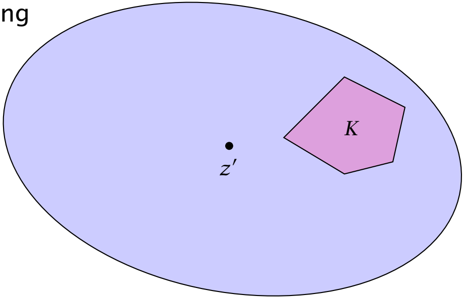
- REPEAT



Issues/Questions:

- How do you choose the first Ellipsoid? What is its volume?
- How do you measure progress? By how much does the volume decrease in each iteration?
- When can you stop? What is the minimum volume of a non-empty polytop?

Definition 3

A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an affine transformation.

Definition 4

A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

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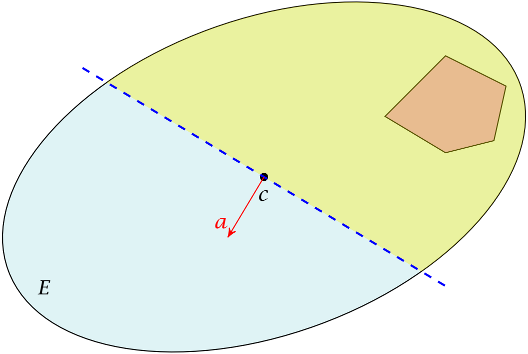
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid



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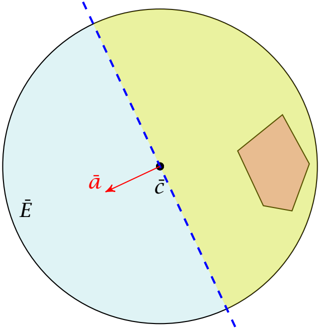
- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



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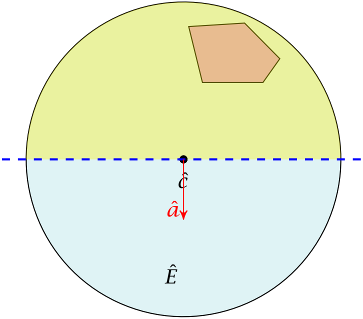

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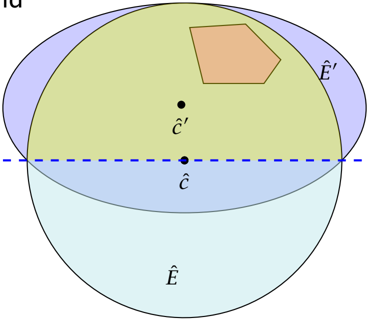

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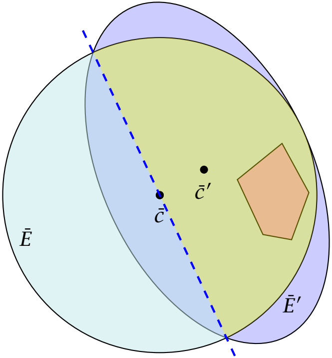

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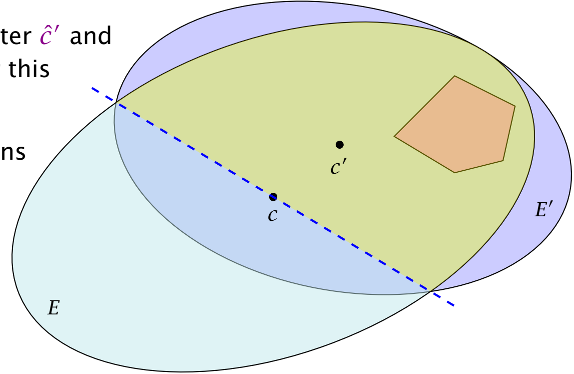

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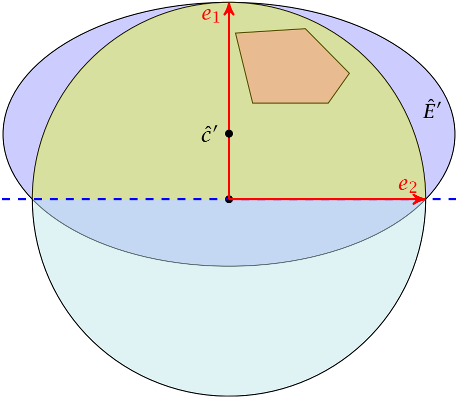

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- The vectors $e_1, e_2, \ldots$ have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1}(e_i - \hat{c}') = 1$.

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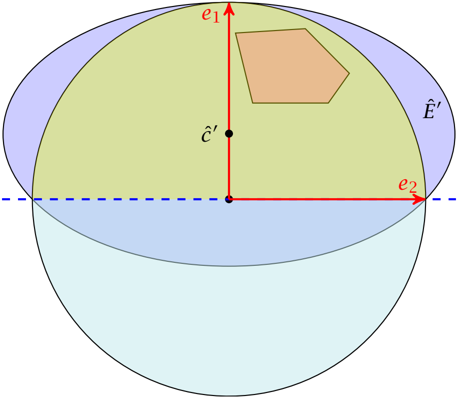

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- To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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  maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

- To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

  maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $(1-t)^2 = a^2$.

The Easy Case

- For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}$$

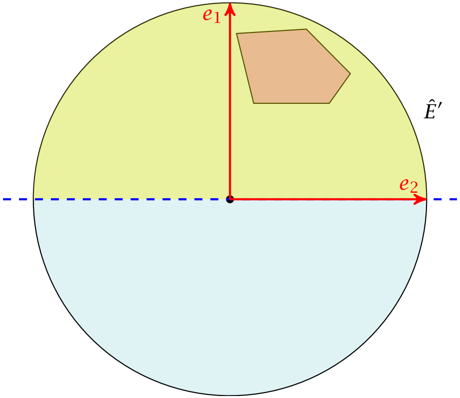
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1-t}{\sqrt{1-2t}}$$

The Easy Case

We still have many choices for t:



Choose t such that the volume of $\hat{E}'$ is minimal!!!

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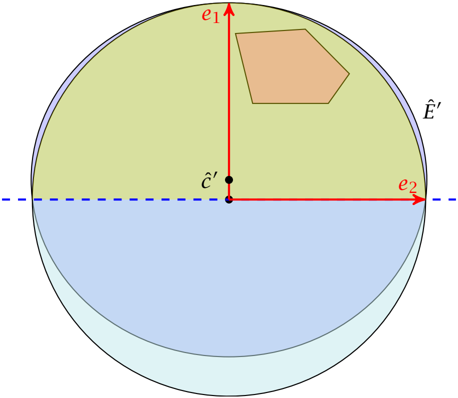

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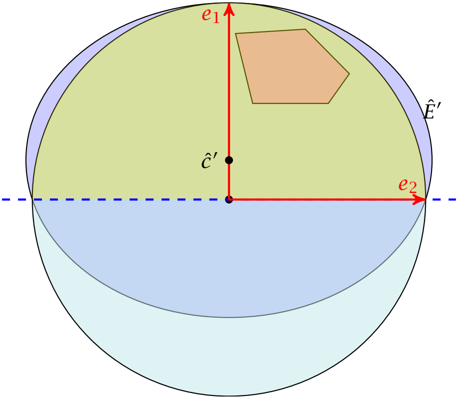

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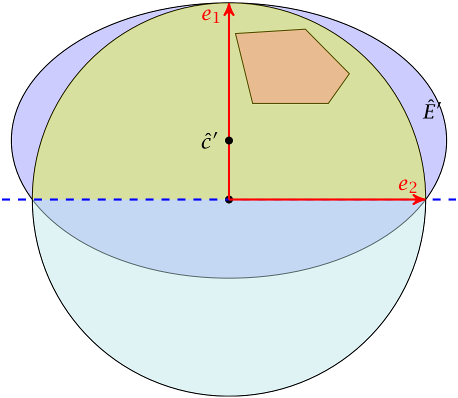

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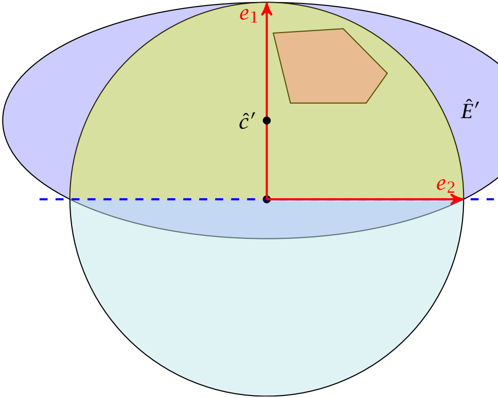

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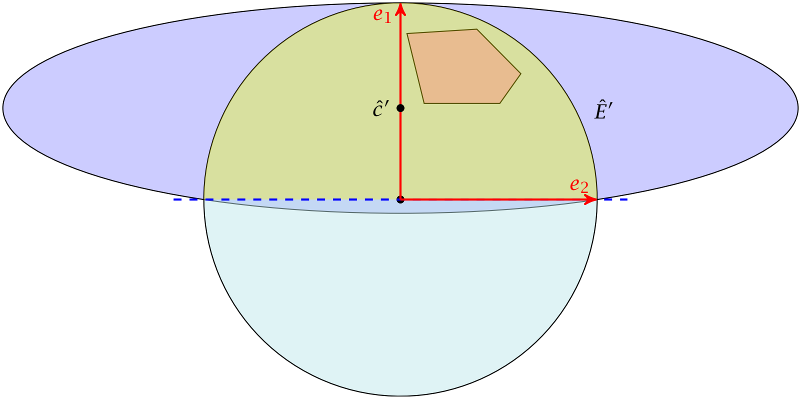

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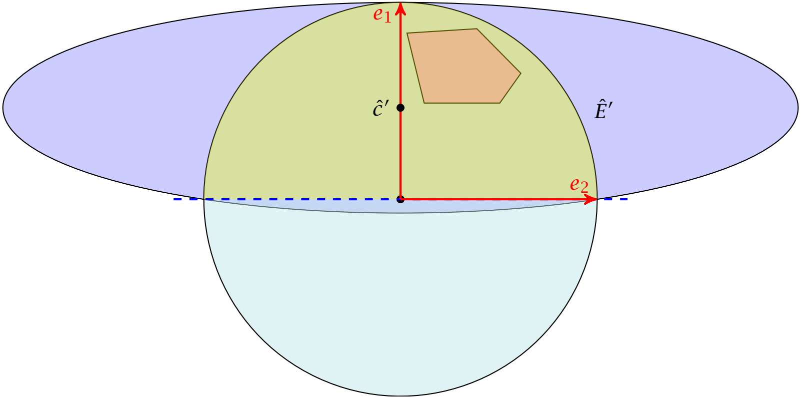

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We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 6

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \enspace .$$

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Lemma 6

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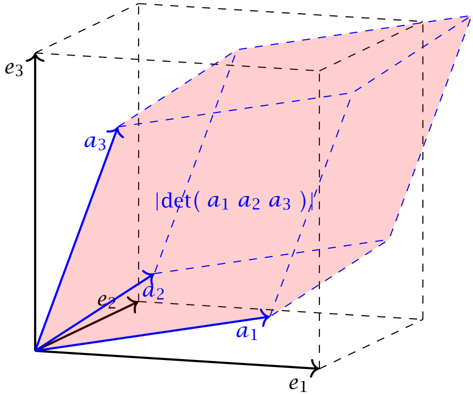
n-dimensional volume



The Easy Case

- We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\operatorname{vol}(\hat{E}') = \operatorname{vol}(B(0,1)) \cdot |\det(\hat{L}')| \ ,$$

- Recall that

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- Note that a and b in the above equations depend on t, by the previous equations.

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The Easy Case

$\mathrm{vol}(\hat{E}')$

The Easy Case

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$$

We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t}$$

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{\Phi}{N^2}$$

N = denominator

The Easy Case

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\ &= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\end{aligned}$$

derivative of numerator

The Easy Case

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right) \\
&= \frac{\Phi}{N^2} \cdot \left((-1) \cdot n(1-t)^{n-1} \cdot \underset{\boxed{\text{denominator}}}{(\sqrt{1-2t})^{n-1}} \right.
\end{aligned}
$$

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$$-(n-1)(\sqrt{1-2t})^{n-2}$$

outer derivative

The Easy Case

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\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot(\sqrt{1-2t})^{n-1}\\
&\quad -(n-1)(\sqrt{1-2t})^{n-2}\cdot\underbrace{\frac{1}{2\sqrt{1-2t}}\cdot(-2)}_{\text{inner derivative}}
\end{aligned}
$$

The Easy Case

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\ &= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot(\sqrt{1-2t})^{n-1}\\ &\qquad -(n-1)(\sqrt{1-2t})^{n-2}\cdot\frac{1}{2\sqrt{1-2t}}\cdot(-2)\cdot\underset{\boxed{\text{numerator}}}{(1-t)^n}\Bigg)\end{aligned}$$

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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
&\qquad\qquad \cdot\Big((n-1)(1-t)-n(1-2t)\Big)
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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}
$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a$$

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The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$\gamma_n^2$$

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$$\begin{aligned}\gamma_n^2 &= \Big(\frac{n}{n+1}\Big)^2\Big(\frac{n^2}{n^2-1}\Big)^{n-1}\\ &= \Big(1-\frac{1}{n+1}\Big)^2\Big(1+\frac{1}{(n-1)(n+1)}\Big)^{n-1}\end{aligned}$$

The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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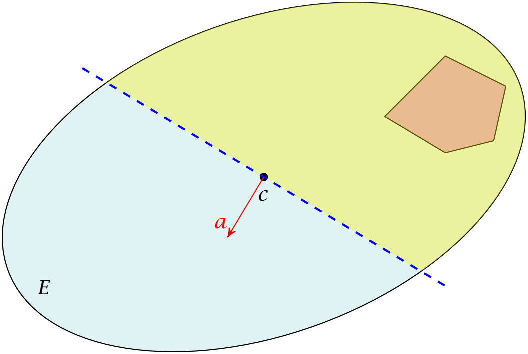
How to Compute the New Ellipsoid



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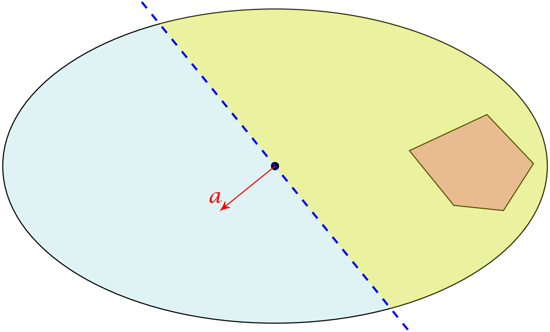
- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



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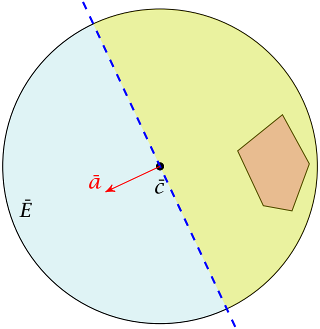

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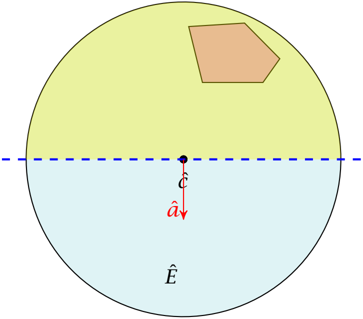

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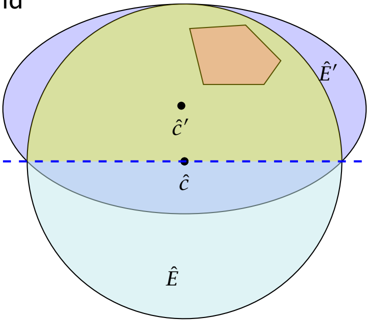

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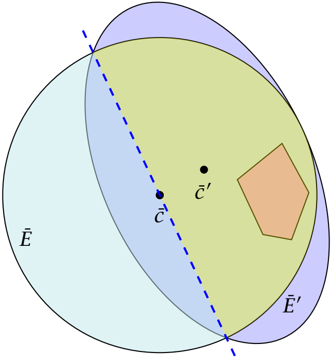

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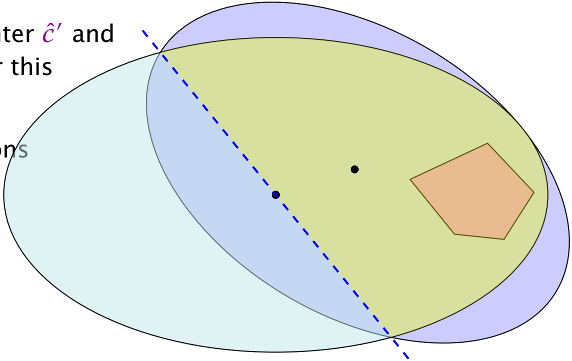

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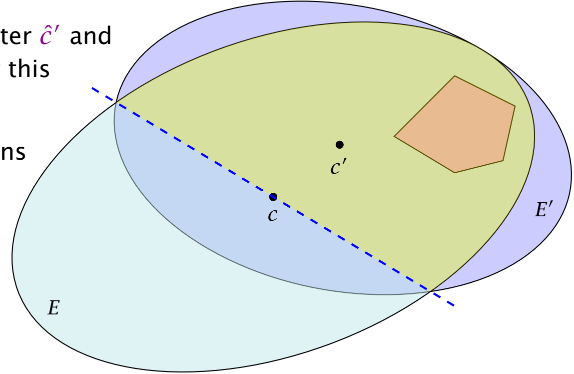

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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

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This means $\bar{a} = L^T a$.

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After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

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For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

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$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1)-2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$\bar{E}'$

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$$\bar{E}' = R(\hat{E}')$$

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$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}\end{aligned}$$

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&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)
\end{aligned}$$

9 The Ellipsoid Algorithm

E'

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Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

```
input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "$K$ is empty"
$Q \leftarrow$ ???
repeat
    if $c \in K$ then return $c$
    else
        choose a violated hyperplane $a$
        $c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$
        $Q \leftarrow \frac{n^2}{n^2-1} \Big( Q - \frac{2}{n+1} \frac{Q a a^T Q}{a^T Q a} \Big)$
    endif
until ???
return "$K$ is empty"
```

Repeat: Size of basic solutions

Lemma 7
Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ *be a bounded polyhedron. Let* $\langle a_{\max}\rangle$ *be the maximum encoding length of an entry in* A, b*. Then every entry* x_j *in a basic solution fulfills* $|x_j| = \frac{D_j}{D}$ *with* $D_j, D \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$*.*

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \leq b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

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Repeat: Size of basic solutions

Proof:

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, b, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by b) can become at most

$$\begin{aligned}\det(\bar{A}_B), \det(\bar{M}_j) &\leq \|\vec{\ell}_{\max}\|^{2n} \\ &\leq (\sqrt{2n} \cdot 2^{\langle a_{\max}\rangle})^{2n} \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} \quad ,\end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \le \delta$.

Hence, P is contained in the cube $-\delta \le x_i \le \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \operatorname{vol}(B(0,1)) \le (n\delta)^n \operatorname{vol}(B(0,1))$.

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When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \right\} ,$$

where $\lambda = \delta^2 + 1$.

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P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

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Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1} b + \frac{1}{\lambda} \bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

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$$(\bar{A}_B^{-1}b)_i < 0 \quad \Longrightarrow \quad (\bar{A}_B^{-1}b)_i \le -\frac{1}{\det(\bar{A}_B)}$$

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Lemma 9

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}} \mathrm{vol}(B(0,1))$.

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Let $\vec{\ell}$ with $\|\vec{\ell}\| \le r$. Then

$$\begin{aligned}(A(x+\vec{\ell}))_i &= (Ax)_i + (A\vec{\ell})_i \le b_i + \vec{a}_i^T \vec{\ell}\\ &\le b_i + \|\vec{a}_i\| \cdot \|\vec{\ell}\| \le b_i + \sqrt{n} \cdot 2^{\langle a_{\max}\rangle} \cdot r\\ &\le b_i + \frac{\sqrt{n}\cdot 2^{\langle a_{\max}\rangle}}{\delta^3} \le b_i + \frac{1}{\delta^2+1} \le b_i + \frac{1}{\lambda}\end{aligned}$$

Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

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Algorithm 1 ellipsoid-algorithm

```
input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii $R$ and $r$
        with $K \subseteq B(c,R)$, and $B(x,r) \subseteq K$ for some $x$
output: point $x \in K$ or "$K$ is empty"
$Q \leftarrow \mathrm{diag}(R^2,\dots,R^2)$ // i.e., $L = \mathrm{diag}(R,\dots,R)$
repeat
    if $c \in K$ then return $c$
    else
        choose a violated hyperplane $a$
        $c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$
        $Q \leftarrow \frac{n^2}{n^2-1} \left( Q - \frac{2}{n+1} \frac{Qaa^TQ}{a^TQa} \right)$
    endif
until $\det(Q) \le r^{2n}$ // i.e., $\det(L) \le r^n$
return "$K$ is empty"
```

Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
- or finds a hyperplane separating x from K.

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
- an initial ball $B(c, R)$ with radius R that contains K,
- a separation oracle for K.

The Ellipsoid algorithm requires $\mathcal{O}(\mathrm{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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