Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source $s$; target $t$
- no edges entering $s$ or leaving $t$
- at least for now: no parallel edges;

**Example 3**

The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$.

Cuts

**Definition 1**
An $(s, t)$-cut in the graph $G$ is given by a set $A \subseteq V$ with $s \in A$ and $t \in V \setminus A$.

**Definition 2**
The capacity of a cut $A$ is defined as

$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e),$$

where $\text{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e., edges leaving $A$).

**Minimum Cut Problem**: Find an $(s, t)$-cut with minimum capacity.

Flows

**Definition 4**
An $(s, t)$-flow is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies

1. For each edge $e$
   $$0 \leq f(e) \leq c(e).$$
   (capacity constraints)
2. For each $v \in V \setminus \{s, t\}$
   $$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e).$$
   (flow conservation constraints)
**Flows**

**Definition 5**
The value of an \((s, t)\)-flow \(f\) is defined as

\[
\text{val}(f) = \sum_{e \in \text{out}(s)} f(e).
\]

**Maximum Flow Problem:** Find an \((s, t)\)-flow with maximum value.

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**Example 6**

The value of the flow is \(\text{val}(f) = 24\).

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**Proof.**

\[
\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)
\]

\[
= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A(s)} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)
\]

\[
= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e).
\]

The last equality holds since every edge with both end-points in \(A\) contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn’t cancel out are edges leaving or entering \(A\).
Corollary 9

Let $f$ be an $(s,t)$-flow and let $A$ be an $(s,t)$-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then $f$ is a maximum flow.

Proof.

Suppose that there is a flow $f'$ with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$

$$= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e)$$

$$\leq \sum_{e \in \text{out}(A)} f'(e)$$

$$\leq \text{cap}(A, V \setminus A)$$