How to find an augmenting path?

Construct an alternating tree.

Case 4:

\[ y \text{ is already contained in } T \text{ as an even vertex} \]

\[ \text{can’t ignore } y \]

The cycle \( w \leftrightarrow y \leftarrow x \leftrightarrow w \) is called a blossom. \( w \) is called the base of the blossom (even node!!!). The path \( u-w \) is called the stem of the blossom.
Flowers and Blossoms

Definition 1
A flower in a graph $G = (V,E)$ w.r.t. a matching $M$ and a (free) root node $r$, is a subgraph with two components:

- A stem is an even length alternating path that starts at the root node $r$ and terminates at some node $w$. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node $w$ of a stem and has no other node in common with the stem. $w$ is called the base of the blossom.
Flowers and Blossoms
Properties:

1. A stem spans $2\ell + 1$ nodes and contains $\ell$ matched edges for some integer $\ell \geq 0$.

2. A blossom spans $2k + 1$ nodes and contains $k$ matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.

3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at $r$).
Flowers and Blossoms

Properties:

4. Every node $x$ in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

5. The even alternating path to $x$ terminates with a matched edge and the odd path with an unmatched edge.
Flowers and Blossoms
Shrinking Blossoms

When during the alternating tree construction we discover a blossom $B$ we replace the graph $G$ by $G' = G/B$, which is obtained from $G$ by contracting the blossom $B$.

- Delete all vertices in $B$ (and its incident edges) from $G$.
- Add a new (pseudo-)vertex $b$. The new vertex $b$ is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from $B$. 
Shrinking Blossoms

- Edges of $T$ that connect a node $u$ not in $B$ to a node in $B$ become tree edges in $T'$ connecting $u$ to $b$.
- Matching edges (there is at most one) that connect a node $u$ not in $B$ to a node in $B$ become matching edges in $M'$.
- Nodes that are connected in $G$ to at least one node in $B$ become connected to $b$ in $G'$.
Shrinking Blossoms

- Edges of $T$ that connect a node $u$ not in $B$ to a node in $B$ become tree edges in $T'$ connecting $u$ to $b$.
- Matching edges (there is at most one) that connect a node $u$ not in $B$ to a node in $B$ become matching edges in $M'$.
- Nodes that are connected in $G$ to at least one node in $B$ become connected to $b$ in $G'$.
Example: Blossom Algorithm

Animation of Blossom Shrinking algorithm is only available in the lecture version of the slides.
Correctness

Assume that in $G$ we have a flower w.r.t. matching $M$. Let $r$ be the root, $B$ the blossom, and $w$ the base. Let graph $G' = G/B$ with pseudonode $b$. Let $M'$ be the matching in the contracted graph.

**Lemma 2**

*If $G'$ contains an augmenting path $P'$ starting at $r$ (or the pseudo-node containing $r$) w.r.t. the matching $M'$ then $G$ contains an augmenting path starting at $r$ w.r.t. matching $M$.***
Correctness

Proof.

If $P'$ does not contain $b$ it is also an augmenting path in $G$.

Case 1: non-empty stem

Next suppose that the stem is non-empty.
Correctness

- After the expansion $\ell$ must be incident to some node in the blossom. Let this node be $k$.
- If $k \neq w$ there is an alternating path $P_2$ from $w$ to $k$ that ends in a matching edge.
- $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.
Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom, $w = r$.

▶ The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
Correctness

Lemma 3

If $G$ contains an augmenting path $P$ from $r$ to $q$ w.r.t. matching $M$ then $G'$ contains an augmenting path from $r$ (or the pseudo-node containing $r$) to $q$ w.r.t. $M'$. 
Correctness

Proof.

▶ If $P$ does not contain a node from $B$ there is nothing to prove.

▶ We can assume that $r$ and $q$ are the only free nodes in $G$.

Case 1: empty stem

Let $i$ be the last node on the path $P$ that is part of the blossom. $P$ is of the form $P_1 \circ (i, j) \circ P_2$, for some node $j$ and $(i, j)$ is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.
Correctness

Illustration for Case 1:
Correctness

Case 2: non-empty stem

Let $P_3$ be alternating path from $r$ to $w$; this exists because $r$ and $w$ are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In $M_+$, $r$ is matched and $w$ is unmatched.

$G$ must contain an augmenting path w.r.t. matching $M_+$, since $M$ and $M_+$ have same cardinality.

This path must go between $w$ and $q$ as these are the only unmatched vertices w.r.t. $M_+$.

For $M'_+$ the blossom has an empty stem. Case 1 applies.

$G'$ has an augmenting path w.r.t. $M'_+$. It must also have an augmenting path w.r.t. $M'$, as both matchings have the same cardinality.

This path must go between $r$ and $q$. 
Algorithm 24 search($r$, $found$)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
2: $found \leftarrow$ false
3: unlabel all nodes;
4: give an even label to $r$ and initialize $list \leftarrow \{r\}$
5: while $list \neq \emptyset$ do
6: delete a node $i$ from $list$
7: examine($i$, $found$)
8: if $found = true$ then return

Search for an augmenting path starting at $r$. 
Algorithm 25 examine\((i, \text{found})\)

1: \textbf{for} all \(j \in \bar{A}(i)\) \textbf{do}

2: \quad \textbf{if} \(j\) is even \textbf{then} contract\((i, j)\) and \textbf{return}

3: \quad \textbf{if} \(j\) is unmatched \textbf{then}

4: \quad \quad q \leftarrow j;

5: \quad \quad \text{pred}(q) \leftarrow i;

6: \quad \quad \text{found} \leftarrow \text{true};

7: \quad \quad \textbf{return}

8: \quad \textbf{if} \(j\) is matched and unlabeled \textbf{then}

9: \quad \quad \text{pred}(j) \leftarrow i;

10: \quad \quad \text{pred}(\text{mate}(j)) \leftarrow j;

11: \quad \text{add mate}(j) \text{ to list}

The lecture slides contain a step by step explanation.

Examine the neighbours of a node \(i\)
Algorithm 26 contract\((i, j)\)

1: trace pred-indices of \(i\) and \(j\) to identify a blossom \(B\)
2: create new node \(b\) and set \(\tilde{A}(b) \leftarrow \bigcup_{x \in B} \tilde{A}(x)\)
3: label \(b\) even and add to list
4: update \(\tilde{A}(j) \leftarrow \tilde{A}(j) \cup \{b\}\) for each \(j \in \tilde{A}(b)\)
5: form a circular double linked list of nodes in \(B\)
6: delete nodes in \(B\) from the graph

Contract blossom identified by nodes \(i\) and \(j\)
Algorithm 26 contract($i, j$)

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \bigcup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to list
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Get all nodes of the blossom.

Time: $O(m)$
Algorithm 26 contract\((i, j)\)

1: trace pred-indices of \(i\) and \(j\) to identify a blossom \(B\)

2: create new node \(b\) and set \(\bar{A}(b) \leftarrow \bigcup_{x \in B} \bar{A}(x)\)

3: label \(b\) even and add to list

4: update \(\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}\) for each \(j \in \bar{A}(b)\)

5: form a circular double linked list of nodes in \(B\)

6: delete nodes in \(B\) from the graph

Identify all neighbours of \(b\).

Time: \(\Theta(m)\) (how?)
Algorithm 26 contract\((i, j)\)

1. trace pred-indices of \(i\) and \(j\) to identify a blossom \(B\)
2. create new node \(b\) and set \(\bar{A}(b) \leftarrow \bigcup_{x \in B} \bar{A}(x)\)
3. label \(b\) even and add to list
4. update \(\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}\) for each \(j \in \bar{A}(b)\)
5. form a circular double linked list of nodes in \(B\)
6. delete nodes in \(B\) from the graph

\(b\) will be an even node, and it has unexamined neighbours.
Algorithm 26 \texttt{contract}(i, j)

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \bigcup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to list
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Every node that was adjacent to a node in $B$ is now adjacent to $b$
Algorithm 26 contract($i, j$)

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \bigcup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to list
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{ b \}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Only for making a blossom expansion easier.
Algorithm 26 contract\((i, j)\)

1: trace pred-indices of \(i\) and \(j\) to identify a blossom \(B\)
2: create new node \(b\) and set \(\tilde{A}(b) \leftarrow \bigcup_{x \in B} \tilde{A}(x)\)
3: label \(b\) even and add to list
4: update \(\tilde{A}(j) \leftarrow \tilde{A}(j) \cup \{b\}\) for each \(j \in \tilde{A}(b)\)
5: form a circular double linked list of nodes in \(B\)
6: delete nodes in \(B\) from the graph

Only delete links from nodes not in \(B\) to \(B\). When expanding the blossom again we can recreate these links in time \(O(m)\).
Analysis

- A contraction operation can be performed in time $O(m)$. Note, that any graph created will have at most $m$ edges.
- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $O(m)$.
- There are at most $n$ contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $O(n)$. There are at most $n$ of them.
- In total the running time is at most

$$n \cdot (O(mn) + O(n)) = O(mn^2).$$
Example: Blossom Algorithm

Animation of Blossom Shrinking algorithm is only available in the lecture version of the slides.