4 Modelling Issues

What do you measure?

- Memory requirement
- Running time
- Number of comparisons
- Number of multiplications
- Number of hard-disc accesses
- Program size
- Power consumption
- ...
4 Modelling Issues

How do you measure?

- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like “this algorithm always runs in time $\mathcal{O}(n^2)$”.
  - Typically focuses on the worst case.
  - Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.
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Input length
The theoretical bounds are usually given by a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be
- the size of the input (number of bits)
- the number of arguments

Example 1
Suppose \( n \) numbers from the interval \( \{1, \ldots, N\} \) have to be sorted. In this case we usually say that the input length is \( n \) instead of e.g. \( n \log N \), which would be the number of bits required to encode the input.
How to measure performance

1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .

2. Calculate number of certain basic operations: comparisons, multiplications, hard disc accesses, . . .

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.
Turing Machine

- Very simple model of computation.
- Only the “current” memory location can be altered.
- Very good model for discussing computability, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form $xx$, where $x$ is a string, have quadratic lower bound.

⇒ Not a good model for developing efficient algorithms.
Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.
- Registers hold integers.
- Indirect addressing.

Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.
Random Access Machine (RAM)

Operations

▶ input operations (input tape → \(R[i]\))
  ▶ READ \(i\)

▶ output operations (\(R[i]\) → output tape)
  ▶ WRITE \(i\)

▶ register-register transfers
  ▶ \(R[j] := R[i]\)
  ▶ \(R[j] := 4\)

▶ indirect addressing
  ▶ \(R[j] := R[R[i]]\)
    loads the content of the \(R[i]\)-th register into the \(j\)-th register
  ▶ \(R[R[i]] := R[j]\)
    loads the content of the \(j\)-th into the \(R[i]\)-th register
Random Access Machine (RAM)

Operations

- branching (including loops) based on comparisons
  - jump \( x \)
    jumps to position \( x \) in the program;
    sets instruction counter to \( x \);
    reads the next operation to perform from register \( R[x] \)
  - jumpz \( x R[i] \)
    jump to \( x \) if \( R[i] = 0 \)
    if not the instruction counter is increased by 1;
  - jumpi \( i \)
    jump to \( R[i] \) (indirect jump);

- arithmetic instructions: +, −, ×, /
  - \( R[i] := R[j] + R[k] \);
  - \( R[i] := -R[k] \);

The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.
Model of Computation

- **uniform** cost model
  Every operation takes time 1.

- **logarithmic** cost model
  The cost depends on the content of memory cells:
  - The time for a step is equal to the largest operand involved;
  - The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

**Bounded word RAM model:** cost is uniform but the largest value stored in a register may not exceed $2^w$, where usually $w = \log_2 n$.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size $n$ must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.
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Example 2

Algorithm 1 RepeatedSquaring(n)

1: \( r \leftarrow 2; \)
2: \textbf{for} \( i = 1 \rightarrow n \) \textbf{do}
3: \quad \( r \leftarrow r^2 \)
4: \textbf{return} \( r \)

- running time:
  - uniform model: \( n \) steps
  - logarithmic model: \( 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1 = \Theta(2^n) \)

- space requirement:
  - uniform model: \( \Theta(1) \)
  - logarithmic model: \( \Theta(2^n) \)
There are different types of complexity bounds:

- **best-case complexity:**
  \[ C_{bc}(n) := \min \{ C(x) \mid |x| = n \} \]
  Usually easy to analyze, but not very meaningful.

- **worst-case complexity:**
  \[ C_{wc}(n) := \max \{ C(x) \mid |x| = n \} \]
  Usually moderately easy to analyze; sometimes too pessimistic.

- **average case complexity:**
  \[ C_{avg}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x) \]
  more general: probability measure \( \mu \)
  \[ C_{avg}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x) \]
There are different types of complexity bounds:

- **amortized complexity:**
  The average cost of data structure operations over a worst case sequence of operations.

- **randomized complexity:**
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input $x$. Then take the worst-case over all $x$ with $|x| = n$. 

<table>
<thead>
<tr>
<th>$C(x)$</th>
<th>cost of instance $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$I_n$</td>
<td>set of instances of length $n$</td>
</tr>
</tbody>
</table>
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Bibliography

[MS08] Kurt Mehlhorn, Peter Sanders:

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:

Chapter 2.1 and 2.2 of [MS08] and Chapter 2 of [CLRS90] are relevant for this section.