8 Priority Queues

A Priority Queue $S$ is a dynamic set data structure that supports the following operations:

- $S.\text{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.
- $S.\text{insert}(x)$: Adds element $x$ to the data-structure.
- $\text{element } S.\text{minimum}()$: Returns an element $x \in S$ with minimum key-value $\text{key}[x]$.
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An addressable Priority Queue also supports:

- **handle $S.\text{ insert}(x)$**: Adds element $x$ to the data-structure, and returns a handle to the object for future reference.
- **$S.\text{ delete}(h)$**: Deletes element specified through handle $h$.
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Dijkstra’s Shortest Path Algorithm

**Algorithm 14 Shortest-Path** \((G = (V, E, d), s \in V)\)

1. **Input:** weighted graph \(G = (V, E, d)\); start vertex \(s\);
2. **Output:** key-field of every node contains distance from \(s\);
3. \(S\).build(); // build empty priority queue
4. **for all** \(v \in V \setminus \{s\} \text{ do}\)
   5. \(v\).key \(\leftarrow \infty\);
   6. \(h_v \leftarrow S.insert(v)\);
   7. \(s\).key \(\leftarrow 0\); \(S.insert(s)\);
8. **while** \(S.is-empty() = \text{false} \text{ do}\)
   9. \(v \leftarrow S.delete-min()\);
10. **for all** \(x \in V \text{ s.t. } (v, x) \in E \text{ do}\)
   11. **if** \(x\).key > \(v\).key + \(d(v, x)\) **then**
   12. \(S.decrease-key(h_x, v\text{.key }+ d(v, x))\);
   13. \(x\).key \(\leftarrow v\text{.key }+ d(v, x)\);
**Prim’s Minimum Spanning Tree Algorithm**

**Algorithm 15** Prim-MST\((G = (V, E, d), s \in V)\)

1: **Input:** weighted graph \(G = (V, E, d)\); start vertex \(s\);
2: **Output:** pred-fields encode MST;
3: \(S\).build(); // build empty priority queue
4: **for all** \(v \in V \setminus \{s\}\) **do**
5: \(v\).key ← ∞;
6: \(h_v \leftarrow S\).insert\((v)\);
7: \(s\).key ← 0; \(S\).insert\((s)\);
8: **while** \(S\).is-empty() = false **do**
9: \(v \leftarrow S\).delete-min();
10: **for all** \(x \in V\) s.t. \(\{v, x\} \in E\)** **do**
11: **if** \(x\).key > \(d(v, x)\) **then**
12: \(S\).decrease-key\((h_x,d(v, x))\);
13: \(x\).key ← \(d(v, x)\);
14: \(x\).pred ← \(v\);
Analysis of Dijkstra and Prim

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
- $|V|$ delete-min() operations
- $|V|$ is-empty() operations
- $|E|$ decrease-key() operations

How good a running time can we obtain?
Analysis of Dijkstra and Prim

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How good a running time can we obtain?
8 Priority Queues

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<tr>
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Note that most applications use `build()` only to create an empty heap which then costs time $1$.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an amortized guarantee.
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Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$. 
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![Binary Heap Diagram]
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Maintain a pointer to the last element $x$.

- We can compute the predecessor of $x$ (last element when $x$ is deleted) in time $\Theta(\log n)$.

- Go up until the last edge used was a right edge.
- Go left; go right until you reach a leaf.
- If you hit the root on the way up, go to the rightmost element.
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  - Go up until the last edge used was a left edge.
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  - Insert a new element as a left child.
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![Binary Heap Diagram]
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1. Insert element at successor of $x$.

2. Exchange with parent until heap property is fulfilled.

Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.
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2. Restore the heap-property for the element $e$.

At its new position $e$ may either travel up or down in the tree (but not both directions).
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- **insert(k)**: insert at successor of $x$ and bubble up. Time $O(\log n)$.
- **delete(h)**: swap with $x$ and bubble up or sift-down. Time $O(\log n)$. 
Build Heap

We can build a heap in linear time:

\[
\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_{i} i2^{h-i} = O(2^h) = O(n)
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We can build a heap in linear time:

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\[ \sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i2^{h-i} = \Theta(2^h) = \Theta(n) \]
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8.1 Binary Heaps
Build Heap

We can build a heap in linear time:

\[ \sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i2^{h-i} = \Theta(2^h) = \Theta(n) \]
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We can build a heap in linear time:

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\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = \Theta(2^h) = \Theta(n)
\]
Build Heap

We can build a heap in linear time:

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\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = O(2^h) = O(n)
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\sum_{\ell} 2^\ell \cdot (h - \ell) = \sum_i i2^{h-i} = \Theta(2^h) = \Theta(n)
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We can build a heap in linear time:

\[
\sum \ 2^\ell \cdot (h - \ell) = \sum i2^{h-i} = \Theta(2^h) = \Theta(n)
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We can build a heap in linear time:

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We can build a heap in linear time:

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\sum_{\ell \text{ levels of } h} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = O(2^h) = O(n)
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\]
Binary Heaps

Operations:

- **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert($k$)**: Insert at $x$ and bubble up. Time $\mathcal{O}(\log n)$.
- **delete($h$)**: Swap with $x$ and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \ldots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$. 
The standard implementation of binary heaps is via arrays. Let $A[0,\ldots,n−1]$ be an array

- The parent of $i$-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- The left child of $i$-th element is at position $2i + 1$.
- The right child of $i$-th element is at position $2i + 2$.

Finding the successor of $x$ is much easier than in the description on the previous slide. Simply increase or decrease $x$.

The resulting binary heap is not addressable. The elements don’t maintain their positions and therefore there are no stable handles.
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### 8.2 Binomial Heaps

<table>
<thead>
<tr>
<th>Operation</th>
<th>Binary Heap</th>
<th>BST</th>
<th>Binomial Heap</th>
<th>Fibonacci Heap*</th>
</tr>
</thead>
<tbody>
<tr>
<td>build</td>
<td>$n$</td>
<td>$n \log n$</td>
<td>$n \log n$</td>
<td>$n$</td>
</tr>
<tr>
<td>minimum</td>
<td>1</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>1</td>
</tr>
<tr>
<td>is-empty</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>insert</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>1</td>
</tr>
<tr>
<td>delete</td>
<td>$\log n^{**}$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
</tr>
<tr>
<td>delete-min</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
</tr>
<tr>
<td>decrease-key</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>1</td>
</tr>
<tr>
<td>merge</td>
<td>$n$</td>
<td>$n \log n$</td>
<td>$\log n$</td>
<td>1</td>
</tr>
</tbody>
</table>
Binomial Trees

$B_0 \quad B_1 \quad B_2 \quad B_3 \quad B_4

B_{t-1} \quad B_t

8.2 Binomial Heaps
Properties of Binomial Trees

- $B_k$ has $2^k$ nodes.
- $B_k$ has height $k$.
- The root of $B_k$ has degree $k$.
- $B_k$ has $\binom{k}{\ell}$ nodes on level $\ell$.
- Deleting the root of $B_k$ gives trees $B_0, B_1, \ldots, B_{k-1}$. 
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- Deleting the root of $B_k$ gives trees $B_0, B_1, \ldots, B_{k-1}$. 
Deleting the root of $B_5$ leaves sub-trees $B_4$, $B_3$, $B_2$, $B_1$, and $B_0$. 
Deleting the leaf furthest from the root (in $B_5$) leaves a path that connects the roots of sub-trees $B_4$, $B_3$, $B_2$, $B_1$, and $B_0$. 
The number of nodes on level $\ell$ in tree $B_k$ is therefore

\[
\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}
\]
The binomial tree $B_k$ is a sub-graph of the hypercube $H_k$. The parent of a node with label $b_n, \ldots, b_1, b_0$ is obtained by setting the least significant 1-bit to 0. The $\ell$-th level contains nodes that have $\ell$ 1’s in their label.
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How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.
- A child-pointer points to an arbitrary node within the list.
- A parent-pointer points to the parent node.
- Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of $x$ (if $x$ does not have siblings then $x.\text{left} = x.\text{right} = x$).
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Given a pointer to a node $x$ we can splice out the sub-tree rooted at $x$ in constant time.

We can add a child-tree $T$ to a node $x$ in constant time if we are given a pointer to $x$ and a pointer to the root of $T$. 
In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property.

There is at most one tree for every dimension/order. For example the above heap contains trees $B_0$, $B_1$, and $B_4$. 

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Given the number $n$ of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the $k_i$ are all distinct this means that the $k_i$ define the non-zero bit-positions in the binary representation of $n$. 
Binomial Heap: Merge

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Binomial Heap

Properties of a heap with $n$ keys:

- Let $n = b_d b_{d-1} \ldots b_0$ denote binary representation of $n$.
- The heap contains tree $B_i$ iff $b_i = 1$.
- Hence, at most $\lceil \log n \rceil + 1$ trees.
- The minimum must be contained in one of the roots.
- The height of the largest tree is at most $\lfloor \log n \rfloor$.
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The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.
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\( S_1. \text{merge}(S_2): \)

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- Time is proportional to the number of trees in both heaps.
- Time: \( \Theta(\log n) \).
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8.2 Binomial Heaps

All other operations can be reduced to merge().

\textbf{S. insert}(x):

\begin{itemize}
  \item Create a new heap \( S' \) that contains just the element \( x \).
  \item Execute \( S.\text{merge}(S') \).
  \item Time: \( \Theta(\log n) \).
\end{itemize}
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- Time: \( \Theta(\log n) \).
8.2 Binomial Heaps

S. minimum():

- Find the minimum key-value among all roots.
- Time: $\Theta(\log n)$. 
8.2 Binomial Heaps

S. delete-min():

- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\text{min}}$ from the heap.
- Create a new heap $S'$ that contains the trees obtained from $T_{\text{min}}$ after deleting the root (note that these are just $O(\log n)$ trees).
- Compute $S.\text{merge}(S')$.
- Time: $O(\log n)$. 
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  \item Find the minimum key-value among all roots.
  \item Remove the corresponding tree \(T_{\text{min}}\) from the heap.
  \item Create a new heap \(S'\) that contains the trees obtained from \(T_{\text{min}}\) after deleting the root (note that these are just \(\mathcal{O}(\log n)\) trees).
  \item Compute \(S.\text{merge}(S')\).
  \item Time: \(\mathcal{O}(\log n)\).
\end{itemize}
8.2 Binomial Heaps

\textbf{S. delete-min()}: 

\begin{itemize}
  \item Find the minimum key-value among all roots.
  \item Remove the corresponding tree $T_{\text{min}}$ from the heap.
  \item Create a new heap $S'$ that contains the trees obtained from $T_{\text{min}}$ after deleting the root (note that these are just $O(\log n)$ trees).
  \item Compute $S.\text{merge}(S')$.
  \item Time: $O(\log n)$.
\end{itemize}
8.2 Binomial Heaps

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- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\text{min}}$ from the heap.
- Create a new heap $S'$ that contains the trees obtained from $T_{\text{min}}$ after deleting the root (note that these are just $O(\log n)$ trees).
- Compute $S.\text{merge}(S')$.

Time: $\Theta(\log n)$. 
8.2 Binomial Heaps

**S. delete-min()**:  
- Find the minimum key-value among all roots.  
- Remove the corresponding tree $T_{\text{min}}$ from the heap.  
- Create a new heap $S'$ that contains the trees obtained from $T_{\text{min}}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).  
- Compute $S.\text{merge}(S')$.  
- Time: $\mathcal{O}(\log n)$.  

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8.2 Binomial Heaps

**S. decrease-key(handle h):**

- Decrease the key of the element pointed to by \( h \).
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: \( \Theta(\log n) \) since the trees have height \( \Theta(\log n) \).
8.2 Binomial Heaps

S. decrease-key(handle $h$):

- Decrease the key of the element pointed to by $h$.
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: $O(\log n)$ since the trees have height $O(\log n)$. 
8.2 Binomial Heaps

S. decrease-key(handle h):

- Decrease the key of the element pointed to by h.
- Bubble the element up in the tree until the heap property is fulfilled.

- Time: $O(\log n)$ since the trees have height $O(\log n)$. 


8.2 Binomial Heaps
8.2 Binomial Heaps

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- Bubble the element up in the tree until the heap property is fulfilled.
- Time: \( \Theta(\log n) \) since the trees have height \( \Theta(\log n) \).
8.2 Binomial Heaps

\( S. \text{ delete}(\text{handle } h) \):

- Execute \( S. \text{ decrease-key}(h, -\infty) \).
- Execute \( S. \text{ delete-min()} \).
- Time: \( \Theta(\log n) \).
8.2 Binomial Heaps

S. delete(handle $h$):

- Execute $S. \text{decrease-key}(h, -\infty)$.
- Execute $S. \text{delete-min}()$.
- Time: $\Theta(\log n)$. 
8.2 Binomial Heaps

S. delete(handle h):

▶ Execute S. decrease-key(h, −∞).
▶ Execute S. delete-min().

Time: \( \Theta(\log n) \).
8.2 Binomial Heaps

\( S. \text{delete}(\text{handle } h) : \)

- Execute \( S. \text{decrease-key}(h, -\infty) \).
- Execute \( S. \text{delete-min()} \).
- Time: \( \mathcal{O}(\log n) \).
8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.
Additional implementation details:

- Every node \( x \) stores its degree in a field \( x.\text{degree} \). Note that this can be updated in constant time when adding a child to \( x \).
- Every node stores a boolean value \( x.\text{marked} \) that specifies whether \( x \) is marked or not.
8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.

The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$. 
8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use $c$ to denote the amount of work that a unit of potential can pay for.
8.3 Fibonacci Heaps

**S. minimum()**

- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$. 
8.3 Fibonacci Heaps

\textit{S. merge(} \textit{S'}\textit{)}

\begin{itemize}
  \item Merge the root lists.
  \item Adjust the min-pointer
\end{itemize}

\begin{itemize}
  \item Running time:
    \begin{itemize}
      \item Actual cost \textit{O}\textit{(1)}.
      \item No change in potential.
      \item Hence, amortized cost is \textit{O}\textit{(1)}.
    \end{itemize}
\end{itemize}

\begin{figure}
\centering
\begin{tikzpicture}
  \node (root) at (0,0) {3};
  \node (7) at (-2,-1) {7};
  \node (18) at (-3,-2) {18};
  \node (39) at (-3,-3) {39};
  \node (41) at (-1,-2) {41};
  \node (44) at (-1,-3) {44};
  \node (23) at (2,-1) {23};
  \node (52) at (1,-2) {52};
  \node (min) at (0,-3) {min};

  \draw[->] (root) -- (7);
  \draw[->] (root) -- (18);
  \draw[->] (root) -- (39);
  \draw[->] (root) -- (41);
  \draw[->] (root) -- (44);
  \draw[->] (root) -- (23);
  \draw[->] (root) -- (52);

  \draw[->] (7) -- (18);
  \draw[->] (18) -- (39);
  \draw[->] (18) -- (41);
  \draw[->] (18) -- (44);
  \draw[->] (18) -- (23);
  \draw[->] (18) -- (52);

  \draw[->] (23) -- (52);

  \node (root2) at (2,0) {5};
  \node (24) at (3,-1) {24};
  \node (26) at (3,-2) {26};
  \node (35) at (3,-3) {35};
  \node (46) at (5,-2) {46};
  \node (30) at (5,-3) {30};
  \node (17) at (7,-2) {17};
  \node (11) at (8,-3) {11};
  \node (min2) at (5,-3) {min};

  \draw[->] (root2) -- (24);
  \draw[->] (root2) -- (26);
  \draw[->] (root2) -- (35);
  \draw[->] (root2) -- (46);
  \draw[->] (root2) -- (30);
  \draw[->] (root2) -- (17);
  \draw[->] (root2) -- (11);

  \draw[->] (24) -- (26);
  \draw[->] (24) -- (35);
  \draw[->] (24) -- (46);
  \draw[->] (24) -- (30);
  \draw[->] (24) -- (17);
  \draw[->] (24) -- (11);

  \draw[->] (26) -- (35);
  \draw[->] (26) -- (46);
  \draw[->] (26) -- (30);
  \draw[->] (26) -- (17);
  \draw[->] (26) -- (11);

  \draw[->] (30) -- (17);
  \draw[->] (30) -- (11);

  \end{tikzpicture}
\end{figure}
### 8.3 Fibonacci Heaps

#### $S$. merge($S'$)
- Merge the root lists.
- Adjust the min-pointer

![Diagram of Fibonacci Heaps](image)

**Running time:**
- Actual cost $\Theta(1)$.
8.3 Fibonacci Heaps

S. merge($S'$)

- Merge the root lists.
- Adjust the min-pointer

Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
8.3 Fibonacci Heaps

\textbf{S. merge}(S')

\begin{itemize}
  \item Merge the root lists.
  \item Adjust the min-pointer
\end{itemize}

\begin{itemize}
  \item Running time:
    \begin{itemize}
      \item Actual cost $\mathcal{O}(1)$.
      \item No change in potential.
      \item Hence, amortized cost is $\mathcal{O}(1)$.
    \end{itemize}
\end{itemize}
8.3 Fibonacci Heaps

**S. insert(\(x\))**

- Create a new tree containing \(x\).
- Insert \(x\) into the root-list.
- Update min-pointer, if necessary.

Running time:
- Actual cost \(O(1)\).
- Change in potential is \(+1\).
- Amortized cost is \(c + O(1) = O(1)\).
8.3 Fibonacci Heaps

\textbf{S. insert}(x)

\begin{itemize}
  \item Create a new tree containing \( x \).
  \item Insert \( x \) into the root-list.
  \item Update min-pointer, if necessary.
\end{itemize}
8.3 Fibonacci Heaps

**S. insert(\(x\))**
- Create a new tree containing \(x\).
- Insert \(x\) into the root-list.
- Update min-pointer, if necessary.

Running time:
- Actual cost \(O(1)\).
- Change in potential is +1.
- Amortized cost is \(c + O(1) = O(1)\).
8.3 Fibonacci Heaps

S. delete-min(x)
8.3 Fibonacci Heaps

**S. delete-min(x)**

- Delete minimum; add child-trees to heap;
  
time: $D(\text{min}) \cdot \mathcal{O}(1)$.

![Diagram of a Fibonacci heap](image-url)
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\text{min}) \cdot \mathcal{O}(1)$.
- Update min-pointer; time: $(t + D(\text{min})) \cdot \mathcal{O}(1)$.
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\text{min}) \cdot O(1)$.
- Update min-pointer; time: $(t + D(\text{min})) \cdot O(1)$.

![Diagram of Fibonacci Heap]

Consolidate root-list so that no roots have the same degree.
Time $t \cdot O(1)$ (see next slide).
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\text{min}) \cdot O(1)$.
- Update min-pointer; time: $(t + D(\text{min})) \cdot O(1)$.

- Consolidate root-list so that no roots have the same degree. Time $t \cdot O(1)$ (see next slide).
Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:

$$\text{current}$$

$$\text{min}$$
8.3 Fibonacci Heaps

Consolidate:
Consolidate:
8.3 Fibonacci Heaps

Consolidate:

![Diagram of Fibonacci Heaps]

- CURRENT
- MIN

0 1 2 3

Numbers in the diagram represent nodes in the Fibonacci heap structure.
Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

current

\[
\begin{array}{cccc}
7 & 18 & 52 & 23 \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

min

\[
\begin{array}{cccc}
7 & 18 & 52 & 23 \\
41 & 39 & \circ & \circ \\
44 & 39 & \circ & \circ \\
26 & 46 & 35 & 30 \\
\end{array}
\]

\[
\begin{array}{cccc}
24 & 17 & \circ & \circ \\
24 & 17 & \circ & \circ \\
\end{array}
\]
Consolidate:
8.3 Fibonacci Heaps

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Consolidate:
Consolidate:
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Consolidate:

![Diagram of Fibonacci Heaps with nodes and connections]
8.3 Fibonacci Heaps

Consolidate:
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $Dn + t$ elements in root-list before consolidate.
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $O(1) \cdot (D_n + t)$. Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$. 
8.3 Fibonacci Heaps

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Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
8.3 Fibonacci Heaps

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Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
Actual cost for delete-min()

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Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
8.3 Fibonacci Heaps

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- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $O(1) \cdot (D_n + t)$.

Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

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- The amortized cost is
8.3 Fibonacci Heaps

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- Therefore $\Delta\Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is
  
  $$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most \(D_n + t\) elements in root-list before consolidate.
- Actual cost for a delete-min is at most \(\mathcal{O}(1) \cdot (D_n + t)\).

Hence, there exists \(c_1\) s.t. actual cost is at most \(c_1 \cdot (D_n + t)\).

Amortized cost for delete-min()

- \(t' \leq D_n + 1\) as degrees are different after consolidating.
- Therefore \(\Delta \Phi \leq D_n + 1 - t\);
- We can pay \(c \cdot (t - D_n - 1)\) from the potential decrease.
- The amortized cost is

\[
c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)
\leq (c_1 + c)D_n + (c_1 - c)t + c
\]
8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.
▶ Actual cost for a delete-min is at most $O(1) \cdot (D_n + t)$. Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
▶ Therefore $\Delta \Phi \leq D_n + 1 - t$;
▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
▶ The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$

$$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1)$$
8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $O(1) \cdot (D_n + t)$.
  Hence, there exists $c_1$ s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

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- $t' \leq D_n + 1$ as degrees are different after consolidating.
- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is
  $$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
  $$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq O(D_n)$$
8.3 Fibonacci Heaps

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Amortized cost for delete-min()

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- Therefore $\Delta \Phi \leq D_n + 1 - t$;
- We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- The amortized cost is
  
  \[
  c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)
  \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq O(D_n)
  \]
  for $c \geq c_1$. 

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$. 
8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$. 
**Fibonacci Heaps: decrease-key(handle \( h, \nu \))**

### Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by \( h \).
- Nothing else to do.
Fibonacci Heaps: decrease-key(handle $h$, $v$)

Case 1: decrease-key does not violate heap-property
  - Just decrease the key-value of element referenced by $h$.
  - Nothing else to do.
Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by \( h \).
  Nothing else to do.
Fibonacci Heaps: decrease-key(handle $h$, $v$)

Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by $h$. Nothing else to do.
Fibonacci Heaps: decrease-key(handle $h$, $v$)

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element $x$ reference by $h$.
- If the heap-property is violated, cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Mark the (previous) parent of $x$ (unless it’s a root).
Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element $x$ reference by $h$.
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Fibonacci Heaps: decrease-key(handle $h, v$)

Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element $x$ reference by $h$.
- Cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Continue cutting the parent until you arrive at an unmarked node.
Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element $x$ reference by $h$.
- Cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Continue cutting the parent until you arrive at an unmarked node.
Fibonacci Heaps: decrease-key\((\text{handle } h, v)\)

Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element \(x\) reference by \(h\).
- Cut the parent edge of \(x\), and make \(x\) into a root.
- Adjust min-pointers, if necessary.
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- Adjust min-pointers, if necessary.
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Fibonacci Heaps: decrease-key\((h, v)\)

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Fibonacci Heaps: decrease-key(handle $h, v$)

Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element $x$ reference by $h$.
- Cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Execute the following:
  
  $p ← \text{parent}[x]$;
  
  while ($p$ is marked)
  
  $pp ← \text{parent}[p]$;
  
  cut of $p$; make it into a root; unmark it;
  
  $p ← pp$;
  
  if $p$ is unmarked and not a root mark it;
Fibonacci Heaps: decrease-key(handle $h, v$)

**Actual cost:**
- Constant cost for decreasing the value.
- Constant cost for each of $\ell$ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant $c_2$.

**Amortized cost:**
- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta \Phi \leq \ell + 2 (4 - \ell + 2) = 4 - \ell$.
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8.3 Fibonacci Heaps
Fibonacci Heaps: \texttt{decrease-key}(\texttt{handle} \: h, v)

**Actual cost:**

- Constant cost for decreasing the value.
- Constant cost for each of \( \ell \) cuts.
- Hence, cost is at most \( c_2 \cdot (\ell + 1) \), for some constant \( c_2 \).

**Amortized cost:**

- \( t' = t + \ell \), as every cut creates one new root.
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Delete node

$H. \text{delete}(x)$:

- decrease value of $x$ to $-\infty$.
- delete-min.

Amortized cost: $O(D_n)$

- $O(1)$ for decrease-key.
- $O(D_n)$ for delete-min.
8.3 Fibonacci Heaps

Lemma 1

Let $x$ be a node with degree $k$ and let $y_1, \ldots, y_k$ denote the children of $x$ in the order that they were linked to $x$. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$
8.3 Fibonacci Heaps

Proof

- When $y_i$ was linked to $x$, at least $y_1, \ldots, y_{i-1}$ were already linked to $x$.
- Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- Since, then $y_i$ has lost at most one child.
- Therefore, $\text{degree}(y_i) \geq i - 2$. 
8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

Let $s_k$ be the minimum possible size of a sub-tree rooted at a node of degree $k$ that can occur in a Fibonacci heap.
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- $s_k$ monotonically increases with $k$.
- $s_0 = 1$ and $s_1 = 2$. 

Let $x$ be a degree $k$ node of size $s_k$ and let $y_1, \ldots, y_k$ be its children.
8.3 Fibonacci Heaps

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Let $x$ be a degree $k$ node of size $s_k$ and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$
Let $s_k$ be the minimum possible size of a sub-tree rooted at a node of degree $k$ that can occur in a Fibonacci heap.

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\[
s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i) \geq 2 + \sum_{i=2}^{k} s_{i-2}
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8.3 Fibonacci Heaps

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= 2 + \sum_{i=0}^{k-2} s_i
\]
8.3 Fibonacci Heaps

Definition 2
Consider the following non-standard Fibonacci type sequence:

\[
F_k = \begin{cases} 
1 & \text{if } k = 0 \\
2 & \text{if } k = 1 \\
F_{k-1} + F_{k-2} & \text{if } k \geq 2
\end{cases}
\]

\(\phi = \frac{1}{2}(1 + \sqrt{5})\) denotes the golden ratio. Note that \(\phi^2 = 1 + \phi\).

Facts:

1. \(F_k \geq \phi^k\).
2. For \(k \geq 2\): \(F_k = 2 + \sum_{i=0}^{k-2} F_i\).

The above facts can be easily proved by induction. From this it follows that \(s_k \geq F_k \geq \phi^k\), which gives that the maximum degree in a Fibonacci heap is logarithmic.
k=0: \[ 1 = F_0 \geq \Phi^0 = 1 \]

k=1: \[ 2 = F_1 \geq \Phi^1 \approx 1.61 \]

k-2,k-1 \rightarrow k: \[ F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}(\Phi + 1) = \Phi^k \]

k=2: \[ 3 = F_2 = 2 + 1 = 2 + F_0 \]

k-1 \rightarrow k: \[ F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i \]