Algorithm 2 mergesort(list $L$)

1: $n \leftarrow \text{size}(L)$
2: if $n \leq 1$ return $L$
3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
4: $L_2 \leftarrow L[\lceil \frac{n}{2} \rceil + 1 \cdots n]$
5: mergesort($L_1$)
6: mergesort($L_2$)
7: $L \leftarrow \text{merge}(L_1, L_2)$
8: return $L$
Algorithm 2 mergesort(list $L$)

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7: $L \leftarrow \text{merge}(L_1, L_2)$
8: return $L$

This algorithm requires

$$T(n) = T\left(\lceil \frac{n}{2} \rceil \right) + T\left(\lfloor \frac{n}{2} \rfloor \right) + O(n) \leq 2T\left(\lceil \frac{n}{2} \rceil \right) + O(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$. 
How do we bring the expression for the number of comparisons (≈ running time) into a **closed form**?

For this we need to **solve** the recurrence.
How do we bring the expression for the number of comparisons (≈ running time) into a closed form?

For this we need to solve the recurrence.
1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.
Methods for Solving Recurrences

4. Generating Functions
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.
First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 
2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\
0 & \text{otherwise}
\end{cases}$$

Informal way:
First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 
2T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn & n \geq 2 \\
0 & \text{otherwise}
\end{cases}$$

**Informal way:**
Assume that instead we have

$$T(n) \leq \begin{cases} 
2T\left(\frac{n}{2} \right) + cn & n \geq 2 \\
0 & \text{otherwise}
\end{cases}$$
6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

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**Informal way:**
Assume that instead we have

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.
6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. 
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$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$
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$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(d \frac{n}{2} \log \frac{n}{2}\right) + cn$$
Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(d \frac{n}{2} \log \frac{n}{2}\right) + cn$$

$$= dn(\log n - 1) + cn$$
Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2 \left(d \frac{n}{2} \log \frac{n}{2}\right) + cn$$

$$= dn (\log n - 1) + cn$$

$$= dn \log n + (c - d)n$$
Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(d \frac{n}{2} \log \frac{n}{2}\right) + cn$$

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$$= dn \log n + (c - d)n$$

$$\leq dn \log n$$

if we choose $d \geq c$. 
Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + cn \\
\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\
= dn(\log n - 1) + cn \\
= dn \log n + (c - d)n \\
\leq dn \log n
\]

if we choose $d \geq c$.

Formally, this is not correct if $n$ is not a power of 2. Also even in this case one would need to do an induction proof.
How do we get a result for all values of $n$?
6.1 Guessing+Induction

How do we get a result for all values of $n$?

We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 
2T([\frac{n}{2}]) + cn & n \geq 16 \\
b & \text{otherwise}
\end{cases}$$
How do we get a result for all values of $n$?

We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 2T([\frac{n}{2}]) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant ($b$ in the above case).
6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n)$$
We also make a guess of $T(n) \leq d n \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn$$
6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn$$

$$\leq 2 \left( d \left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn$$
We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil \right) + cn$$

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} + 1$$
We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn$$

$$\leq 2\left( d \left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn$$

\[\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} + 1 \leq 2\left( d \left( \frac{n}{2} + 1 \right) \log(\frac{n}{2} + 1) \right) + cn\]
We also make a guess of \( T(n) \leq dn \log n \) and get

\[
T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\
\leq 2\left( d\left\lceil \frac{n}{2} \right\rceil \log\left\lceil \frac{n}{2} \right\rceil \right) + cn \\
\leq 2\left( d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right) \right) + cn
\]

\[\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1\] 

\[\frac{n}{2} + 1 \leq \frac{9}{16} n\]
We also make a guess of $T(n) \leq dn \log n$ and get

\[
T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\
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\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\
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We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2(d(n/2 + 1) \log(n/2 + 1)) + cn$$

$$\leq dn \log \left(\frac{9}{16} n\right) + 2d \log n + cn$$

$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$

$\frac{n}{2} + 1 \leq \frac{9}{16} n$

$\log \frac{9}{16} n = \log n + (\log 9 - 4)$
We also make a guess of $T(n) \leq dn \log n$ and get

\[
T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn
\]

\[
\leq 2 \left( d \left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil \right) + cn
\]

\[
\leq 2 \left( d \left( \frac{n}{2} + 1 \right) \log \left( \frac{n}{2} + 1 \right) \right) + cn
\]

\[
\leq d n \log \left( \frac{9}{16} n \right) + 2d \log n + cn
\]

\[
= d n \log n + (\log 9 - 4)dn + 2d \log n + cn
\]
We also make a guess of $T(n) \leq dn \log n$ and get

$$
T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn
\leq 2 \left( d \left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn
\leq 2 \left( d \left( \frac{n}{2} + 1 \right) \log \left( \frac{n}{2} + 1 \right) \right) + cn
\leq dn \log \left( \frac{9}{16} n \right) + 2d \log n + cn
$$

log $\frac{9}{16} n = \log n + (\log 9 - 4)$

$\log n \leq \frac{n}{4}$
6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

\[
T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn \\
\leq 2\left( d \left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn \\
\leq 2\left( d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right) \right) + cn \\
\leq dn \log \left(\frac{9}{16}n\right) + 2d \log n + cn \\
= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\
\leq dn \log n + (\log 9 - 3.5)dn + cn
\]
6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

\[
T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn
\]

\[
\leq 2\left(d\left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn
\]

\[
\leq 2\left(d(n/2 + 1) \log(n/2 + 1) \right) + cn
\]

\[
\leq dn \log \left(\frac{9}{16} n\right) + 2d \log n + cn
\]

\[
= dn \log n + (\log 9 - 4)dn + 2d \log n + cn
\]

\[
\leq dn \log n + (\log 9 - 3.5)dn + cn
\]

\[
\leq dn \log n - 0.33dn + cn
\]
We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\leq dn \log \left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of $d$. 
6.2 Master Theorem

Lemma 1

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Case 1.

If $f(n) = \Theta(n^{\log_b(a) - \epsilon})$ then $T(n) = \Theta(n^{\log_b(a)})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a) \log^k n})$ then $T(n) = \Theta(n^{\log_b(a) \log^{k+1} n})$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a) + \epsilon})$ and for sufficiently large $n$ $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$. 
We prove the Master Theorem for the case that \( n \) is of the form \( b^\ell \), and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

\[ n \]
The Recursion Tree

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![Recursion Tree Diagram](image-url)
The Recursion Tree

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\[ f(n) \]

\[ a f\left(\frac{n}{b}\right) \]

6.2 Master Theorem
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

\[ f(n) \]

\[ a f \left( \frac{n}{b} \right) \]

\[ a^2 f \left( \frac{n}{b^2} \right) \]
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

$$f(n)$$

$$a f\left(\frac{n}{b}\right)$$

$$a^2 f\left(\frac{n}{b^2}\right)$$

$$a^{\log_b n}$$
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

\[ f(n) = a f\left(\frac{n}{b}\right) \]

\[ a^2 f\left(\frac{n}{b^2}\right) \]

\[ a \log_b n = n \log_b a \]
6.2 Master Theorem

This gives

\[ T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right). \]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$. 

\[ T(n) = n \log_b a - 1 \sum_{i=0}^{n-1} a^i f(n^b) \leq c n \log_b a - \epsilon \log_b n - \sum_{i=0}^{n-1} a^i (n^b \epsilon) \log_b a - \epsilon = c n \log_b a - \epsilon \frac{\log_b n - 1}{\log_b a - \epsilon} \]

Hence,

\[ T(n) \leq \left( \frac{c b \epsilon - 1 + 1}{c b \epsilon - 1} \right) n \log_b a \]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a}$$
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a - \epsilon} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left( \frac{n}{b^i} \right)
\]
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a - \epsilon} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a - \epsilon} \).

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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]

\[
b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}
\]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a-\varepsilon} \).

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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

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\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon}
\]

\[
b^{-i(\log_b a-\varepsilon)} = b^{\varepsilon i} (b^{\log_b a})^{-i} = b^{\varepsilon i} a^{-i}
\]

\[
\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}
\]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

$$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$$

$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n - 1}) / (b^{\epsilon} - 1)$$
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]

\[
b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}
\]

\[
\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}
\]

\[
= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i
\]

\[
= cn^{\log_b a - \epsilon} \frac{(b^{\epsilon \log_b n} - 1)}{(b^{\epsilon} - 1)}
\]

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Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

$$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$$

$$= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon} (n^\epsilon - 1) / (b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a} (n^\epsilon - 1) / (n^\epsilon)$$
Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}$$

$$= c n^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i$$

$$= c n^{\log_b a-\epsilon} \frac{b^\epsilon \log_b n - 1}{b^\epsilon - 1}$$

$$= c n^{\log_b a-\epsilon} (n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a} (n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}$$
Case 1. Now suppose that $f(n) \leq c n^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

$$\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$$

$$= c n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$= c n^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

$$= c n^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1)$$

$$= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1) / (n^{\epsilon})$$

Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$. 
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$$T(n) - n^{\log_b a}$$
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f \left( \frac{n}{b^i} \right)
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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1$$

$$= cn^{\log_b a} \log_b n$$
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1
\]

\[
= cn^{\log_b a} \log_b n
\]

Hence,

\[
T(n) = O(n^{\log_b a} \log_b n)
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \Theta(n^{\log_b a} \log_b n) \Rightarrow T(n) = \Theta(n^{\log_b a} \log n).$$
Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$. 
Case 2. Now suppose that \( f(n) \geq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a}
\]
Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$
Case 2. Now suppose that \( f(n) \geq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\]
Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\]

Hence, $T(n) = \Omega(n^{\log_b a})$. 

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Case 2. Now suppose that \( f(n) \geq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\]

\[
= cn^{\log_b a} \log_b n
\]
Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$
Case 2. Now suppose that \( f(n) \geq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
= cn^{\log_b a} \log_b n
\]

Hence,

\[
T(n) = \Omega(n^{\log_b a} \log_b n) \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$. 
Case 2. Now suppose that $f(n) \leq cn^\log_b a (\log_b (n))^k$.

\[ T(n) - n^\log_b a \]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b (n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} (\log_b(n))^k \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k
\]
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} (\log_b (n))^k \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f \left( \frac{n}{b^i} \right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left( \frac{n}{b^i} \right)^{\log_b a} \cdot \left( \log_b \left( \frac{n}{b^i} \right) \right)^k
\]

\[
n = b^\ell \Rightarrow \ell = \log_b n
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b (n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k$$

$n = b^\ell \Rightarrow \ell = \log_b n$

$$= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a}(\log_b(n))^k \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k
\]

\( n = b^\ell \Rightarrow \ell = \log_b n \)
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a (\log_b (n))^k} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k
\]

\[
n = b^\ell \Rightarrow \ell = \log_B n = cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_B \left(\frac{b^\ell}{b^i}\right)\right)^k
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k
\]

\[
= cn^{\log_b a} \sum_{i=1}^{\ell} i^k
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b (n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k$$

$n = b^\ell \Rightarrow \ell = \log_b n$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} (\log_b(n))^k \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k
\]

\[
n = b^\ell \Rightarrow \ell = \log_b n
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
\]

\[
= cn^{\log_b a} \sum_{i=1}^{\ell} i^k
\]

\[
\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\]
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a (\log_b (n))} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k
\]

\[
= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^{\ell}}{b^i}\right)\right)^k
\]

\[
= c n^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
\]

\[
= c n^{\log_b a} \sum_{i=1}^{\ell} i^k
\]

\[
\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\]

\[
\Rightarrow T(n) = \Theta(n^{\log_b a \log^{k+1} n}).
\]
Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$. 
Case 3. Now suppose that \( f(n) \geq dn^{\log_b a + \epsilon} \), and that for sufficiently large \( n \): \( af(n/b) \leq cf(n) \), for \( c < 1 \).

From this we get \( a^i f(n/b^i) \leq c^i f(n) \), where we assume that \( n/b^{i-1} \geq n_0 \) is still sufficiently large.
Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$
Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\log_b n-1} c^i f(n) + O(n^{\log_b a})$$
Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + O\left(n^{\log_b a}\right)$$

$q < 1: \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$
Case 3. Now suppose that \( f(n) \geq dn^{\log_b a + \epsilon} \), and that for sufficiently large \( n \): \( af(n/b) \leq cf(n) \), for \( c < 1 \).

From this we get \( a^i f(n/b^i) \leq c^i f(n) \), where we assume that \( n/b^{i-1} \geq n_0 \) is still sufficiently large.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + O(n^{\log_b a})
\]

**q < 1:** \[
\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}
\]

\[
\leq \frac{1}{1-c} f(n) + O(n^{\log_b a})
\]
Case 3. Now suppose that \( f(n) \geq dn^{\log_b a + \epsilon} \), and that for sufficiently large \( n \): \( af(n/b) \leq cf(n) \), for \( c < 1 \).

From this we get \( a^i f(n/b^i) \leq c^i f(n) \), where we assume that \( n/b^{i-1} \geq n_0 \) is still sufficiently large.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + O(n^{\log_b a})
\]

\[
q < 1: \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}
\]

Hence,

\[
T(n) \leq O(f(n))
\]
Case 3. Now suppose that \( f(n) \geq dn^{\log_b a + \epsilon} \), and that for sufficiently large \( n \): \( af(n/b) \leq cf(n) \), for \( c < 1 \).

From this we get \( a^i f(n/b^i) \leq c^i f(n) \), where we assume that \( n/b^{i-1} \geq n_0 \) is still sufficiently large.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + O(n^{\log_b a})
\]

\[q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}\]

Hence,

\[
T(n) \leq O(f(n)) \Rightarrow T(n) = \Theta(f(n)).
\]
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.
Example: Multiplying Two Integers

Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):
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Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 
Example: Multiplying Two Integers

Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{c}
11011011 \\
11001000 \\
001001101 \\
11011001
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

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Example: Multiplying Two Integers

Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

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\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
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For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

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Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

6.2 Master Theorem 11. Apr. 2018

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Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{cccccc}
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccc}
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

6.2 Master Theorem
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{c}
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{c}
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
\end{array}
\end{array}
\]
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For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
0 & 1 & 0 & 0 & 0 & 0
\end{array}
\]
Example: Multiplying Two Integers

Suppose we want to multiply two \(n\)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \(A\) and \(B\):

\[
\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
0 & 0 & 1 & 0 & 0 & 0
\end{array}
\]
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\begin{align*}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & & & & & 0 0 1 0 0 0 0
\end{align*}
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
\]
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

$$
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

This gives that two $n$-bit integers can be added in time $O(n)$. 6.2 Master Theorem 11. Apr. 2018

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Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

6.2 Master Theorem

Ernst Mayr, Harald Räcke
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\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array} \]

This gives that two $n$-bit integers can be added in time $O(n)$. 
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\[
\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( \mathcal{O}(n) \).
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\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{array}
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\begin{array}{cccccccc}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

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Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \((m \leq n)\).
Example: Multiplying Two Integers

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\[
\begin{array}{c}
1 \ 0 \ 0 \ 0 \ 1 \\
\times \ 1 \ 0 \ 1 \ 1 \\
\hline
\end{array}
\]

Time requirement:
- Computing intermediate results: \( O(nm) \).
- Adding \( m \) numbers of length \( \leq 2n \): \( O((m+n)m) = O(nm) \).
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \((m \leq n)\).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \ 1 \\
\times & 1 & 0 & 1 \ 1 \\
\hline
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
\end{array}
\]

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Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1
\end{array}
\]

Time requirement:

- Computing intermediate results: $O(nm)$.
- Adding $m$ numbers of length $\leq 2n$: $O((m+n)m) = O(nm)$.
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \( (m \leq n) \).

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1
\end{array}
\]

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\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\times & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
0 & & & & \\
\end{array}
\]

Time requirement:

\[
\text{Computing intermediate results: } O(nm).
\]

\[
\text{Adding } m \text{ numbers of length } \leq 2n: \quad O((m+n)m) = O(nm).
\]
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

Time requirement:

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Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c c c c c}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & \boxed{0} & 1 & 1 \\
\hline
& 1 & 0 & 0 & 0 & 1 \\
& 1 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

Time requirement:

- Computing intermediate results: $O(nm)$.
- Adding $m$ numbers of length $\leq 2n$: $O((m+n)m) = O(nm)$.
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & & & \\
1 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0
\end{array}
\]

Time requirement:
- Computing intermediate results: $O(nm)$.
- Adding $m$ numbers of length $\leq 2n$: $O((m+n)^2) = O(nm)$.
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & & & & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\times & & & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Time requirement:

- Computing intermediate results: \( O(nm) \).
- Adding \( m \) numbers of length \( \leq 2n \): \( O((m+n)m) = O(nm) \).
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \((m \leq n)\).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\times & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Time requirement:

- **Computing intermediate results:** \( O(nm) \).
- **Adding** \( m \) numbers of length \( \leq 2n \): \( O((m+n)m) = O(nm) \).
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Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & & & & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

Time requirement:
- Computing intermediate results: \( O(nm) \).
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Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

Time requirement:

- Computing intermediate results: $O(nm)$.
- Adding $m$ numbers of length $\leq 2n$: $O((m+n)m) = O(nm)$.
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{c}
1 \; 0 \; 0 \; 0 \; 1 \\
\times \\
1 \; 0 \; 1 \; 1 \; 1
\end{array}
\]

\[
\begin{array}{c}
1 \; 0 \; 0 \; 0 \; 0 \; 1 \\
1 \; 0 \; 0 \; 0 \; 1 \; 0 \\
0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \\
1 \; 0 \; 0 \; 0 \; 1 \; 0 \; 0 \; 0 \\
\hline
1 \; 0 \; 1 \; 1 \; 1 \; 0 \; 1 \; 1
\end{array}
\]

Time requirement:
- Computing intermediate results: \( O(nm) \).
- Adding \( m \) numbers of length \( \leq 2n \): \( O((m+n)m) = O(nm) \).
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\times & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 \\
\end{array}
\]

Time requirement:
**Example: Multiplying Two Integers**

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
\]

**Time requirement:**
- Computing intermediate results: $O(nm)$. 

Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

Time requirement:

- Computing intermediate results: \( \Theta(nm) \).
- Adding \( m \) numbers of length \( \leq 2n \):
  \[
  \Theta((m + n)m) = \Theta(nm).
  \]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 

\[
\begin{array}{c}
B \\
\times \\
A
\end{array}
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

\[
\begin{array}{cccccc}
  b_{n-1} & \cdots & b_0 \\
\end{array}
\times
\begin{array}{cccccc}
  a_{n-1} & \cdots & a_0 \\
\end{array}
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 

\[
\begin{array}{cccccccc}
  b_{n-1} & \cdots & b_n & b_{n-1} & \cdots & b_0 \\
\end{array}
\times
\begin{array}{cccccccc}
  a_{n-1} & \cdots & a_n & a_{n-1} & \cdots & a_0 \\
\end{array}
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 

\[
\begin{array}{c}
B_1 & B_0 \\
\times & \\
A_1 & A_0 \\
\end{array}
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1B_1 \cdot 2^n + (A_1B_0 + A_0B_1) \cdot 2^{\frac{n}{2}} + A_0B_0$$
Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $|A| = |B| = 1$ then
2: return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

We get the following recurrence:

$$T(n) = \begin{cases} 4T\left(\frac{n}{2}\right) + O(n) & \text{if } n \geq 2 \\ \cdot & \text{if } n = 1 \end{cases}$$
Example: Multiplying Two Integers

**Algorithm 3** \( \text{mult}(A, B) \)

1. \textbf{if} \(|A| = |B| = 1\) \textbf{then}
2. \textbf{return} \(a_0 \cdot b_0\)
3. split \(A\) into \(A_0\) and \(A_1\)
4. split \(B\) into \(B_0\) and \(B_1\)
5. \(Z_2 \leftarrow \text{mult}(A_1, B_1)\)
6. \(Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)\)
7. \(Z_0 \leftarrow \text{mult}(A_0, B_0)\)
8. \textbf{return} \(Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0\)

\(\mathcal{O}(1)\)

We get the following recurrence:

\[ T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \]
Example: Multiplying Two Integers

Algorithm 3 \(\text{mult}(A, B)\)

1: if \(|A| = |B| = 1\) then
2: \text{return } a_0 \cdot b_0
3: split \(A\) into \(A_0\) and \(A_1\)
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5: \(Z_2 \leftarrow \text{mult}(A_1, B_1)\)
6: \(Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)\)
7: \(Z_0 \leftarrow \text{mult}(A_0, B_0)\)
8: \text{return } Z_2 \cdot 2^n + Z_1 \cdot 2^\frac{n}{2} + Z_0

\(\mathcal{O}(1)\)

We get the following recurrence:

\[ T(n) = 4 \cdot T\left(\frac{n}{2}\right) + \mathcal{O}(n) \]
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A, B)

1: \textbf{if} |A| = |B| = 1 \textbf{then}
2: \hspace{1em} \textbf{return} \ a_0 \cdot b_0
3: \hspace{1em} \text{split} \ A \text{ into} \ A_0 \text{ and} \ A_1
4: \hspace{1em} \text{split} \ B \text{ into} \ B_0 \text{ and} \ B_1
5: \hspace{1em} Z_2 \leftarrow \text{mult}(A_1, B_1)
6: \hspace{1em} Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)
7: \hspace{1em} Z_0 \leftarrow \text{mult}(A_0, B_0)
8: \hspace{1em} \textbf{return} \ Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0

\text{\(O(1)\)}

\text{\(O(1)\)}

\text{\(O(n)\)}
Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: \textbf{if} $|A| = |B| = 1$ \textbf{then} \hspace{1cm} $O(1)$
2: \hspace{1cm} \textbf{return} $a_0 \cdot b_0$ \hspace{1cm} $O(1)$
3: \hspace{1cm} \text{split} $A$ into $A_0$ and $A_1$ \hspace{1cm} $O(n)$
4: \hspace{1cm} \text{split} $B$ into $B_0$ and $B_1$ \hspace{1cm} $O(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$ \hspace{1cm} $O(n)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$ \hspace{1cm} $O(n)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$ \hspace{1cm} $O(n)$
8: \textbf{return} $Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0$
Example: Multiplying Two Integers

Algorithm 3 \texttt{mult}(A, B)

1: \textbf{if } |A| = |B| = 1 \textbf{ then} \hfill \mathcal{O}(1)
2: \quad \textbf{return } a_0 \cdot b_0 \hfill \mathcal{O}(1)
3: \quad \text{split } A \text{ into } A_0 \text{ and } A_1 \hfill \mathcal{O}(n)
4: \quad \text{split } B \text{ into } B_0 \text{ and } B_1 \hfill \mathcal{O}(n)
5: \quad Z_2 \leftarrow \texttt{mult}(A_1, B_1) \hfill T\left(\frac{n}{2}\right)
6: \quad Z_1 \leftarrow \texttt{mult}(A_1, B_0) + \texttt{mult}(A_0, B_1) 
7: \quad Z_0 \leftarrow \texttt{mult}(A_0, B_0)
8: \quad \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0 

We get the following recurrence:

\[ T(n) = 4 T\left(\frac{n}{2}\right) + \mathcal{O}(n) \]
Example: Multiplying Two Integers

Algorithm 3 \texttt{mult}(A, B)

1: \textbf{if} $|A| = |B| = 1$ \textbf{then}
2: \hspace{1em} \textbf{return} $a_0 \cdot b_0$
3: \texttt{split} $A$ into $A_0$ and $A_1$
4: \texttt{split} $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \texttt{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \texttt{mult}(A_1, B_0) + \texttt{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \texttt{mult}(A_0, B_0)$
8: \textbf{return} $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

$O(1)$
$O(1)$
$O(n)$
$O(n)$
$T\left(\frac{n}{2}\right)$
$2T\left(\frac{n}{2}\right) + O(n)$

6.2 Master Theorem

11. Apr. 2018
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A, B)

1: \textbf{if } \vert A \vert = \vert B \vert = 1 \textbf{ then }
2: \quad \textbf{return } a_0 \cdot b_0
3: \quad \text{split } A \text{ into } A_0 \text{ and } A_1
4: \quad \text{split } B \text{ into } B_0 \text{ and } B_1
5: \quad Z_2 \leftarrow \text{mult}(A_1, B_1)
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8: \quad \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0

\begin{align*}
\mathcal{O}(1) & \quad \mathcal{O}(1) \\
\mathcal{O}(n) & \quad \mathcal{O}(n) \\
T\left(\frac{n}{2}\right) & \quad 2T\left(\frac{n}{2}\right) + \mathcal{O}(n) \\
T\left(\frac{n}{2}\right) & \quad T\left(\frac{n}{2}\right)
\end{align*}

We get the following recurrence:

\[ T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \]
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A, B)

1: \textbf{if} $|A| = |B| = 1$ \textbf{then}
2: \quad \textbf{return} $a_0 \cdot b_0$
3: \quad split $A$ into $A_0$ and $A_1$
4: \quad split $B$ into $B_0$ and $B_1$
5: \quad $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: \quad $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
7: \quad $Z_0 \leftarrow \text{mult}(A_0, B_0)$
8: \quad \textbf{return} $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

$\Theta(1)$
$\Theta(1)$
$\Theta(n)$
$\Theta(n)$
$T(\frac{n}{2})$
$2T(\frac{n}{2}) + \Theta(n)$
$T(\frac{n}{2})$
$\Theta(n)$
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A,B)

1: \textbf{if } |A| = |B| = 1 \textbf{ then}
2: \quad \textbf{return } a_0 \cdot b_0
3: \quad \text{split } A \text{ into } A_0 \text{ and } A_1
4: \quad \text{split } B \text{ into } B_0 \text{ and } B_1
5: \quad Z_2 \leftarrow \text{mult}(A_1, B_1)
6: \quad Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)
7: \quad Z_0 \leftarrow \text{mult}(A_0, B_0)
8: \quad \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n).$$
Example: Multiplying Two Integers

Master Theorem: Recurrence: \( T[n] = aT\left(\frac{n}{b}\right) + f(n) \).

- **Case 1:** \( f(n) = \Theta(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
- **Case 2:** \( f(n) = \Theta(n^{\log_b a \log^k n}) \) \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
- **Case 3:** \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) \( T(n) = \Theta(f(n)) \)
Example: Multiplying Two Integers

Master Theorem: Recurrence: \( T[n] = aT\left(\frac{n}{b}\right) + f(n) \).

- **Case 1**: \( f(n) = \Theta(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
- **Case 2**: \( f(n) = \Theta(n^{\log_b a + \log^k n}) \) \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
- **Case 3**: \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) \( T(n) = \Theta(f(n)) \)

In our case \( a = 4 \), \( b = 2 \), and \( f(n) = \Theta(n) \). Hence, we are in Case 1, since \( n = \Theta(n^{2-\epsilon}) = \Theta(n^{\log_b a - \epsilon}) \).
Example: Multiplying Two Integers

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\[ \Rightarrow \text{Not better than the “school method”}. \]
Example: Multiplying Two Integers

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Hence,

**Algorithm 4 mult($A, B$)**

1: if $|A| = |B| = 1$ then
2: return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
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Hence,

\[\textbf{Algorithm 4} \text{ mult}(A, B)\]

\begin{enumerate}
  \item if $|A| = |B| = 1$ then
  \item return $a_0 \cdot b_0$
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$O(1)$
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$O(1)$

$O(n)$

$O(n)$

$T(\frac{n}{2})$
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6.2 Master Theorem 11. Apr. 2018
Ernst Mayr, Harald Räcke
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\[ T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n) \]

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- Case 1: \( f(n) = \Theta(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
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Again we are in Case 1. We get a running time of \( \Theta(n^{\log_2 3}) \approx \Theta(n^{1.59}) \).

A huge improvement over the “school method”.
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6.2 Master Theorem
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- **Case 2:** $f(n) = \Theta(n^{\log_b a \log^k n})$ \quad $T(n) = \Theta(n^{\log_b a \log^{k+1} n})$
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Ernst Mayr, Harald Räcke
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6.2 Master Theorem
Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

This is the general form of a linear recurrence relation of order \( k \) with constant coefficients \((c_0, c_k \neq 0)\).

- \( T(n) \) only depends on the \( k \) preceding values. This means the recurrence relation is of order \( k \).
- The recurrence is linear as there are no products of \( T(n) \)’s.
- If \( f(n) = 0 \) then the recurrence relation becomes a linear, homogenous recurrence relation of order \( k \).

Note that we ignore boundary conditions for the moment.
6.3 The Characteristic Polynomial

Consider the recurrence relation:

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Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any $k$ consecutive values completely determine the solution.
- $k$ non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.
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▶ First consider the homogenous case.
The Homogenous Case

The solution space

\[ S = \left\{ T = T[1], T[2], T[3], \ldots \mid T \text{ fulfills recurrence relation} \right\} \]

is a vector space. This means that if \( T_1, T_2 \in S \), then also \( \alpha T_1 + \beta T_2 \in S \), for arbitrary constants \( \alpha, \beta \).

How do we find a non-trivial solution?
We guess that the solution is of the form \( \lambda^n, \lambda \neq 0 \), and see what happens. In order for this guess to fulfill the recurrence we need

\[ c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0 \]

for all \( n \geq k \).
The Homogenous Case

The solution space

\[ S = \left\{ T = T[1], T[2], T[3], \ldots \mid T \text{ fulfills recurrence relation} \right\} \]

is a vector space. This means that if \( T_1, T_2 \in S \), then also \( \alpha T_1 + \beta T_2 \in S \), for arbitrary constants \( \alpha, \beta \).

How do we find a non-trivial solution?

We guess that the solution is of the form \( \lambda^n, \lambda \neq 0 \), and see what happens. In order for this guess to fulfill the recurrence we need

\[ c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0 \]

for all \( n \geq k \).
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The Homogenous Case

Dividing by $\lambda^{n-k}$ gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if $\lambda_i$ is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the $k$ (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n$$

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The Homogenous Case

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Lemma 2

Assume that the characteristic polynomial has $k$ distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n.$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.
The Homogenous Case

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Assume that the characteristic polynomial has \( k \) distinct roots \( \lambda_1, \ldots, \lambda_k \). Then all solutions to the recurrence relation are of the form

\[
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The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:
The Homogenous Case

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Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \cdots + \alpha_k \cdot \lambda_k = T[1]$$
The Homogenous Case

Proof (cont.).
Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:

\[
\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \cdots + \alpha_k \cdot \lambda_k = T[1]
\]
\[
\alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \cdots + \alpha_k \cdot \lambda_k^2 = T[2]
\]
Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i'$s such that these conditions are met:

\[
\begin{align*}
\alpha_1 \cdot \lambda_1 & \quad + \quad \alpha_2 \cdot \lambda_2 \quad + \quad \cdots \quad + \quad \alpha_k \cdot \lambda_k 
= \quad T[1] \\
\alpha_1 \cdot \lambda_1^2 & \quad + \quad \alpha_2 \cdot \lambda_2^2 \quad + \quad \cdots \quad + \quad \alpha_k \cdot \lambda_k^2 
= \quad T[2] \\
& \quad \vdots
\end{align*}
\]
The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:

\[
\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \cdots + \alpha_k \cdot \lambda_k = T[1]
\]
\[
\alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \cdots + \alpha_k \cdot \lambda_k^2 = T[2]
\]
\[
\vdots
\]
\[
\alpha_1 \cdot \lambda_1^k + \alpha_2 \cdot \lambda_2^k + \cdots + \alpha_k \cdot \lambda_k^k = T[k]
\]
The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:

$$
\begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{pmatrix}
=
\begin{pmatrix}
T[1] \\
T[2] \\
\vdots \\
T[k]
\end{pmatrix}
$$
Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$
\begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_k \\
\lambda_2 & \lambda_2 & \cdots & \lambda_k \\
\vdots & & & \\
\lambda_k & \lambda_k & \cdots & \lambda_k \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k \\
\end{pmatrix}
= 
\begin{pmatrix}
T[1] \\
T[2] \\
\vdots \\
T[k] \\
\end{pmatrix}
$$

We show that the column vectors are linearly independent. Then the above equation has a solution.
Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i
\]
Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \\
\end{vmatrix}
\]

6.3 The Characteristic Polynomial

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### Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k
\end{vmatrix}
= \prod_{i=1}^{k} \lambda_i 
\]

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1
\end{vmatrix}
\]

\[
= \prod_{i=1}^{k} \lambda_i 
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_{k-2} & \lambda_{k-1} \\
1 & \lambda_2 & \cdots & \lambda_{k-2} & \lambda_{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_{k-2} & \lambda_{k-1}
\end{vmatrix}
= 
\begin{vmatrix}
1 & \lambda_1 \\
1 & \lambda_2 \\
\vdots & \vdots \\
1 & \lambda_k
\end{vmatrix}
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1}
\end{vmatrix}
= \\
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix}
\]
### Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \ldots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \ldots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \ldots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix}
= \ldots
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix}
= 
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix}
\]

6.3 The Characteristic Polynomial
Computing the Determinant

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) & 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) & 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix} =
\]
Computing the Determinant

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{pmatrix} = \\
\prod_{i=2}^{k} (\lambda_i - \lambda_1) \cdot 
\begin{pmatrix}
1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2}
\end{pmatrix}
\]
Computing the Determinant

Repeating the above steps gives:

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)
\]

Hence, if all \(\lambda_i\)'s are different, then the determinant is non-zero.
The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root $\lambda_i$ with multiplicity (Vielfachheit) at least 2. Then not only is $\lambda_i^n$ a solution to the recurrence but also $n\lambda_i^n$. To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since $\lambda_i$ is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root $\lambda_i$. 
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Calculating the derivative gives a polynomial that still has root $\lambda_i$. 
This means

\[ c_0 n\lambda_i^{n-1} + c_1 (n - 1)\lambda_i^{n-2} + \cdots + c_k (n - k)\lambda_i^{n-k-1} = 0 \]

Hence,

\[ \underbrace{c_0 n\lambda_i^n}_{T[n]} + \underbrace{c_1 (n - 1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + \underbrace{c_k (n - k)\lambda_i^{n-k}}_{T[n-k]} = 0 \]
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\[ \underbrace{T[n]}_{T[n]} + \underbrace{T[n-1]}_{T[n-1]} + \underbrace{T[n-k]}_{T[n-k]} = 0 \]
The Homogeneous Case

Suppose $\lambda_i$ has multiplicity $j$. We know that

$$c_0 n \lambda_i^n + c_1 (n - 1) \lambda_i^{n-1} + \cdots + c_k (n - k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n - 1)^2 \lambda_i^{n-1} + \cdots + c_k (n - k)^2 \lambda_i^{n-k} = 0$$

We can continue $j - 1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$. 
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Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$. 
The Homogeneous Case

Suppose $\lambda_i$ has multiplicity $j$. We know that

$$c_0 n \lambda_i^n + c_1 (n - 1) \lambda_i^{n-1} + \cdots + c_k (n - k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n - 1)^2 \lambda_i^{n-1} + \cdots + c_k (n - k)^2 \lambda_i^{n-k} = 0$$

We can continue $j - 1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$. 
Lemma 3

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n - 1] + \cdots + c_k T[n - k] = 0$$

Let $\lambda_i, i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities $\ell_i$. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$'s is a solution to the recurrence.
Example: Fibonacci Sequence

\[ T[0] = 0 \]
\[ T[1] = 1 \]
\[ T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2 \]

The characteristic polynomial is

\[ \lambda^2 - \lambda - 1 \]

Finding the roots, gives

\[ \lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \left(1 \pm \sqrt{5}\right) \]
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Hence, the solution is of the form

\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
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Hence, the solution is of the form

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\[ T[0] = 0 \text{ gives } \alpha + \beta = 0. \]
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\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}} \]
Example: Fibonacci Sequence

Hence, the solution is

\[
\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]
The Inhomogeneous Case

Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

with \( f(n) \neq 0 \).

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.
The Inhomogeneous Case

The general solution of the recurrence relation is

\[ T(n) = T_h(n) + T_p(n), \]

where \( T_h \) is any solution to the homogeneous equation, and \( T_p \) is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.
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The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$. 
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Then the solution is of the form

\[ T[n] = \alpha_1 n + \beta n \]

\[ T[0] = 1 \] gives \( \alpha = 1 \).

\[ T[1] = 2 \] gives \( 1 + \beta = 2 \Rightarrow \beta = 1 \).
The Inhomogeneous Case

Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2$$

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$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$
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\[
T[1] = 2 \text{ gives } 1 + \beta = 2 \implies \beta = 1.
\]
The Inhomogeneous Case

If \( f(n) \) is a polynomial of degree \( r \) this method can be applied \( r + 1 \) times to obtain a homogeneous equation:

\[
T[n] = T[n - 1] + n^2
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Shift:

\[
T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1
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Difference:

\[
\]

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\[ T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2 \]

and so on...
6.4 Generating Functions

Definition 4 (Generating Function)

Let \((a_n)_{n \geq 0}\) be a sequence. The corresponding
generating function (Erzeugendenfunktion) is

\[ F(z) := \sum_{n \geq 0} a_n z^n; \]

exponential generating function (exponentielle Erzeugendenfunktion) is

\[ F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n. \]
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Example 5

1. The generating function of the sequence \((1, 0, 0, \ldots)\) is

\[
F(z) = 1.
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6.4 Generating Functions

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There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let \( f = \sum_{n \geq 0} a_n z^n \) and \( g = \sum_{n \geq 0} b_n z^n \).

- **Equality:** \( f \) and \( g \) are equal if \( a_n = b_n \) for all \( n \).
- **Addition:** \( f + g := \sum_{n \geq 0} (a_n + b_n) z^n \).
- **Multiplication:** \( f \cdot g := \sum_{n \geq 0} c_n z^n \) with \( c_n = \sum_{p=0}^{n} a_p b_{n-p} \).

There are no convergence issues here.
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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \rightarrow \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.
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We view a power series as a function \( f : \mathbb{C} \to \mathbb{C} \).

Then, it is important to think about convergence/convergence radius etc.
What does \( \sum_{n \geq 0} z^n = \frac{1}{1-z} \) mean in the \textit{algebraic view}?

It means that the power series \( 1 - z \) and the power series \( \sum_{n \geq 0} z^n \) are invers, i.e.,

\[
(1-z) \cdot \left( \sum_{n \geq 0} z^n \right) = 1.
\]

This is well-defined.
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We can compute the derivative:

\[ \sum_{n \geq 1} n z^{n-1} = \frac{1}{(1 - z)^2}. \]
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Hence, the generating function of the sequence \(a_n = n + 1\) is \(1/(1 - z)^2\).
We can repeat this
6.4 Generating Functions

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\[ \sum_{n \geq 0} (n + 1)z^n = \frac{1}{(1 - z)^2} . \]
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Derivative:

\[ \sum_{n \geq 1} n(n + 1) z^{n-1} = \frac{2}{(1 - z)^3} . \]
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Hence, the generating function of the sequence \( a_n = (n + 1)(n + 2) \) is \( \frac{2}{(1 - z)^3} \).
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Hence, the generating function of the sequence \(a_n = (n + 1)(n + 2)\) is \(\frac{2}{(1-z)^3}\).
Computing the $k$-th derivative of $\sum z^n$. 

\[
\sum_{n\geq k} n(n-1)\cdots(n-k+1)z^n = \sum_{n\geq 0} (n+k)\cdots(n+1)z^n = k! \left(1 - z\right)^{-k-1}.
\]

Hence:

\[
\sum_{n\geq 0} \frac{(n+k)\cdots(n+1)}{k!}z^n = \frac{1}{1 - z}^{-k-1}.
\]
Computing the $k$-th derivative of $\sum z^n$.

\[ \sum_{n \geq k} n(n - 1) \cdots (n - k + 1) z^{n-k} \]
Computing the $k$-th derivative of $\sum z^n$.

$$\sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1)z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \ldots \cdot (n+1)z^n$$
Computing the $k$-th derivative of $\sum z^n$.

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\[
= \frac{k!}{(1 - z)^{k+1}}.
\]
6.4 Generating Functions

Computing the $k$-th derivative of $\sum z^n$.

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$$= \frac{k!}{(1-z)^{k+1}}.$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}.$$
Computing the $k$-th derivative of $\sum z^n$.

\[
\sum_{n\geq k} n(n-1) \cdots (n-k+1) z^{n-k} = \sum_{n\geq 0} (n+k) \cdots (n+1) z^n = \frac{k!}{(1-z)^{k+1}}.
\]

Hence:

\[
\sum_{n\geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}.
\]

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$. 

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6.4 Generating Functions

\[ \sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n + 1) z^n - \sum_{n \geq 0} z^n \]

The generating function of the sequence \( a_n = n \) is \( z \left( \frac{1}{1 - z} \right)^2 \).
6.4 Generating Functions

\[ \sum_{n \geq 0} nz^n = \sum_{n \geq 0} (n + 1)z^n - \sum_{n \geq 0} z^n \]

\[ = \frac{1}{(1 - z)^2} - \frac{1}{1 - z} \]
The generating function of the sequence \( a_n = n \) is

\[
\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n + 1) z^n - \sum_{n \geq 0} z^n
\]

\[
= \frac{1}{(1 - z)^2} - \frac{1}{1 - z} + \frac{z}{(1 - z)^2}
\]
The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$. 

\[
\sum_{n \geq 0} nz^n = \sum_{n \geq 0} (n + 1)z^n - \sum_{n \geq 0} z^n = \frac{1}{(1-z)^2} - \frac{1}{1-z} = \frac{1}{(1-z)^2} - \frac{z}{(1-z)^2} 
\]
6.4 Generating Functions

We know

\[ \sum_{n \geq 0} y^n = \frac{1}{1 - y} \]

Hence,

\[ \sum_{n \geq 0} a^n z^n = \frac{1}{1 - az} \]

The generating function of the sequence \( f_n = a^n \) is \( \frac{1}{1 - az} \).
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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$
Example: \( a_n = a_{n-1} + 1, \ a_0 = 1 \)

Suppose we have the recurrence \( a_n = a_{n-1} + 1 \) for \( n \geq 1 \) and \( a_0 = 1 \).

\[
A(z) = \sum_{n \geq 0} a_n z^n
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$$A(z) = \sum_{n \geq 0} a_n z^n$$

$$= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n$$
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\[
A(z) = \sum_{n \geq 0} a_n z^n \\
= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\
= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n
\]
Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

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= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n \\
= zA(z) + \sum_{n \geq 0} z^n
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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

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$$= zA(z) + \frac{1}{1 - z}$$
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Solving for $A(z)$ gives
Example: $a_n = a_{n-1} + 1, \ a_0 = 1$

Solving for $A(z)$ gives

$$A(z) = \frac{1}{(1 - z)^2}$$
Example: \( a_n = a_{n-1} + 1, \ a_0 = 1 \)

Solving for \( A(z) \) gives

\[
\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1 - z)^2}
\]
Example: $a_n = a_{n-1} + 1, a_0 = 1$

Solving for $A(z)$ gives

$$
\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1 - z)^2} = \sum_{n \geq 0} (n + 1) z^n
$$
Example: $a_n = a_{n-1} + 1, \ a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1 - z)^2} = \sum_{n \geq 0} (n + 1) z^n$$

Hence, $a_n = n + 1$. 
## Some Generating Functions

<table>
<thead>
<tr>
<th>$n$-th sequence element</th>
<th>generating function</th>
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<tbody>
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<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{1 - z}$</td>
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- **n-th sequence element**
- **generating function**

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</tr>
<tr>
<td>$\binom{n+k}{k}$</td>
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### Some Generating Functions

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<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{1 - z} )</td>
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<td>( n + 1 )</td>
<td>( \frac{1}{(1 - z)^2} )</td>
</tr>
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<td>( \binom{n+k}{k} )</td>
<td>( \frac{1}{(1 - z)^{k+1}} )</td>
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<tr>
<td>( n )</td>
<td>( \frac{z}{(1 - z)^2} )</td>
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<tr>
<td>$n$</td>
<td>$\frac{z}{(1 - z)^2}$</td>
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<tr>
<td>$a^n$</td>
<td>$\frac{1}{1 - az}$</td>
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<tr>
<td>$(n+k)$</td>
<td>$\frac{1}{(1 - z)^{k+1}}$</td>
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<td>$n$</td>
<td>$\frac{z}{(1 - z)^2}$</td>
</tr>
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<td>$a^n$</td>
<td>$\frac{1}{1 - az}$</td>
</tr>
<tr>
<td>$n^2$</td>
<td>$\frac{z(1 + z)}{(1 - z)^3}$</td>
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<tr>
<td>( a^n )</td>
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<tbody>
<tr>
<td>$c f_n$</td>
<td>$cF$</td>
</tr>
</tbody>
</table>

**Notes:**
- $c f_n$ represents the $n$-th sequence element.
- $cF$ represents the generating function associated with $c f_n$. 

---

**Further阅读：**

- Ernst Mayr, Harald Räcke
- 6.4 Generating Functions
- 11. Apr. 2018

---

**Table:**
- **Column 1:** $n$-th sequence element
- **Column 2:** generating function

**Row:**
- $c f_n$ in the first column
- $cF$ in the second column
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<thead>
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<tbody>
<tr>
<td>$c f_n$</td>
<td>$c F$</td>
</tr>
<tr>
<td>$f_n + g_n$</td>
<td>$F + G$</td>
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<td>$\sum_{i=0}^{n} f_i g_{n-i}$</td>
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</tr>
<tr>
<td>( f_{n-k} ) (( n \geq k )); 0 othew.</td>
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<tr>
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<td>$\frac{F(z)}{1-z}$</td>
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<td>$nf_n$</td>
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Solving Recursions with Generating Functions

1. Set \( A(z) = \sum_{n \geq 0} a_n z^n \).
Solving Recursions with Generating Functions

1. Set \( A(z) = \sum_{n\geq 0} a_n z^n \).

2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
Solving Recursions with Generating Functions

1. Set \( A(z) = \sum_{n \geq 0} a_n z^n \).

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3. Do further transformations so that the infinite sums on the right hand side can be replaced by \( A(z) \).
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4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
Solving Recursions with Generating Functions

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5. Write \( f(z) \) as a formal power series.

Techniques:

- Partial fraction decomposition (Partialbruchzerlegung)
- Lookup in tables
Solving Recursions with Generating Functions

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4. Solving for \( A(z) \) gives an equation of the form \( A(z) = f(z) \), where hopefully \( f(z) \) is a simple function.

5. Write \( f(z) \) as a formal power series. Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables

6. The coefficients of the resulting power series are the \( a_n \).
Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:
Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$
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1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:
Example: \(a_n = 2a_{n-1}, \ a_0 = 1\)

1. Set up generating function:

\[
A(z) = \sum_{n \geq 0} a_n z^n
\]

2. Transform right hand side so that recurrence can be plugged in:

\[
A(z) = a_0 + \sum_{n \geq 1} a_n z^n
\]
Example: \( a_n = 2a_{n-1}, a_0 = 1 \)

1. Set up generating function:

\[
A(z) = \sum_{n \geq 0} a_n z^n
\]

2. Transform right hand side so that recurrence can be plugged in:

\[
A(z) = a_0 + \sum_{n \geq 1} a_n z^n
\]

2. Plug in:
Example: \(a_n = 2a_{n-1}, \ a_0 = 1\)

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\[
A(z) = \sum_{n \geq 0} a_n z^n
\]

2. Transform right hand side so that recurrence can be plugged in:

\[
A(z) = a_0 + \sum_{n \geq 1} a_n z^n
\]

2. Plug in:

\[
A(z) = 1 + \sum_{n \geq 1} (2a_{n-1}) z^n
\]
Example: $a_n = 2a_{n-1}, \ a_0 = 1$
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3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.
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Example: $a_n = 3a_{n-1} + n, \ a_0 = 1$

1. Set up generating function:

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$$= 1 + 3z A(z) + \frac{z}{(1 - z)^2}$$
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A(z) = \frac{(1 - z)^2 + z}{(1 - 3z)(1 - z)^2} = \frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2}
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We use partial fraction decomposition:
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= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z)
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\[
= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C)
\]
Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

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This leads to the following conditions:

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\begin{align*}
A + B + C &= 1 \\
2A + 4B + 3C &= 1 \\
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which gives

\[
\begin{align*}
A &= \frac{7}{4} \\
B &= -\frac{1}{4} \\
C &= -\frac{1}{2}
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6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$.
6.5 Transformation of the Recurrence

Example 6

\[ f_0 = 1 \]
\[ f_1 = 2 \]
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Define

\[ g_n := \log f_n. \]
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\[ f_n = 2^{F_n} \]
Example 7

\[ f_1 = 1 \]
\[ f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1; \]
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Then:

\[ g_0 = 1 \]
\[ g_k = 3 g_{k-1} + 2^k, \ k \geq 1 \]
6 Recurrences

We get

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\[ = 3^2 \left[ 3g_{k-3} + 2^{k-2} \right] + 32^{k-1} + 2^k \]
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\[ = 2^k \cdot \sum_{i=0}^{k} \left( \frac{3}{2} \right)^i \]
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\[ = 3^3 g_{k-3} + 32^{k-2} + 32^{k-1} + 2^k \]

\[ = 2^k \cdot \sum_{i=0}^{k} \left( \frac{3}{2} \right)^i \]

\[ = 2^k \cdot \frac{\left( \frac{3}{2} \right)^{k+1} - 1}{\frac{1}{2}} \]
6 Recurrences

We get

\[ g_k = 3 \left[ g_{k-1} \right] + 2^k \]
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\[ = 3^2 \left[ 3g_{k-3} + 2^{k-2} \right] + 32^{k-1} + 2^k \]
\[ = 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \]
\[ = 2^k \cdot \sum_{i=0}^{k} \left( \frac{3}{2} \right)^i \]
\[ = 2^k \cdot \left( \frac{3}{2} \right)^{k+1} - 1 \]
\[ \frac{1}{\sqrt{2}} = 3^{k+1} - 2^{k+1} \]
6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence }$$

$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$
Let \( n = 2^k \):

\[
g_k = 3^{k+1} - 2^{k+1}, \text{ hence } \\
f_n = 3 \cdot 3^k - 2 \cdot 2^k \\
= 3 \left(2^{\log_3} \right)^k - 2 \cdot 2^k
\]
Let $n = 2^k$:

\[ g_k = 3^{k+1} - 2^{k+1}, \text{ hence} \]
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\[ = 3(2^k \log_3) - 2 \cdot 2^k \]
\[ = 3n \log_3 - 2n . \]