6 Recurrences

Algorithm 2 mergesort(list L)

1: \( n \leftarrow \text{size}(L) \)
2: \textbf{if} \( n \leq 1 \) return \( L \)
3: \( L_1 \leftarrow L[1 \cdots \lceil \frac{n}{2} \rceil] \)
4: \( L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n] \)
5: mergesort\((L_1)\)
6: mergesort\((L_2)\)
7: \( L \leftarrow \text{merge}(L_1, L_2) \)
8: return \( L \)

This algorithm requires \( T(n) = T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + O(n) \leq 2T(\lceil \frac{n}{2} \rceil) + O(n) \) comparisons when \( n > 1 \) and \( 0 \) comparisons when \( n \leq 1 \).
6 Recurrences

This algorithm requires

\[ T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil \right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil \right) + \mathcal{O}(n) \]

comparisons when \( n > 1 \) and 0 comparisons when \( n \leq 1 \).
How do we bring the expression for the number of comparisons (≈ running time) into a closed form?

For this we need to solve the recurrence.
How do we bring the expression for the number of comparisons (≈ running time) into a closed form?

For this we need to solve the recurrence.
Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.
4. Generating Functions
A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence
Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.
6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 
2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\
0 & \text{otherwise}
\end{cases}$$

Informal way:
First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

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**Informal way:** Assume that instead we have

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Assume that instead we have

$$T(n) \leq \begin{cases} 
2T\left(\frac{n}{2} \right) + cn & n \geq 2 \\
0 & \text{otherwise}
\end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.
6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. 

Formally, this is not correct if $n$ is not a power of 2. Also even in this case one would need to do an induction proof.
Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$
6.1 Guessing+Induction

Suppose we guess \( T(n) \leq dn \log n \) for a constant \( d \). Then

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + cn
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\[
\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn
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Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + cn \\
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= dn (\log n - 1) + cn
\]
6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + cn \]
\[ \leq 2\left(d \frac{n}{2} \log \frac{n}{2}\right) + cn \]
\[ = dn(\log n - 1) + cn \]
\[ = dn \log n + (c - d)n \]
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Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

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$$\leq 2\left(d \frac{n}{2} \log \frac{n}{2}\right) + cn$$

$$= dn(\log n - 1) + cn$$

$$= dn \log n + (c - d)n$$

$$\leq dn \log n$$

if we choose $d \geq c$. 
Suppose we guess $T(n) \leq dn \log n$ for a constant $d$. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(\frac{d}{2} \log \frac{n}{2}\right) + cn$$

$$= dn (\log n - 1) + cn$$

$$= dn \log n + (c - d)n$$

$$\leq dn \log n$$

if we choose $d \geq c$.

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How do we get a result for all values of $n$?
6.1 Guessing+Induction

How do we get a result for all values of $n$?

We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 
2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\
b & \text{otherwise}
\end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant ($b$ in the above case).
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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant ($b$ in the above case).
We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n)$$
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$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$
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We also make a guess of $T(n) \leq dn \log n$ and get

\begin{equation}
T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn
\end{equation}

\begin{equation}
\leq 2 \left( d \left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil \right) + cn
\end{equation}
We also make a guess of $T(n) \leq dn \log n$ and get

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$$\leq 2\left( d\left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn$$

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} + 1$$
6.1 Guessing+Induction

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T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\
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\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1
\leq 2\left( d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right) \right) + cn
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$$\leq 2 \left( d \left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil \right) + cn$$

$$\leq 2 \left( d \left( n/2 + 1 \right) \log \left( n/2 + 1 \right) \right) + cn$$

$$\leq dn \log \left( \frac{9}{16} n \right) + 2d \log n + cn$$

\[\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1\]

\[\frac{n}{2} + 1 \leq \frac{9}{16} n\]
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We also make a guess of $T(n) \leq dn \log n$ and get

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$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$
6.1 Guessing+Induction

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$$\leq dn \log \left( \frac{9}{16} n \right) + 2d \log n + cn$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

\(\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1\)

\(\frac{n}{2} + 1 \leq \frac{9}{16} n\)

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\leq 2\left( d\left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn
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\[
\leq 2\left( d\left(\frac{n}{2} + 1\right) \log \left(\frac{n}{2} + 1\right) \right) + cn
\]

\[
\leq d\left(\frac{n}{2} + 1\right) \log \left(\frac{9}{16} n\right) + 2d \log n + cn
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= dn \log n + (\log 9 - 4)dn + 2d \log n + cn
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\[
\leq dn \log n + (\log 9 - 3.5)dn + cn
\]
6.1 Guessing+Induction

We also make a guess of \( T(n) \leq dn \log n \) and get

\[
T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn
\]

\[
\leq 2\left(d\left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor \right) + cn
\]

\[
\leq 2(d(n/2 + 1) \log(n/2 + 1)) + cn
\]

\[
\leq dn \log \left(\frac{9}{16} n\right) + 2d \log n + cn
\]

\[
= dn \log n + (\log 9 - 4)dn + 2d \log n + cn
\]

\[
\leq dn \log n + (\log 9 - 3.5)dn + cn
\]

\[
\leq dn \log n - 0.33dn + cn
\]

\[\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} + 1\]

\[
\frac{n}{2} + 1 \leq \frac{9}{16} n
\]

\[\log \frac{9}{16} n = \log n + (\log 9 - 4)\]

\[\log n \leq \frac{n}{4}\]
6.1 Guessing+Induction

We also make a guess of \( T(n) \leq dn \log n \) and get

\[
T(n) \leq 2T(\lceil \frac{n}{2} \rceil) + cn
\]
\[
\leq 2 \left( d \lceil \frac{n}{2} \rceil \log \lceil \frac{n}{2} \rceil \right) + cn
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\[
\leq 2 \left( d \left( \frac{n}{2} + 1 \right) \log \left( \frac{n}{2} + 1 \right) \right) + cn
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= dn \log n + (\log 9 - 4)dn + 2d \log n + cn
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\leq dn \log n + (\log 9 - 3.5)dn + cn
\]
\[
\leq dn \log n - 0.33dn + cn
\]
\[
\leq dn \log n
\]

for a suitable choice of \( d \).
Lemma 1

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.
If $f(n) = \Theta(n^{\log_b a - \epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b a} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.
If $f(n) = \Omega(n^{\log_b a + \epsilon})$ and for sufficiently large $n$

$$af\left(\frac{n}{b}\right) \leq cf(n)$$

for some constant $c < 1$ then $T(n) = \Theta(f(n))$.
6.2 Master Theorem

We prove the Master Theorem for the case that $n$ is of the form $b^\ell$, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:
The Recursion Tree

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![Recursion Tree Diagram]
The Recursion Tree

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\[ f(n) \]

\[ a \frac{n}{b^2} \]

\[ a f\left( \frac{n}{b^2} \right) \]
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

\[ f(n) \]

\[ a f \left( \frac{n}{b} \right) \]

\[ a^2 f \left( \frac{n}{b^2} \right) \]
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

\[ f(n) = a f(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + \ldots + a^i f(\frac{n}{b^i}) + \ldots + n^{\log_b a} \]

6.2 Master Theorem
This gives

\[ T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right). \]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$. 

Hence, 

$$
T(n) \leq \left( c b^{\epsilon} - 1 + 1 \right) n \log_b a \left( n^{\epsilon} - 1 \right) / \left( n^{\epsilon} \right)
$$
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a - \epsilon} \).

\[
T(n) - n^{\log_b a}
\]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$
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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$= c n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

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Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]

\[
b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}
\]

\[
\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}
\]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

$$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$$

$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a - \epsilon} \).

\[
T(n) = n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]

\[
b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}
\]

\[
\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}
\]

Hence,

\[
T(n) \leq c n^{\log_b a - \epsilon} \left(\frac{b^{\epsilon \log_b n} - 1}{b^{\epsilon} - 1}\right)
\]

6.2 Master Theorem
Case 1. Now suppose that \( f(n) \leq cn^{\log_b a - \epsilon} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]
\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]

\[
b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}
\]

\[
\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}
\]

\[
= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)
\]
\[
= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1)
\]
\[
= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1) / (n^{\epsilon})
\]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\]

\[
b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}
\]

\[
\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}
\]

\[
= cn^{\log_b a - \epsilon} \left(b^{\epsilon \log_b n} - 1\right) / (b^{\epsilon} - 1)
\]

\[
= cn^{\log_b a - \epsilon} (n^\epsilon - 1) / (b^{\epsilon} - 1)
\]

\[
= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^\epsilon - 1) / (n^\epsilon)
\]

Hence,

\[
T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)}
\]
Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i$$

$$= cn^{\log_b a - \epsilon} (b^\epsilon \log_b n - 1) / (b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon} (n^\epsilon - 1) / (b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a} (n^\epsilon - 1) / (n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$. 

\[ T(n) = n \log_b n - \sum_{i=0}^{\log_b n} a_i f(n^{b^i}) \leq c n \log_b n - \sum_{i=0}^{\log_b n} a_i (n^{b^i}) \log_b a = cn \log_b a \log_b n - \sum_{i=0}^{\log_b n} a_i = cn \log_b a \log_b n. \]

Hence, $T(n) = O(n \log_b a \log_b n)$. 

6.2 Master Theorem
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a}
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$
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\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
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$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1$$

Hence, $T(n) = O(n^{\log_b a} \log_b n)$.
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f \left( \frac{n}{b^i} \right) \\
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left( \frac{n}{b^i} \right)^{\log_b a} \\
= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
= cn^{\log_b a} \log_b n
\]
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$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$
Case 2. Now suppose that \( f(n) \leq cn^{\log_b a} \).

\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
= cn^{\log_b a} \log_b n
\]

Hence,

\[
T(n) = \Theta(n^{\log_b a} \log_b n) \quad \Rightarrow \quad T(n) = \Theta(n^{\log_b a} \log n).
\]
Case 2. Now suppose that \( f(n) \geq cn^{\log_b a} \).
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\[
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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\]
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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$
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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
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= cn^{\log_b a} \log_b n
\]

Hence,

\[
T(n) = \Omega(n^{\log_b a} \log_b n) \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).
\]
Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$. 
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$$T(n) - n^{\log_b a}$$
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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f \left( \frac{n}{b^i} \right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left( \frac{n}{b^i} \right)^{\log_b a} \cdot \left( \log_b \left( \frac{n}{b^i} \right) \right)^k$$
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\[n = b^\ell \Rightarrow \ell = \log_b n\]
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$n = b^\ell \Rightarrow \ell = \log_b n$

\[ = cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k \]
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\[
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\]

$n = b^\ell \Rightarrow \ell = \log_b n$

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\]

\[
= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}
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6.2 Master Theorem
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T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f \left( \frac{n}{b^i} \right)
\]

\[
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\[ \leq c \sum_{i=0}^{\log_b n-1} a^i \left( \frac{n}{b^i} \right)^{\log_b a} \cdot \left( \log_b \left( \frac{n}{b^i} \right) \right)^k \]

$n = b^\ell \Rightarrow \ell = \log_b n$

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\[ = cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \]

\[ = cn^{\log_b a} \sum_{i=1}^{\ell} i^k \]

\[ \approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \]

$\Rightarrow T(n) = O(n^{\log_b a \log^{k+1} n})$. 

6.2 Master Theorem
Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$.
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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?
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$$\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

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\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f \left( \frac{n}{b^i} \right)
\]

\[
\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + O(n^{\log_b a})
\]

\[
q < 1: \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}
\]

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\]

\[
\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\]

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\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\]

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q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}
\]

Hence,

\[
T(n) \leq \mathcal{O}(f(n))
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\[
T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)
\]

\[
\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + O(n^{\log_b a})
\]

\[
q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}
\]

Hence,

\[
T(n) \leq O(f(n)) \Rightarrow T(n) = \Theta(f(n)).
\]

Where did we use \( f(n) \geq \Omega(n^{\log_b a + \epsilon}) \)?
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.
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For this we first need to be able to add two integers $A$ and $B$:
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For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 
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\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
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\( A \)

\( B \)
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6.2 Master Theorem
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```
1 1 0 1 1 0 1 0 1  A
1 0 0 0 1 0 0 1 1  B
```

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Example: Multiplying Two Integers

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\[
\begin{array}{c}
11011011 \\
110010001 \\
\end{array}
\begin{array}{c}
00 \\
01001101 \\
11 \\
1011001 \\
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
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\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
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6.2 Master Theorem 11. Apr. 2018
Ernst Mayr, Harald Räcke
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1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
& & & & & A \\
& & & & & \\
0 & 1 & 0 & 1 & 1 & \\
0 & 0 & 1 & 1 & \\
\hline
0 & 0 & 0 & 0 & & B \\
\end{array}
\]

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
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\[
\begin{array}{c}
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{c}
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1
\end{array}
\quad
\begin{array}{c}
A \\
B
\end{array}
\]

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1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0
\end{array}
\begin{array}{c}
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\hline
0 & 1 & 0 & 0 & 0
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A \\
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\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
0 & 1 & 0 & 0 & 0 & 0
\end{array}
$$
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\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 
Example: Multiplying Two Integers

Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
Example: Multiplying Two Integers

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For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\]
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
\]
Example: Multiplying Two Integers

Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
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For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Example: Multiplying Two Integers

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For this we first need to be able to add two integers $A$ and $B$: 

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1
\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

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Ernst Mayr, Harald Räcke
Example: Multiplying Two Integers

Suppose we want to multiply two \( n \)-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers \( A \) and \( B \):

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

This gives that two \( n \)-bit integers can be added in time \( O(n) \).
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\]

This gives that two $n$-bit integers can be added in time $O(n)$. 

6.2 Master Theorem

Ernst Mayr, Harald Räcke
Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

\[
\begin{array}{cccccccc}
    1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\quad A
\]

\[
\begin{array}{cccccccc}
    1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\quad B
\]

\[
\begin{array}{cccccccc}
    1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
    \hline
    1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

This gives that two $n$-bit integers can be added in time $\mathcal{O}(n)$. 
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

$\begin{array}{cccc} 1 & 0 & 0 & 0 \ 1 \end{array} \times \begin{array}{cccc} 1 & 0 & 1 & 1 \end{array}$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c}
1 \\
0 \\
0 \\
0 \\
1
\end{array}
\times
\begin{array}{c}
1 \\
0 \\
1
\end{array}
\]

- This is also known as the “school method” for multiplying integers.
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Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \((m \leq n)\).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1
\end{array}
\]

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• Note that the intermediate numbers that are generated can have at most \( m + n \leq 2n \) bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & & & 1 & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 1
\end{array}
\]

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Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \((m \leq n)\).

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 & \times \ 1 & 0 & \underline{1} & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 & \\
0 & \ \\
1 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

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• Note that the intermediate numbers that are generated can have at most \( m + n \leq 2n \) bits.
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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
1\ 0\ 0\ 0\ 1 \times 1\ 0\ 0\ 1\ 1
\]

\[
1\ 0\ 0\ 0\ 1 \\
1\ 0\ 0\ 0\ 1\ 0
\]

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\times & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

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\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 & \times \ 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 &
\end{array}
\]

- This is also known as the “school method” for multiplying integers.
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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

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Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\multicolumn{5}{c}{\times} \\
1 & 0 & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most \( m + n \leq 2n \) bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & & & & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

• This is also known as the “school method” for multiplying integers.

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\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & & & & 1 & 0 & 1 & 1 \\
\hline & & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

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- Note that the intermediate numbers that are generated can have at most \( m + n \leq 2n \) bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & & & & \\
& 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\]

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• Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ $(m \leq n)$.

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) (\( m \leq n \)).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & & & & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
& & & & 1 & 0 & 0 & 0 & 1 & 0 \\
& & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most \( m + n \leq 2n \) bits.

Time requirement:
Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

\[
\begin{array}{c}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

• This is also known as the "school method" for multiplying integers.
• Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:
• Computing intermediate results: $O(nm)$. 
Example: Multiplying Two Integers

Suppose that we want to multiply an \( n \)-bit integer \( A \) and an \( m \)-bit integer \( B \) \((m \leq n)\).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\times & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1
\end{array}
\]

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most \( m + n \leq 2n \) bits.

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
\]

Time requirement:
- Computing intermediate results: \( \mathcal{O}(nm) \).
- Adding \( m \) numbers of length \( \leq 2n \):
  \[
  \mathcal{O}((m + n)m) = \mathcal{O}(nm).
  \]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 

\[ A \times B = \ldots \]
A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 

\[ A \times B = A_1 \cdot 2^{n/2} + A_0 \text{ and } B = B_1 \cdot 2^{n/2} + B_0 \]

Hence,
\[ AB = A_1 B_1 \cdot 2^{n/2} + (A_1 B_0 + A_0 B_1) \cdot 2^{n/2} + A_0 B_0 \]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers \( A \) and \( B \) are of length \( n = 2^k \), for some \( k \).

\[
\begin{array}{c}
  b_{n-1} & \cdots & b_0 \\
  a_{n-1} & \cdots & a_0 \\
\end{array}
\]

Then it holds that
\[
A = A_1 \cdot 2^{n_2} + A_0 \quad \text{and} \quad B = B_1 \cdot 2^{n_2} + B_0
\]

Hence,
\[
A \cdot B = A_1 B_1 \cdot 2^{n_2} + (A_1 B_0 + A_0 B_1) \cdot 2^{n_2} + A_0 B_0
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers \( A \) and \( B \) are of length \( n = 2^k \), for some \( k \).

\[
\begin{array}{c c c c}
\quad & b_{n-1} & \cdots & b_n & b_{n-1} & \cdots & b_0 \\
\times & a_{n-1} & \cdots & a_n & a_{n-1} & \cdots & a_0 \\
\end{array}
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$. 

\[
\begin{array}{c|c}
B_1 & B_0 \\
\hline
A_1 & A_0 \\
\end{array}
\]
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \quad \text{and} \quad B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$
Example: Multiplying Two Integers

A recursive approach:
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$
Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $|A| = |B| = 1$ then
2: return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^\frac{n}{2} + Z_0$

We get the following recurrence:

$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$. 

6.2 Master Theorem
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A, B)

1: \textbf{if } |A| = |B| = 1 \textbf{ then}
2: \quad \textbf{return } a_0 \cdot b_0
3: \quad \text{split } A \text{ into } A_0 \text{ and } A_1
4: \quad \text{split } B \text{ into } B_0 \text{ and } B_1
5: \quad Z_2 \leftarrow \text{mult}(A_1, B_1)
6: \quad Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)
7: \quad Z_0 \leftarrow \text{mult}(A_0, B_0)
8: \quad \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0

\mathcal{O}(1)
Example: Multiplying Two Integers

**Algorithm 3** \texttt{mult}(A, B)

1: \textbf{if } \lvert A \rvert = \lvert B \rvert = 1 \textbf{ then}
2: \hspace{1em} \textbf{return } a_0 \cdot b_0
3: \hspace{1em} \text{split } A \text{ into } A_0 \text{ and } A_1
4: \hspace{1em} \text{split } B \text{ into } B_0 \text{ and } B_1
5: \hspace{1em} Z_2 \leftarrow \text{mult}(A_1, B_1)
6: \hspace{1em} Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)
7: \hspace{1em} Z_0 \leftarrow \text{mult}(A_0, B_0)
8: \hspace{1em} \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{n\over 2} + Z_0

\text{\textit{O}(1)}

\text{\textit{O}(1)}
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A, B)

1: \textbf{if} \ |A| = |B| = 1 \textbf{then}
2: \quad \textbf{return} \ a_0 \cdot b_0
3: \quad \text{split } A \text{ into } A_0 \text{ and } A_1
4: \quad \text{split } B \text{ into } B_0 \text{ and } B_1
5: \quad Z_2 \leftarrow \text{mult}(A_1, B_1)
6: \quad Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)
7: \quad Z_0 \leftarrow \text{mult}(A_0, B_0)
8: \quad \textbf{return} \ Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0

\mathcal{O}(1)
\mathcal{O}(1)
\mathcal{O}(n)

We get the following recurrence:
\[ T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n). \]
Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $|A| = |B| = 1$ then
2: \hspace{1em} return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$
Example: Multiplying Two Integers

Algorithm 3 \( \text{mult}(A, B) \)

1: if \(|A| = |B| = 1\) then
2: \hspace{1em} \text{return } a_0 \cdot b_0
3: split \(A\) into \(A_0\) and \(A_1\)
4: split \(B\) into \(B_0\) and \(B_1\)
5: \(Z_2 \leftarrow \text{mult}(A_1, B_1)\)
6: \(Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)\)
7: \(Z_0 \leftarrow \text{mult}(A_0, B_0)\)
8: \text{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0

\(O(1)\) \hspace{1em} \(O(1)\) \hspace{1em} \(O(n)\) \hspace{1em} \(O(n)\) \hspace{1em} \(T\left(\frac{n}{2}\right)\)

We get the following recurrence:

\[
T(n) = 4 \cdot T\left(\frac{n}{2}\right) + O(n)
\]
Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $|A| = |B| = 1$ then
2: return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

$O(1)$
$O(1)$
$O(n)$
$O(n)$
$T(\frac{n}{2})$
$2T(\frac{n}{2}) + O(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$
Example: Multiplying Two Integers

**Algorithm 3** \( \text{mult}(A, B) \)

1. **if** \( |A| = |B| = 1 \) **then**
2. **return** \( a_0 \cdot b_0 \)
3. split \( A \) into \( A_0 \) and \( A_1 \)
4. split \( B \) into \( B_0 \) and \( B_1 \)
5. \( Z_2 \leftarrow \text{mult}(A_1, B_1) \)
6. \( Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1) \)
7. \( Z_0 \leftarrow \text{mult}(A_0, B_0) \)
8. **return** \( Z_2 \cdot 2^n + Z_1 \cdot 2^\frac{n}{2} + Z_0 \)

\( \mathcal{O}(1) \)  
\( \mathcal{O}(1) \)  
\( \mathcal{O}(n) \)  
\( \mathcal{O}(n) \)  
\( T(n) \)  
\( 2T(\frac{n}{2}) + \mathcal{O}(n) \)  
\( T(\frac{n}{2}) \)
Example: Multiplying Two Integers

Algorithm 3 \text{mult}(A, B)

1: \textbf{if } |A| = |B| = 1 \textbf{ then} \hspace{1cm} \mathcal{O}(1)
2: \hspace{1cm} \textbf{return } a_0 \cdot b_0 \hspace{1cm} \mathcal{O}(1)
3: \hspace{1cm} \text{split } A \text{ into } A_0 \text{ and } A_1 \hspace{1cm} \mathcal{O}(n)
4: \hspace{1cm} \text{split } B \text{ into } B_0 \text{ and } B_1 \hspace{1cm} \mathcal{O}(n)
5: \hspace{1cm} Z_2 \leftarrow \text{mult}(A_1, B_1) \hspace{1cm} T\left(\frac{n}{2}\right)
6: \hspace{1cm} Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1) \hspace{1cm} 2T\left(\frac{n}{2}\right) + \mathcal{O}(n)
7: \hspace{1cm} Z_0 \leftarrow \text{mult}(A_0, B_0) \hspace{1cm} T\left(\frac{n}{2}\right)
8: \hspace{1cm} \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0 \hspace{1cm} \mathcal{O}(n)

We get the following recurrence:
\[ T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \]
Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $|A| = |B| = 1$ then
  
2:   return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$
Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT\left(\frac{n}{b}\right) + f(n)$.

- **Case 1:** $f(n) = \Theta(n^{\log_b a - \epsilon})$  
  $T(n) = \Theta(n^{\log_b a})$

- **Case 2:** $f(n) = \Theta(n^{\log_b a \log^k n})$  
  $T(n) = \Theta(n^{\log_b a \log^{k+1} n})$

- **Case 3:** $f(n) = \Omega(n^{\log_b a + \epsilon})$  
  $T(n) = \Theta(f(n))$
Master Theorem: Recurrence: $T[n] = aT(n^{\frac{1}{b}}) + f(n)$.

▶ Case 1: $f(n) = O(n^{\log_b a - \epsilon})$  $\quad T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a \log^k n})$  $\quad T(n) = \Theta(n^{\log_b a \log^{k+1} n})$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$  $\quad T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = O(n^{2-\epsilon}) = O(n^{\log_b a - \epsilon})$. 

We get a running time of $O(n^2)$ for our algorithm. ⇒ Not better than the “school method.”
Example: Multiplying Two Integers

**Master Theorem**: Recurrence: \( T[n] = aT\left(\frac{n}{b}\right) + f(n) \).

- **Case 1**: \( f(n) = \Theta(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
- **Case 2**: \( f(n) = \Theta(n^{\log_b a \log^k n}) \) \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
- **Case 3**: \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) \( T(n) = \Theta(f(n)) \)

In our case \( a = 4, \ b = 2, \) and \( f(n) = \Theta(n) \). Hence, we are in Case 1, since \( n = \Theta(n^{2 - \epsilon}) = \Theta(n^{\log_b a - \epsilon}) \).

We get a running time of \( \Theta(n^2) \) for our algorithm.
Example: Multiplying Two Integers

**Master Theorem:** Recurrence: \( T[n] = aT\left(\frac{n}{b}\right) + f(n) \).

- **Case 1:** \( f(n) = \Theta(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
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In our case \( a = 4 \), \( b = 2 \), and \( f(n) = \Theta(n) \). Hence, we are in Case 1, since \( n = \Theta(n^{2-\epsilon}) = \Theta(n^{\log_b a - \epsilon}) \).

We get a running time of \( \Theta(n^2) \) for our algorithm.

⇒ Not better then the “school method”.

6.2 Master Theorem
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1 B_0 + A_0 B_1 = (A_0 + A_1) \cdot (B_0 + B_1) - A_1 B_1 - A_0 B_0$$

Hence,

Algorithm 4 $mult(A,B)$

1: if $|A| = |B| = 1$ then
2: return $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow mult(A_1, B_1)$
6: $Z_0 \leftarrow mult(A_0, B_0)$
7: $Z_1 \leftarrow mult(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^n + Z_0$

A more precise (correct) analysis would say that computing $Z_1$ needs time $T(\frac{n}{2} + 1) + \Theta(n)$. 
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1 B_0 + A_0 B_1$$

A more precise (correct) analysis would say that computing $Z_1$ needs time $T\left(\frac{n}{2} + 1\right) + \mathcal{O}(n)$. 

6.2 Master Theorem
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1B_0 + A_0B_1$$

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A more precise (correct) analysis would say that computing $Z_1$ needs time $T\left(\frac{n}{2} + 1\right) + O(n)$.
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1B_0 + A_0B_1 = Z_2 = Z_0$$

$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0$$

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6.2 Master Theorem

Ernst Mayr, Harald Räcke
Example: Multiplying Two Integers

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Hence,

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Hence,

```
Algorithm 4 mult(A, B)
1: if |A| = |B| = 1 then
2:   return $a_0 \cdot b_0$
3: else
4:     split A into $A_0$ and $A_1$
5:     split B into $B_0$ and $B_1$
6:     $Z_2 \leftarrow$ mult($A_1, B_1$)
7:     $Z_0 \leftarrow$ mult($A_0, B_0$)
8:     $Z_1 \leftarrow$ mult($A_0 + A_1, B_0 + B_1$) - $Z_2 - Z_0$
9: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{n \frac{3}{2}} + Z_0$
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A more precise (correct) analysis would say that computing $Z_1$ needs time $T(\frac{n}{2} + 1) + O(n)$.  

6.2 Master Theorem
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1B_0 + A_0B_1 = Z_2 = Z_0$$

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Hence,

```
Algorithm 4 mult(A, B)
1: if |A| = |B| = 1 then
2:    return $a_0 \cdot b_0$
3: split A into $A_0$ and $A_1$
4: split B into $B_0$ and $B_1$
5: $Z_2 \leftarrow$ mult($A_1, B_1$)
6: $Z_0 \leftarrow$ mult($A_0, B_0$)
7: $Z_1 \leftarrow$ mult($A_0 + A_1, B_0 + B_1$) - $Z_2 - Z_0$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0$
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A more precise (correct) analysis would say that computing $Z_1$ needs time $T(n/2 + 1) + O(n)$. 

6.2 Master Theorem
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

\[
Z_1 = A_1 B_0 + A_0 B_1
= (A_0 + A_1) \cdot (B_0 + B_1) - A_1 B_1 - A_0 B_0
\]

Hence,

\[
\text{Algorithm 4 } \text{mult}(A, B)
\]

1: \textbf{if } |A| = |B| = 1 \textbf{ then} \hfill \mathcal{O}(1)
2: \textbf{return } a_0 \cdot b_0 \hfill \mathcal{O}(1)
3: \text{split } A \text{ into } A_0 \text{ and } A_1
4: \text{split } B \text{ into } B_0 \text{ and } B_1
5: Z_2 \leftarrow \text{mult}(A_1, B_1)
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8: \textbf{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0

A more precise (correct) analysis would say that computing $Z_1$ needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$. 

6.2 Master Theorem

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We can use the following identity to compute $Z_1$:

$$Z_1 = A_1B_0 + A_0B_1 = Z_2 = Z_0$$

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Hence,

Algorithm 4 \texttt{mult}(A, B)

1: \textbf{if} $|A| = |B| = 1$ \textbf{then} \hspace{1cm} \mathcal{O}(1)
2: \textbf{return} $a_0 \cdot b_0$ \hspace{1cm} \mathcal{O}(1)
3: split $A$ into $A_0$ and $A_1$ \hspace{1cm} \mathcal{O}(n)
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$
8: \textbf{return} $Z_2 \cdot 2^n + Z_1 \cdot 2^{n \frac{n}{2}} + Z_0$

A more precise (correct) analysis would say that computing $Z_1$ needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$. 
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

\[
Z_1 = A_1B_0 + A_0B_1 = Z_2 = Z_0
= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0
\]

Hence,

\[
\begin{align*}
Z_2 \leftarrow \text{mult}(A_1, B_1) \\
Z_0 \leftarrow \text{mult}(A_0, B_0) \\
Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0 \\
\text{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{n \frac{1}{2}} + Z_0
\end{align*}
\]

A more precise (correct) analysis would say that computing $Z_1$ needs time $T\left(\frac{n}{2} + 1\right) + O(n)$. 

Algorithm 4 \text{mult}(A, B)

1: if $|A| = |B| = 1$ then \hspace{1cm} $O(1)$
2: return $a_0 \cdot b_0$ \hspace{1cm} $O(1)$
3: split $A$ into $A_0$ and $A_1$ \hspace{1cm} $O(n)$
4: split $B$ into $B_0$ and $B_1$ \hspace{1cm} $O(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
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$$Z_1 = A_1B_0 + A_0B_1$$

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Hence,

```
Algorithm 4 mult(A, B)
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2:     return $a_0 \cdot b_0$
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```

A more precise (correct) analysis would say that computing $Z_1$ needs time $T\left(\frac{n}{2} + 1\right) + \mathcal{O}(n)$. 

$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T\left(\frac{n}{2}\right)$
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1B_0 + A_0B_1 = Z_2 = Z_0$$

$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0$$

Hence,

**Algorithm 4** \texttt{mult}(A, B)

1: \textbf{if} $|A| = |B| = 1$ \textbf{then} $O(1)$
2: \quad \textbf{return} $a_0 \cdot b_0$ \quad $O(1)$
3: split $A$ into $A_0$ and $A_1$ $O(n)$
4: split $B$ into $B_0$ and $B_1$ $O(n)$
5: $Z_2 \leftarrow \texttt{mult}(A_1, B_1)$ $T\left(\frac{n}{2}\right)$
6: $Z_0 \leftarrow \texttt{mult}(A_0, B_0)$ $T\left(\frac{n}{2}\right)$
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A more precise (correct) analysis would say that computing $Z_1$ needs time $T\left(\frac{n}{2} + 1\right) + \mathcal{O}(n)$.
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1B_0 + A_0B_1 = Z_2 = Z_0$$

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Hence,

$$Z_1 = A_1B_0 + A_0B_1$$

$$= Z_2 = Z_0$$

$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0$$

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Algorithm 4 \text{mult}(A, B)

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2: \hspace{1cm} \text{return } a_0 \cdot b_0
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8: \text{return } Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0

O(1) \hspace{1cm} O(1)

O(n) \hspace{1cm} O(n)

T\left(\frac{n}{2}\right) \hspace{1cm} T\left(\frac{n}{2}\right)

T\left(\frac{n}{2}\right) + O(n)
Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$Z_1 = A_1 B_0 + A_0 B_1 = Z_2 = Z_0$$

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6: $Z_0 \leftarrow \texttt{mult}(A_0, B_0)$ \hspace{1cm} $T(n/2)$
7: $Z_1 \leftarrow \texttt{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$ \hspace{1cm} $T(n/2) + O(n)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{n/2} + Z_0$ \hspace{1cm} $O(n)$

A more precise (correct) analysis would say that computing $Z_1$ needs time $T(n/2 + 1) + O(n)$. 

6.2 Master Theorem
Example: Multiplying Two Integers

We get the following recurrence:

\[ T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n) . \]

Master Theorem: Recurrence: \( T[n] = aT\left(\frac{n}{b}\right) + f(n) . \)

- Case 1: \( f(n) = \Theta(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
- Case 2: \( f(n) = \Theta(n^{\log_b a \log k n}) \) \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
- Case 3: \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) \( T(n) = \Theta(f(n)) \)

Again we are in Case 1. We get a running time of \( \Theta(n^{\log_2 3}) \approx \Theta(n^{1.59}) . \)

A huge improvement over the “school method”.

6.2 Master Theorem
Example: Multiplying Two Integers

We get the following recurrence:

\[ T(n) = 3T\left( \frac{n}{2} \right) + O(n) . \]

Master Theorem: Recurrence: \( T[n] = aT\left( \frac{n}{b} \right) + f(n) \).

- **Case 1:** \( f(n) = O(n^{\log_b a - \epsilon}) \) \( T(n) = \Theta(n^{\log_b a}) \)
- **Case 2:** \( f(n) = \Theta(n^{\log_b a \log^k n}) \) \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
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Again we are in Case 1. We get a running time of \( \Theta(n^{\log_2 3}) \approx \Theta(n^{1.59}) \).

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6.3 The Characteristic Polynomial

Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

This is the general form of a linear recurrence relation of order \( k \) with constant coefficients \((c_0, c_k \neq 0)\).

\( T(n) \) only depends on the \( k \) preceding values. This means the recurrence relation is of order \( k \).

The recurrence is linear as there are no products of \( T(n) \)’s.

If \( f(n) = 0 \) then the recurrence relation becomes a linear, homogenous recurrence relation of order \( k \).

Note that we ignore boundary conditions for the moment.
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Consider the recurrence relation:

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This is the general form of a linear recurrence relation of order \( k \) with constant coefficients \((c_0, c_k \neq 0)\).

- \( T(n) \) only depends on the \( k \) preceding values. This means the recurrence relation is of order \( k \).
- The recurrence is linear as there are no products of \( T[n] \)’s.
- If \( f(n) = 0 \) then the recurrence relation becomes a linear, homogenous recurrence relation of order \( k \).

Note that we ignore boundary conditions for the moment.
6.3 The Characteristic Polynomial

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Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any $k$ consecutive values completely determine the solution.
- $k$ non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
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The Homogenous Case

The solution space

\[ S = \left\{ \mathcal{T} = T[1], T[2], T[3], \ldots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\} \]

is a vector space. This means that if \( \mathcal{T}_1, \mathcal{T}_2 \in S \), then also \( \alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S \), for arbitrary constants \( \alpha, \beta \).

How do we find a non-trivial solution?

We guess that the solution is of the form \( \lambda^n, \lambda \neq 0 \), and see what happens. In order for this guess to fulfill the recurrence we need

\[ c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0 \]

for all \( n \geq k \).
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The Homogenous Case

Dividing by $\lambda^{n-k}$ gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if $\lambda_i$ is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the $k$ (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n$$

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6.3 The Characteristic Polynomial 11. Apr. 2018
Ernst Mayr, Harald Räcke
The Homogenous Case

Lemma 2
Assume that the characteristic polynomial has $k$ distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n.$$

Proof.
There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.
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Assume that the characteristic polynomial has \( k \) distinct roots \( \lambda_1, \ldots, \lambda_k \). Then all solutions to the recurrence relation are of the form

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Proof (cont.).
Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:
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Proof (cont.).
Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:

$$\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \cdots + \alpha_k \cdot \lambda_k = T[1]$$
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Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i$'s such that these conditions are met:

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$$\alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \cdots + \alpha_k \cdot \lambda_k^2 = T[2]$$
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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:

\[
\begin{align*}
\alpha_1 \cdot \lambda_1 & \quad + \quad \alpha_2 \cdot \lambda_2 & \quad + \quad \cdots \quad + \quad \alpha_k \cdot \lambda_k & \quad = \quad T[1] \\
\alpha_1 \cdot \lambda_1^2 & \quad + \quad \alpha_2 \cdot \lambda_2^2 & \quad + \quad \cdots \quad + \quad \alpha_k \cdot \lambda_k^2 & \quad = \quad T[2] \\
& \quad \vdots
\end{align*}
\]
The Homogenous Case

Proof (cont.).

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\alpha_1 \cdot \lambda_1^2 & + \alpha_2 \cdot \lambda_2^2 + \cdots + \alpha_k \cdot \lambda_k^2 = T[2] \\
\vdots & \\
\alpha_1 \cdot \lambda_1^k & + \alpha_2 \cdot \lambda_2^k + \cdots + \alpha_k \cdot \lambda_k^k = T[k]
\end{align*}
\]
The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_{i}'s$ such that these conditions are met:

$$
\begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_k \\
\lambda_2^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\
\vdots & & & \\
\lambda_k^1 & \lambda_k^2 & \cdots & \lambda_k^k
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{pmatrix}
= 
\begin{pmatrix}
T[1] \\
T[2] \\
\vdots \\
T[k]
\end{pmatrix}
$$

We show that the column vectors are linearly independent. Then the above equation has a solution.
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\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\
\vdots & \ddots & \ddots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{pmatrix}
=
\begin{pmatrix}
T[1] \\
T[2] \\
\vdots \\
T[k]
\end{pmatrix}
$$

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Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k
\end{vmatrix} =
\]
Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot 
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \\
\end{vmatrix}
\]
Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix}
= \prod_{i=1}^{k} \lambda_i 
\]

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \\
\end{vmatrix}
= \prod_{i=1}^{k} \lambda_i 
\]

\[
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_{1}^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \cdots & \lambda_{2}^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_{k}^{k-2} & \lambda_k^{k-1} \\
\end{vmatrix}
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Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 & \ldots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \ldots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \ldots & \lambda_k^{k-2} & \lambda_k^{k-1}
\end{vmatrix}
= 
\]
Computing the Determinant

<table>
<thead>
<tr>
<th>1</th>
<th>λ₁</th>
<th>...</th>
<th>λ₁^{k-2}</th>
<th>λ₁^{k-1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>λ₂</td>
<td>...</td>
<td>λ₂^{k-2}</td>
<td>λ₂^{k-1}</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>λₖ</td>
<td>...</td>
<td>λₖ^{k-2}</td>
<td>λₖ^{k-1}</td>
</tr>
</tbody>
</table>

= 

<table>
<thead>
<tr>
<th>1</th>
<th>λ₁ - λ₁ · 1</th>
<th>...</th>
<th>λ₁^{k-2} - λ₁ · λ₁^{k-3}</th>
<th>λ₁^{k-1} - λ₁ · λ₁^{k-2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>λ₂ - λ₁ · 1</td>
<td>...</td>
<td>λ₂^{k-2} - λ₁ · λ₂^{k-3}</td>
<td>λ₂^{k-1} - λ₁ · λ₂^{k-2}</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>λₖ - λ₁ · 1</td>
<td>...</td>
<td>λₖ^{k-2} - λ₁ · λₖ^{k-3}</td>
<td>λₖ^{k-1} - λ₁ · λₖ^{k-2}</td>
</tr>
</tbody>
</table>
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix} =
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix} =

\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix}
\]
## Computing the Determinant

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
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1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix} = 
\]
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\[
\begin{vmatrix}
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1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix} =
\prod_{i=2}^{k} (\lambda_i - \lambda_1) \cdot
\begin{vmatrix}
1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2}
\end{vmatrix}
\]
Computing the Determinant

Repeating the above steps gives:

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell}^{k} (\lambda_i - \lambda_\ell)
\]

Hence, if all \(\lambda_i\)'s are different, then the determinant is non-zero.
The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root $\lambda_i$ with multiplicity (Vielfachheit) at least 2. Then not only is $\lambda^n_i$ a solution to the recurrence but also $n\lambda^n_i$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since $\lambda_i$ is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$.

Calculating the derivative gives a polynomial that still has root $\lambda_i$. 
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This means

\[ c_0 n \lambda_i^{n-1} + c_1 (n - 1) \lambda_i^{n-2} + \cdots + c_k (n - k) \lambda_i^{n-k-1} = 0 \]

Hence,

\[ \underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n - 1) \lambda_i^{n-1}}_{T[n-1]} + \cdots + \underbrace{c_k (n - k) \lambda_i^{n-k}}_{T[n-k]} = 0 \]
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Suppose $\lambda_i$ has multiplicity $j$. We know that

\[ c_0 n \lambda_i^n + c_1 (n - 1) \lambda_i^{n-1} + \cdots + c_k (n - k) \lambda_i^{n-k} = 0 \]

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

\[ c_0 n^2 \lambda_i^n + c_1 (n - 1)^2 \lambda_i^{n-1} + \cdots + c_k (n - k)^2 \lambda_i^{n-k} = 0 \]

We can continue $j - 1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$. 
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We can continue $j - 1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$. 
The Homogeneous Case

Lemma 3

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n - 1] + \cdots + c_kT[n - k] = 0$$

Let $\lambda_i, \ i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities $\ell_i$. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$’s is a solution to the recurrence.
Example: Fibonacci Sequence

\[ T[0] = 0 \]
\[ T[1] = 1 \]
\[ T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2 \]

The characteristic polynomial is

\[ \lambda^2 - \lambda - 1 \]

Finding the roots, gives

\[ \lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \left(1 \pm \sqrt{5}\right) \]
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Hence, the solution is of the form

\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
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\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}} \]
Example: Fibonacci Sequence

Hence, the solution is

\[
\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]
The Inhomogeneous Case

Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

with \( f(n) \neq 0 \).

While we have a fairly general technique for solving \textbf{homogeneous}, linear recurrence relations the inhomogeneous case is different.
The Inhomogeneous Case

The general solution of the recurrence relation is

\[ T(n) = T_h(n) + T_p(n) , \]

where \( T_h \) is any solution to the homogeneous equation, and \( T_p \) is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.
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The Inhomogeneous Case

Example:

\[ T[n] = T[n - 1] + 1 \quad T[0] = 1 \]

Then,

\[ T[n - 1] = T[n - 2] + 1 \quad (n \geq 2) \]

Subtracting the first from the second equation gives,

\[ T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2) \]

or

\[ T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2) \]

I get a completely determined recurrence if I add \( T[0] = 1 \) and \( T[1] = 2 \).
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Example: Characteristic polynomial:

\[ \lambda^2 - 2\lambda + 1 = 0 \]
The Inhomogeneous Case

Example: Characteristic polynomial:

\[
\lambda^2 - 2\lambda + 1 = 0
\]

Then the solution is of the form

\[
T[n] = \alpha_1 n + \beta n
\]

\[
T[0] = 1 \text{ gives } \alpha = 1.
\]

\[
T[1] = 2 \text{ gives } 1 + \beta = 2 \implies \beta = 1.
\]
The Inhomogeneous Case

**Example:** Characteristic polynomial:

\[
\lambda^2 - 2\lambda + 1 = 0
\]

\[
(\lambda-1)^2
\]

Then the solution is of the form

\[
T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n
\]
The Inhomogeneous Case

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**Example:** Characteristic polynomial:

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\frac{\lambda^2 - 2\lambda + 1}{(\lambda-1)^2} = 0
\]

Then the solution is of the form

\[
T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n
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\[T[0] = 1 \text{ gives } \alpha = 1.\]

\[T[1] = 2 \text{ gives } 1 + \beta = 2 \implies \beta = 1.\]
The Inhomogeneous Case

If $f(n)$ is a polynomial of degree $r$ this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

Difference:


$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$
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difference:

\[
\]

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T[n] = 2T[n - 1] - T[n - 2] + 2n - 1
\]
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Shift:

\[ T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \]
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**Shift:**

\[ T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \]
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\[ - 2T[n - 2] + T[n - 3] - 2n + 3 \]

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and so on...
Definition 4 (Generating Function)

Let \((a_n)_{n \geq 0}\) be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

\[
F(z) := \sum_{n \geq 0} a_n z^n;
\]

- exponential generating function (exponentielle Erzeugendenfunktion) is

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Example 5

1. The generating function of the sequence \((1, 0, 0, \ldots)\) is

\[ F(z) = 1. \]

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6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

- **Equality:** $f$ and $g$ are equal if $a_n = b_n$ for all $n$.
- **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.
- **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.
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There are no convergence issues here.
There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

- Equality: $f$ and $g$ are equal if $a_n = b_n$ for all $n$.
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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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What does $\sum_{n \geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

$$(1 - z) \cdot \left( \sum_{n \geq 0} z^n \right) = 1.$$ 

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Hence, the generating function of the sequence \( a_n = n + 1 \)
is \( 1/(1 - z)^2 \).
6.4 Generating Functions

We can repeat this
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Computing the $k$-th derivative of $\sum z^n$. 

\[ \sum_{n \geq k} n(n-1) \cdots (n-k+1) z^n = \sum_{n \geq 0} (n+k) \cdots k z^n = \frac{k!}{(1-z)^{k+1}}. \]

Hence:

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\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n + 1) z^n - \sum_{n \geq 0} z^n
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The generating function of the sequence $a_n = n$ is $z \frac{1}{(1-z)^2} - \frac{1}{1-z}$. 

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The generating function of the sequence \( a_n = n \) is \( \frac{z}{(1 - z)^2} \).
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We know

\[ \sum_{n \geq 0} y^n = \frac{1}{1 - y} \]

Hence,

\[ \sum_{n \geq 0} a^n z^n = \frac{1}{1 - az} \]

The generating function of the sequence \( f_n = a^n \) is \( \frac{1}{1 - az} \).
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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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Hence, \( a_n = n + 1. \)
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<th>( \text{generating function} )</th>
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6.4 Generating Functions

11. Apr. 2018

Ernst Mayr, Harald Räcke
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</tr>
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<td>$f_{n-k}$ ($n \geq k$); 0 otrw.</td>
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Some Generating Functions

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<th>$n$-th sequence element</th>
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</tr>
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<tbody>
<tr>
<td>$c f_n$</td>
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</tr>
<tr>
<td>$f_n + g_n$</td>
<td>$F + G$</td>
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Solving Recursions with Generating Functions

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2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

6.4 Generating Functions 11. Apr. 2018
Ernst Mayr, Harald Räcke
Solving Recursions with Generating Functions

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3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

Techniques:
- partial fraction decomposition (Partialbruchzerlegung)
- lookup in tables

4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

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6. The coefficients of the resulting power series are the $a_n$. 
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$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1}) z^n$$
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$$= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1}$$
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A(z) = \sum_{n \geq 0} a_n z^n = a_0 + \sum_{n \geq 1} a_n z^n = 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n
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$$= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n$$

$$= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n$$
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$$= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n$$

$$= 1 + 3zA(z) + \frac{z}{(1 - z)^2}$$
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**Example:** \( a_n = 3a_{n-1} + n, \ a_0 = 1 \)

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$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \equiv \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$
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\[
z^2 - z + 1 = A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z)
\]

\[
= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z)
\]

\[
= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C)
\]
Example: \( a_n = 3a_{n-1} + n, \; a_0 = 1 \)

5. Write \( f(z) \) as a formal power series:

This leads to the following conditions:

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A + B + C = 1 \\
2A + 4B + 3C = 1 \\
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5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$
Example: \( a_n = 3a_{n-1} + n, \; a_0 = 1 \)

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Example: $a_n = 3a_{n-1} + n, \ a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$
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\[
= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n + 1) z^n
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$$= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2} (n + 1) \right) z^n$$
Example: \( a_n = 3a_{n-1} + n, \ a_0 = 1 \)

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\[
= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4} \right) z^n
\]
Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = 7 \cdot \frac{1}{4} \cdot \frac{1}{1 - 3z} - \frac{1}{4} \cdot \frac{1}{1 - z} - \frac{1}{2} \cdot \frac{1}{(1 - z)^2}$$

$$= 7 \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n + 1) z^n$$

$$= \sum_{n\geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2} (n + 1) \right) z^n$$

$$= \sum_{n\geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4} \right) z^n$$

6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$. 
Example 6

\[ f_0 = 1 \]
\[ f_1 = 2 \]
\[ f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2. \]
6.5 Transformation of the Recurrence

Example 6

\[ f_0 = 1 \]
\[ f_1 = 2 \]
\[ f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2. \]

Define

\[ g_n := \log f_n. \]
Example 6

\[
\begin{align*}
  f_0 &= 1 \\
  f_1 &= 2 \\
  f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2.
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\[ f_n = 2^{F_n} \]
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Example 7

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\begin{align*}
f_1 &= 1 \\
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Example 7

\[
f_1 = 1 \\
f_n = 3f_{n/2} + n; \text{ for } n = 2^k, k \geq 1 ;
\]

Define

\[
g_k := f_{2^k}.
\]

Then:

\[
g_0 = 1 \\
g_k = 3g_{k-1} + 2^k, \quad k \geq 1
\]
We get

\[ g_k = 3 \left[ g_{k-1} \right] + 2^k \]
6 Recurrences

We get

\[ g_k = 3 [g_{k-1}] + 2^k \]
\[ = 3 \left[ 3g_{k-2} + 2^{k-1} \right] + 2^k \]
6 Recurrences

We get

\[ g_k = 3 \left[ g_{k-1} \right] + 2^k \]
\[ = 3 \left[ 3g_{k-2} + 2^{k-1} \right] + 2^k \]
\[ = 3^2 \left[ g_{k-2} \right] + 32^{k-1} + 2^k \]
6 Recurrences

We get

\[ g_k = 3\left[ g_{k-1} \right] + 2^k \]

\[ = 3 \left[ 3g_{k-2} + 2^{k-1} \right] + 2^k \]

\[ = 3^2 \left[ g_{k-2} \right] + 32^{k-1} + 2^k \]

\[ = 3^2 \left[ 3g_{k-3} + 2^{k-2} \right] + 32^{k-1} + 2^k \]
6 Recurrences

We get

\[ g_k = 3 \left[ g_{k-1} \right] + 2^k \]
\[ = 3 \left[ 3g_{k-2} + 2^{k-1} \right] + 2^k \]
\[ = 3^2 \left[ g_{k-2} \right] + 32^{k-1} + 2^k \]
\[ = 3^2 \left[ 3g_{k-3} + 2^{k-2} \right] + 32^{k-1} + 2^k \]
\[ = 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \]
6 Recurrences

We get

\[ g_k = 3 [g_{k-1}] + 2^k \]

\[ = 3 [3g_{k-2} + 2^{k-1}] + 2^k \]

\[ = 3^2 [g_{k-2}] + 32^{k-1} + 2^k \]

\[ = 3^2 [3g_{k-3} + 2^{k-2}] + 32^{k-1} + 2^k \]

\[ = 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \]

\[ = 2^k \cdot \sum_{i=0}^{k} \left( \frac{3}{2} \right)^i \]
6 Recurrences

We get

\[ g_k = 3 \left[ g_{k-1} \right] + 2^k \]
\[ = 3 \left[ 3g_{k-2} + 2^{k-1} \right] + 2^k \]
\[ = 3^2 \left[ g_{k-2} \right] + 32^{k-1} + 2^k \]
\[ = 3^2 \left[ 3g_{k-3} + 2^{k-2} \right] + 32^{k-1} + 2^k \]
\[ = 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \]
\[ = 2^k \cdot \sum_{i=0}^{k} \left( \frac{3}{2} \right)^i \]
\[ = 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} \]
We get

\[ g_k = 3 \left[ g_{k-1} \right] + 2^k \]

\[ = 3 \left[ 3g_{k-2} + 2^{k-1} \right] + 2^k \]

\[ = 3^2 \left[ g_{k-2} \right] + 32^{k-1} + 2^k \]

\[ = 3^2 \left[ 3g_{k-3} + 2^{k-2} \right] + 32^{k-1} + 2^k \]

\[ = 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \]

\[ = 2^k \cdot \sum_{i=0}^{k} \left( \frac{3}{2} \right)^i \]

\[ = 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1} \]
Let \( n = 2^k \):

\[
\begin{align*}
g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\
f_n &= 3 \cdot 3^k - 2 \cdot 2^k
\end{align*}
\]
6 Recurrences

Let $n = 2^k$:

$g_k = 3^{k+1} - 2^{k+1}$, hence

$f_n = 3 \cdot 3^k - 2 \cdot 2^k$

$= 3 (2 \log_3)^k - 2 \cdot 2^k$
Let $n = 2^k$:

\[ g_k = 3^{k+1} - 2^{k+1}, \text{ hence} \]
\[ f_n = 3 \cdot 3^k - 2 \cdot 2^k \]
\[ = 3 \left(2^{\log_3 3}\right)^k - 2 \cdot 2^k \]
\[ = 3(2^k)^{\log_3 3} - 2 \cdot 2^k \]
Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence }$$

$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

$$= 3(2^{\log_3^k}) - 2 \cdot 2^k$$

$$= 3(2^k)^{\log_3} - 2 \cdot 2^k$$

$$= 3n^{\log_3} - 2n.$$