9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- $\mathcal{P}.\text{makeset}(x)$: Given an element $x$, adds $x$ to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for $x$ in the data-structure.

- $\mathcal{P}.\text{find}(x)$: Given a handle for an element $x$; find the set that contains $x$. Returns a representative/identifier for this set.

- $\mathcal{P}.\text{union}(x, y)$: Given two elements $x$, and $y$ that are currently in sets $S_x$ and $S_y$, respectively, the function replaces $S_x$ and $S_y$ by $S_x \cup S_y$ and returns an identifier for the new set.
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Applications:

▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

▶ Kruskals Minimum Spanning Tree Algorithm
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Algorithm 16 Kruskal-MST \(G = (V, E), w\)

1: \(A \leftarrow \emptyset\);
2: for all \(v \in V\) do
3: \(v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})\)
4: sort edges in non-decreasing order of weight \(w\)
5: for all \((u, v) \in E\) in non-decreasing order do
6: if \(\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})\) then
7: \(A \leftarrow A \cup \{(u, v)\}\)
8: \(\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})\)
List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.
- `makeset(x)` can be performed in constant time.
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union($x, y$)

- Determine sets $S_x$ and $S_y$.
- Traverse the smaller list (say $S_y$), and change all backward pointers to the head of list $S_x$.
- Insert list $S_y$ at the head of $S_x$.
- Adjust the size-field of list $S_x$.
- Time: $\min\{|S_x|, |S_y|\}$. 
List Implementation

**union**(*x*, *y*)

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List Implementation

9 Union Find

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List Implementation

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Running times:

- \texttt{find}(x): constant
- \texttt{makeset}(x): constant
- \texttt{union}(x, y): \Theta(n), where \( n \) denotes the number of elements contained in the set system.
Lemma 1

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x) : \Theta(1)$.
- $\text{makeset}(x) : \Theta(\log n)$.
- $\text{union}(x, y) : \Theta(1)$.
The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
  - Initially the balance on all accounts is zero.
  - Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
  - Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
  - If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.
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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the **excess** to the elements involved.
- In total we will charge at most $\Theta(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.
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For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.

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List Implementation

makeset(x): The actual cost is $\Theta(1)$. Due to the cost inflation the amortized cost is $\Theta(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\Theta(1)$.

union(x, y):
- If $S_x = S_y$ the cost is constant; no bank accounts change.
- Otherwise the actual cost is $O(\min\{|S_x|, |S_y|\})$.
- Assume w.l.o.g. that $S_x$ is the smaller set; let $c$ denote the hidden constant, i.e., the actual cost is at most $c |S_x|$.
- Charge $c$ to every element in set $S_x$. 

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List Implementation

**make(x)**: The actual cost is $O(1)$. Due to the cost inflation the amortized cost is $O(\log n)$.

**find(x)**: For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $O(1)$.

**union(x, y)**:
- If $S_x = S_y$ the cost is constant; no bank accounts change.
- Otherwise, the actual cost is $O(\min\{|S_x|, |S_y|\})$.
- Assume wlog. that $S_x$ is the smaller set; let $c$ denote the hidden constant, i.e., the actual cost is at most $c|S_x|$.
- Charge $c$ to every element in set $S_x$. 
List Implementation

**makeset**(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

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- Assume w.l.o.g. that $S_x$ is the smaller set; let $c$ denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.
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List Implementation

Lemma 2

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where $n$ is the total number of elements in the set system.

Proof.

Whenever an element $x$ is charged the number of elements in $x$’s set doubles. This can happen at most $\lfloor \log n \rfloor$ times. \qed
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Implementation via Trees

- Maintain nodes of a set in a tree.
- The root of the tree is the label of the set.
- Only pointer to parent exists; we cannot list all elements of a given set.

Example:

Set system \{2, 5, 10, 12\}, \{3, 6, 7, 8, 9, 14, 17\}, \{16, 19, 23\}. 
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\textbf{makeset}(x)

- Create a singleton tree. Return pointer to the root.
- Time: $O(1)$.

\textbf{find}(x)

- Start at element $x$ in the tree. Go upwards until you reach the root.
- Time: $O(\text{level}(x))$, where $\text{level}(x)$ is the distance of element $x$ to the root in its tree. Not constant.
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To support union we store the size of a tree in its root.
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union\((x, y)\)

- Perform \(a \leftarrow \text{find}(x); b \leftarrow \text{find}(y)\). Then: \(\text{link}(a, b)\).
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\[ \text{union}(x, y) \]

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- Time: constant for link\((a, b)\) plus two find-operations.
Lemma 3

The running time (non-amortized!!!) for \texttt{find}(x) is $O(\log n)$.

Proof.

When we attach a tree with root $c$ to become a child of a tree with root $p$, then $\text{size}(p) \geq 2 \times \text{size}(c)$, where $\text{size}$ denotes the value of the size-field right after the operation.

After that the value of $\text{size}(c)$ stays fixed, while the value of $\text{size}(p)$ may still increase.

Hence, at any point in time a tree fulfills $\text{size}(p) \geq 2 \times \text{size}(c)$, for any pair of nodes $(p,c)$, where $p$ is a parent of $c$. 

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Path Compression

\textbf{find}(x):

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
- Speeds up successive find-operations.

Note that the size-fields now only give an upper bound on the size of a sub-tree.
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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\Theta(\log n)$. 
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Amortized Analysis

Definitions:

- **size** \( (v) \) is the number of nodes that were in the sub-tree rooted at \( v \) when \( v \) became the child of another node (or the number of nodes if \( v \) is the root).

Note that this is the same as the size of \( v \)'s subtree in the case that there are no find-operations.

- **rank** \( (v) \) = \[ \lfloor \log \left( \text{size} (v) \right) \rfloor \).

- \( \implies \) \( \text{size} (v) \geq 2^{\text{rank} (v)} \).

**Lemma 4**

*The rank of a parent must be strictly larger than the rank of a child.*
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Lemma 5

There are at most $n/2^s$ nodes of rank $s$.

Proof.

Let’s say a node $v$ sees node $x$ if $v$ is in $x$’s sub-tree at the time that $v$ becomes a child.

A node $v$ sees at most one node of rank $s$ during the running time of the algorithm.

This holds because the rank-sequence of the roots of the different trees that contain $v$ during the running time of the algorithm is a strictly increasing sequence.

Hence, every node sees at most one rank $s$ node, but every rank $s$ node is seen by at least $2^s$ different nodes.
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We define

\[ \text{tow}(i) := \begin{cases} 
1 & \text{if } i = 0 \\
2 \text{tow}(i-1) & \text{otherwise} 
\end{cases} \]

Theorem 6
Union find with path compression fulfills the following amortized running times:

- \( \text{makeset}(x) : \mathcal{O}(\log^\ast(n)) \)
- \( \text{find}(x) : \mathcal{O}(\log^\ast(n)) \)
- \( \text{union}(x,y) : \mathcal{O}(\log^\ast(n)) \)
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\[
tow(i) = 2^{2^{2^{2^i}}} \text{ } i \text{ times}
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and

\[ \log^*(n) := \min\{i \mid \text{tow}(i) \geq n\} . \]
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Theorem 6

Union find with path compression fulfills the following amortized running times:

- \(\text{makeset}(x) : \Theta(\log^*(n))\)
- \(\text{find}(x) : \Theta(\log^*(n))\)
- \(\text{union}(x, y) : \Theta(\log^*(n))\)
Amortized Analysis

In the following we assume \( n \geq 2 \).

**rank-group:**

- A node with rank \( \text{rank}(v) \) is in rank group \( \log^*(\text{rank}(v)) \).
- The rank group \( g = 0 \) contains only nodes with rank 0 or rank 1.
- A rank group \( g \geq 1 \) contains ranks \( \text{rank}(v) \) for every \( v \in [g] \).
- The maximum non-empty rank group is \( \log^*(\log n) \) (which holds for \( n \geq 2 \)).

Hence, the total number of rank groups is at most \( \log^* n \).
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- Hence, the total number of rank-groups is at most $\log^* n$. 
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Amortized Analysis

Accounting Scheme:

create an account for every find-operation
create an account for every node $v$

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from $v$ to $parent[v]$ as follows:

If $parent[v]$ is the root we charge the cost to the find-account.

If the group-number of rank(rank($v$)) is the same as that of rank(rank($parent[v]$)) (before starting path compression) we charge the cost to the node-account of $v$.

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Observations:

A find account is charged at most \( \log^* n \) times (once for the root and at most \( \log^* n - 1 \) times when increasing the rank-group).

After a node \( v \) is charged its parent edge is re-assigned. The rank of the parent strictly increases.

After some charges to \( v \), the parent will be in a larger rank-group, so \( v \) will never be charged again.

The total charge made to a node in rank-group \( g \) is at most \( t_{ow}(g) - t_{ow}(g - 1) - 1 \leq t_{ow}(g) \).
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What is the total charge made to nodes?

The total charge is at most

\[ \sum_{g} n(g) \cdot \text{tow}(g), \]

where \( n(g) \) is the number of nodes in group \( g \).
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Hence,

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Hence,

\[
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Hence,

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\sum_{g} n(g) tow(g) \leq n(0) tow(0) + \sum_{g \geq 1} n(g) tow(g) \leq n \log^*(n)
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Amortized Analysis

Without loss of generality we can assume that all \texttt{makeset}-operations occur at the start.

This means if we inflate the cost of \texttt{makeset} to $\log^* n$ and add this to the node account of $v$ then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).
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Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\Theta(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of $m$ operations on at most $n$ elements).

There is also a lower bound of $\Omega(\alpha(m, n))$. 
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\[ A(x, y) = \begin{cases} 
  y + 1 & \text{if } x = 0 \\
  A(x - 1, 1) & \text{if } y = 0 \\
  A(x - 1, A(x, y - 1)) & \text{otw.}
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- \( A(0, y) = y + 1 \)
- \( A(1, y) = y + 2 \)
- \( A(2, y) = 2y + 3 \)
- \( A(3, y) = 2^{y+3} - 3 \)
- \( A(4, y) = 2^{2^{2^{y+3}}} - 3 \), \( y+3 \) times