9 Union Find

Union Find Data Structure $P$: Maintains a partition of disjoint sets over elements.

- $P. \text{makeset}(x)$: Given an element $x$, adds $x$ to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for $x$ in the data-structure.

- $P. \text{find}(x)$: Given a handle for an element $x$; find the set that contains $x$. Returns a representative/identifier for this set.

- $P. \text{union}(x, y)$: Given two elements $x$, and $y$ that are currently in sets $S_x$ and $S_y$, respectively, the function replaces $S_x$ and $S_y$ by $S_x \cup S_y$ and returns an identifier for the new set.
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Applications:

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- Kruskal's Minimum Spanning Tree Algorithm
9 Union Find

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▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
▶ Kruskal's Minimum Spanning Tree Algorithm
Algorithm 16 Kruskal-MST($G = (V, E), w$)

1: $A \leftarrow \emptyset$
2: for all $v \in V$ do
3: $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$
4: sort edges in non-decreasing order of weight $w$
5: for all $(u, v) \in E$ in non-decreasing order do
6: if $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$ then
7: $A \leftarrow A \cup \{(u, v)\}$
8: $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$
List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.
▶ The head of the list contains the identifier for the set and a field that stores the size of the set.
▶ \text{makeset}(x)\ can\ be\ performed\ in\ constant\ time.
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List Implementation

union(\(x, y\))

- Determine sets \(S_x\) and \(S_y\).
- Traverse the smaller list (say \(S_y\)), and change all backward pointers to the head of list \(S_x\).
- Insert list \(S_y\) at the head of \(S_x\).
- Adjust the size-field of list \(S_x\).
- Time: \(\min\{|S_x|, |S_y|\}\).
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9 Union Find
List Implementation
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Running times:

- \textbf{find}(x): constant
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- \textbf{union}(x, y): } O(n), where } n } denotes the number of elements contained in the set system.
List Implementation

Lemma 1

The list implementation for the ADT union find fulfills the following amortized time bounds:

- \( \text{find}(x) : \Theta(1) \).
- \( \text{makeset}(x) : \Theta(\log n) \).
- \( \text{union}(x, y) : \Theta(1) \).
The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
  - Initially the balance on all accounts is zero.
  - Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
  - Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
  - If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.
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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\Theta(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.
List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
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List Implementation

**makeset(x):** The actual cost is $O(1)$. Due to the cost inflation the amortized cost is $O(\log n)$.

**find(x):** For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $O(1)$.

**union(x, y):**
- If $x = y$, the cost is constant; no bank accounts change.
- Otherwise, the actual cost is $O(\min\{|S_x|, |S_y|\})$.
- Assume wlog. that $S_x$ is the smaller set; let $c$ denote the hidden constant, i.e., the actual cost is at most $c|S_x|$.
- Charge $c$ to every element in set $S_x$. 
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List Implementation

Lemma 2

An element is charged at most \( \lfloor \log_2 n \rfloor \) times, where \( n \) is the total number of elements in the set system.

Proof.
Whenever an element \( x \) is charged the number of elements in \( x \)'s set doubles. This can happen at most \( \lceil \log n \rceil \) times. \( \square \)
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Implementation via Trees

- Maintain nodes of a set in a tree.
- The root of the tree is the label of the set.
- Only pointer to parent exists; we cannot list all elements of a given set.

Example:

```
Set system {2, 5, 10, 12}, {3, 6, 7, 8, 9, 14, 17}, {16, 19, 23}.
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Example:

```
10
12
5
2
6
9
3
8
14 17
7
16
19 23
```

Set system \{2, 5, 10, 12\}, \{3, 6, 7, 8, 9, 14, 17\}, \{16, 19, 23\}.
Implementation via Trees

**makeset(x)**

- Create a singleton tree. Return pointer to the root.
- Time: $O(1)$.

**find(x)**

- Start at element $x$ in the tree. Go upwards until you reach the root.
- Time: $O(\text{level}(x))$, where $\text{level}(x)$ is the distance of element $x$ to the root in its tree. Not constant.
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Implementation via Trees

To support union we store the size of a tree in its root.

\[
\text{union}(x, y) \quad \text{\texttt{\langle perform find(x); find(y); link(a, b); \rangle}}
\]

\[
\text{\texttt{\langle link(a, b); \}}} \quad \text{attaches the smaller tree as the child of the larger.}
\]

\[
\text{\texttt{\langle in addition it updates the size-field of the new root.}}
\]

\[
\text{\texttt{\langle time: constant for link(a, b) plus two find-operations.}}
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\[ \text{union}(x, y) \]

- Perform \( a \leftarrow \text{find}(x); \ b \leftarrow \text{find}(y) \). Then: \( \text{link}(a, b) \).

Time: constant for \( \text{link}(a, b) \) plus two \( \text{find} \)-operations.
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Lemma 3

The running time (non-amortized!!!) for $\text{find}(x)$ is $O(\log n)$.

Proof.

When we attach a tree with root $c$ to become a child of a tree with root $p$, then $\text{size}(p) \geq 2 \times \text{size}(c)$, where $\text{size}$ denotes the value of the size field right after the operation.

After that the value of $\text{size}(c)$ stays fixed, while the value of $\text{size}(p)$ may still increase.

Hence, at any point in time a tree fulfills $\text{size}(p) \geq 2 \times \text{size}(c)$ for any pair of nodes $(p,c)$, where $p$ is a parent of $c$. 

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Path Compression

\textbf{find}(x):

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
- Speeds up successive find-operations.

Note that the size-fields now only give an upper bound on the size of a sub-tree.
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Path Compression

\textbf{find}(x):

\begin{itemize}
\item Go upward until you find the root.
\item Re-attach all visited nodes as children of the root.
\item Speeds up successive find-operations.
\end{itemize}

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\item Note that the size-fields now only give an upper bound on the size of a sub-tree.
\end{itemize}
Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $O(\log n)$. 
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Amortized Analysis

Definitions:

- \textit{size}(v) ñ the number of nodes that were in the sub-tree rooted at \(v\) when \(v\) became the child of another node (or the number of nodes if \(v\) is the root).
  
  Note that this is the same as the size of \(v\)'s subtree in the case that there are no find-operations.

- \textit{rank}(v) ñ \(\lfloor \log(\text{size}(v)) \rfloor\).
  
  \(\text{size}(v) \geq 2^{\text{rank}(v)}\).

Lemma 4

The rank of a parent must be strictly larger than the rank of a child.
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*The rank of a parent must be strictly larger than the rank of a child.*
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Lemma 5
There are at most $n/2^s$ nodes of rank $s$.

Proof.

Let’s say a node $v$ sees node $x$ if $v$ is in $x$’s sub-tree at the time that $x$ becomes a child.

A node $v$ sees at most one node of rank $s$ during the running time of the algorithm.

This holds because the rank-sequence of the roots of the different trees that contain $v$ during the running time of the algorithm is a strictly increasing sequence.

Hence, every node sees at most one rank $s$ node, but every rank $s$ node is seen by at least $2^s$ different nodes.
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Amortized Analysis

We define

\[
\text{tow}(i) := \begin{cases} 
1 & \text{if } i = 0 \\
2\text{tow}(i-1) & \text{otw.}
\end{cases}
\]
Amortized Analysis

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tow(i) := \begin{cases} 
1 & \text{if } i = 0 \\
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\[
tow(i) = 2^{2^{2^{2^{i}}}} \text{ } i \text{ times}
\]
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and

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**Theorem 6**

*Union find with path compression fulfills the following amortized running times:*

- \(\text{makeset}(x) : \Theta(\log^*(n))\)
- \(\text{find}(x) : \Theta(\log^*(n))\)
- \(\text{union}(x, y) : \Theta(\log^*(n))\)
Amortized Analysis

In the following we assume \( n \geq 2 \).

**rank-group:**

- A node with rank \( \text{rank}(v) \) is in rank group \( \log^*(\text{rank}(v)) \).
- The rank group \( g = 0 \) contains only nodes with rank 0 or rank 1.
- A rank group \( g \geq 1 \) contains ranks \( \text{rank} \text{ow}(g - 1) + 1, \ldots, \text{rank} \text{ow}(g) \).
- The maximum non-empty rank group is \( \log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1 \) (which holds for \( n \geq 2 \)).
- Hence, the total number of rank groups is at most \( \log^* n \).
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- Hence, the total number of rank-groups is at most $\log^* n$. 
Amortized Analysis

Accounting Scheme:

- create an account for every find-operation
- create an account for every node \( v \)

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from \( v \) to \( \text{parent}[v] \) as follows:

- If \( \text{parent}[v] \) is the root we charge the cost to the find-account.
- If the group-number of \( v \) is the same as that of \( \text{parent}[v] \) (before starting path compression) we charge the cost to the node-account of \( v \).
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Amortized Analysis

Observations:

A find account is charged at most \(\log^*(n)\) times (once for the root and at most \(\log^*(n) - 1\) times when increasing the rank-group).

After a node is charged its parent edge is re-assigned. The rank of the parent strictly increases.

After some charges to a node the parent will be in a larger rank-group. It will never be charged again.

The total charge made to a node in rank-group \(g\) is at most \(tow(g) - tow(g - 1) - 1 \leq tow(g)\).
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Amortized Analysis

What is the total charge made to nodes?

- The total charge is at most

\[ \sum_{g} n(g) \cdot \text{tow}(g) , \]

where \( n(g) \) is the number of nodes in group \( g \).
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where \( n(g) \) is the number of nodes in group \( g \).
Amortized Analysis

For \( g \geq 1 \) we have

\[
n(g) \leq \text{tow}(g) + 1
\]

Hence,

\[
\sum_{g} n(g) \text{tow}(g) \leq n(0) \text{tow}(0) + \sum_{g \geq 1} n(g) \text{tow}(g) \leq n \log^*(n)
\]
Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s}$$
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$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$
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\[ = \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} \]
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& = \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} \\
& = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2
\end{align*}
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Hence,

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For $g \geq 1$ we have

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Hence,

\[ \sum_{g} n(g) \cdot tow(g) \leq n(0) \cdot tow(0) + \sum_{g \geq 1} n(g) \cdot tow(g) \]
Amortized Analysis

For $g \geq 1$ we have

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Hence,

$$\sum_{g} n(g) \text{ tow}(g) \leq n(0) \text{ tow}(0) + \sum_{g \geq 1} n(g) \text{ tow}(g) \leq n \log^*(n)$$
Amortized Analysis

Without loss of generality we can assume that all \texttt{makeset}-operations occur at the start.

This means if we inflate the cost of \texttt{makeset} to $\log^* n$ and add this to the node account of \texttt{v} then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).
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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\Theta(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of $m$ operations on at most $n$ elements).

There is also a lower bound of $\Omega(\alpha(m, n))$. 
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Amortized Analysis

\[
A(x, y) = \begin{cases} 
  y + 1 & \text{if } x = 0 \\
  A(x - 1, 1) & \text{if } y = 0 \\
  A(x - 1, A(x, y - 1)) & \text{otherwise}
\end{cases}
\]

\[
\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}
\]

- \(A(0, y) = y + 1\)
- \(A(1, y) = y + 2\)
- \(A(2, y) = 2y + 3\)
- \(A(3, y) = 2^{y+3} - 3\)
- \(A(4, y) = 2^{2^{2^{2^{2^{y+3}}}}} - 3\) with \(y+3\) times
**Amortized Analysis**

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