# 8.2 Binomial Heaps

<table>
<thead>
<tr>
<th>Operation</th>
<th>Binary Heap</th>
<th>BST</th>
<th>Binomial Heap</th>
<th>Fibonacci Heap*</th>
</tr>
</thead>
<tbody>
<tr>
<td>build</td>
<td>$n$</td>
<td>$n \log n$</td>
<td>$n \log n$</td>
<td>$n$</td>
</tr>
<tr>
<td>minimum</td>
<td>$1$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$1$</td>
</tr>
<tr>
<td>is-empty</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>insert</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$1$</td>
</tr>
<tr>
<td>delete</td>
<td>$\log n^{**}$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
</tr>
<tr>
<td>delete-min</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
</tr>
<tr>
<td>decrease-key</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
<td>$1$</td>
</tr>
<tr>
<td>merge</td>
<td>$n$</td>
<td>$n \log n$</td>
<td><strong>$\log n$</strong></td>
<td>$1$</td>
</tr>
</tbody>
</table>
Binomial Trees

\[ B_0 \quad B_1 \quad B_2 \quad B_3 \quad B_4 \]

\[ B_{t-1} \quad B_t \]

8.2 Binomial Heaps
Binomial Trees

Properties of Binomial Trees

- $B_k$ has $2^k$ nodes.
- $B_k$ has height $k$.
- The root of $B_k$ has degree $k$.
- $B_k$ has $\binom{k}{\ell}$ nodes on level $\ell$.
- Deleting the root of $B_k$ gives trees $B_0, B_1, \ldots, B_{k-1}$.
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- $B_k$ has $\binom{k}{\ell}$ nodes on level $\ell$.
- Deleting the root of $B_k$ gives trees $B_0, B_1, \ldots, B_{k-1}$.
Deleting the root of $B_5$ leaves sub-trees $B_4$, $B_3$, $B_2$, $B_1$, and $B_0$. 
Deleting the leaf furthest from the root (in $B_5$) leaves a path that connects the roots of sub-trees $B_4$, $B_3$, $B_2$, $B_1$, and $B_0$. 
The number of nodes on level $\ell$ in tree $B_k$ is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$
The binomial tree $B_k$ is a sub-graph of the hypercube $H_k$.

The parent of a node with label $b_n, \ldots, b_1, b_0$ is obtained by setting the least significant 1-bit to 0.

The $\ell$-th level contains nodes that have $\ell$ 1’s in their label.
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8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a **circular linked list**.
- A child-pointer points to an arbitrary node within the list.
- A parent-pointer points to the parent node.
- Pointers \(x\.left\) and \(x\.right\) point to the left and right sibling of \(x\) (if \(x\) does not have siblings then \(x\.left = x\.right = x\)).
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8.2 Binomial Heaps

▶ Given a pointer to a node $x$ we can splice out the sub-tree rooted at $x$ in constant time.

▶ We can add a child-tree $T$ to a node $x$ in constant time if we are given a pointer to $x$ and a pointer to the root of $T$. 
In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees $B_0$, $B_1$, and $B_4$. 
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Given the number $n$ of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the $k_i$ are all distinct this means that the $k_i$ define the non-zero bit-positions in the binary representation of $n$. 
Binomial Heap: Merge

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Binomial Heap

Properties of a heap with \( n \) keys:

- Let \( n = b_d b_{d-1} \ldots b_0 \) denote binary representation of \( n \).
- The heap contains tree \( B_i \) iff \( b_i = 1 \).
- Hence, at most \( \lceil \log n \rceil + 1 \) trees.
- The minimum must be contained in one of the roots.
- The height of the largest tree is at most \( \lfloor \log n \rfloor \).
- The trees are stored in a single-linked list; ordered by dimension/size.
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5. The height of the largest tree is at most \( \lceil \log n \rceil \).
6. The trees are stored in a single-linked list; ordered by dimension/size.

Diagram of a binomial heap with keys 2, 12, 7, 13, 11, 8, 14, 47, 39, 42, 20, 35, 24, 29, 31, 37, 16, 13, 70, 92.
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Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.
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\( S_1. \text{merge}(S_2): \)

- Analogous to binary addition.
- Time is proportional to the number of trees in both heaps.
- Time: \( \Theta(\log n) \).
S.1. **merge**($S_2$):

- Analogous to binary addition.
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8.2 Binomial Heaps

\( S_1. \text{merge}(S_2): \)

- Analogous to binary addition.
- Time is proportional to the number of trees in both heaps.
- Time: \( \mathcal{O}(\log n) \).
8.2 Binomial Heaps

All other operations can be reduced to merge().

$S.\text{insert}(x)$:

- Create a new heap $S'$ that contains just the element $x$.
- Execute $S.\text{merge}(S')$.
- Time: $\Theta(\log n)$. 
All other operations can be reduced to `merge()`.

**S. insert(x):**

- Create a new heap `S'` that contains just the element `x`.
- Execute `S.merge(S')`.
- Time: $\Theta(\log n)$.
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\textbf{S. insert}(x):

- Create a new heap \( S' \) that contains just the element \( x \).
- Execute \( S. \text{merge}(S') \).
- Time: \( \Theta(\log n) \).
8.2 Binomial Heaps

\textbf{S. minimum():}

- Find the minimum key-value among all roots.
- Time: $\Theta(\log n)$. 
8.2 Binomial Heaps

**S. delete-min():**

- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\text{min}}$ from the heap.
- Create a new heap $S'$ that contains the trees obtained from $T_{\text{min}}$ after deleting the root (note that these are just $O(\log n)$ trees).
- Compute $S.\text{merge}(S')$.
- Time: $O(\log n)$. 

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Ernst Mayr, Harald Räcke

11. Apr. 2018
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Ernst Mayr, Harald Räcke
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S. decrease-key(handle h):

- Decrease the key of the element pointed to by $h$.
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: $\Theta(\log n)$ since the trees have height $\Theta(\log n)$. 
8.2 Binomial Heaps

\textbf{S. decrease-key(handle }h\textbf{):}

\begin{itemize}
\item Decrease the key of the element pointed to by \textit{h}.
\item Bubble the element up in the tree until the heap property is fulfilled.
\item Time: $\Theta(\log n)$ since the trees have height $\Theta(\log n)$.
\end{itemize}
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- Time: \( \Theta(\log n) \) since the trees have height \( \Theta(\log n) \).
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\textbf{S. decrease-key(\text{handle } h)}:

- Decrease the key of the element pointed to by \( h \).
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: \( \mathcal{O}(\log n) \) since the trees have height \( \mathcal{O}(\log n) \).
8.2 Binomial Heaps

$S. \text{delete}(\text{handle } h)$:

- Execute $S. \text{decrease-key}(h, -\infty)$.
- Execute $S. \text{delete-min}()$.
- Time: $\Theta(\log n)$. 
8.2 Binomial Heaps

\( S. \text{ delete(\text{handle } h)}: \)

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- Execute \( S. \text{ delete-min()}. \)
- Time: \( \Theta(\log n). \)
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\textbf{S. delete(handle }h\textbf{):}

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- Execute \textit{S. delete-min}().

Time: \(\Theta(\log n)\).
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- Time: $O(\log n)$. 