How to choose augmenting paths?

▶ We need to find paths efficiently.
▶ We want to guarantee a small number of iterations.

Several possibilities:

▶ Choose path with maximum bottleneck capacity.
▶ Choose path with sufficiently large bottleneck capacity.
▶ Choose the shortest augmenting path.

Capacity Scaling

Intuition:

▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
▶ Don’t worry about finding the exact bottleneck.
▶ Maintain scaling parameter \( \Delta \).
▶ \( G_f(\Delta) \) is a sub-graph of the residual graph \( G_f \) that contains only edges with capacity at least \( \Delta \).

Algorithm 2: maxflow \((G, s, t, c)\)

1: foreach \( e \in E \) do \( f_e \leftarrow 0 \);
2: \( \Delta \leftarrow 2^\lceil \log_2 C \rceil \)
3: while \( \Delta \geq 1 \) do
4: \( G_f(\Delta) \leftarrow \Delta \)-residual graph
5: while there is augmenting path \( P \) in \( G_f(\Delta) \) do
6: \( f \leftarrow \text{augment}(f, c, P) \)
7: update\((G_f(\Delta))\)
8: \( \Delta \leftarrow \Delta/2 \)
9: return \( f \)

Assumption:

All capacities are integers between 1 and \( C \).

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

▶ because of integrality we have \( G_f(1) = G_f \)
▶ therefore after the last phase there are no augmenting paths anymore
▶ this means we have a maximum flow.
Capacity Scaling

**Lemma 1**
There are $\lceil \log C \rceil + 1$ iterations over $\Delta$.

**Proof:** obvious.

**Lemma 2**
Let $f$ be the flow at the end of a $\Delta$-phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.

**Proof:** less obvious, but simple:
- There must exist an $s$-$t$ cut in $G_f(\Delta)$ of zero capacity.
- In $G_f$ this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

**Theorem 4**
We need $O(m \log C)$ augmentations. The algorithm can be implemented in time $O(m^2 \log C)$.

**Lemma 3**
There are at most $2m$ augmentations per scaling-phase.

**Proof:**
- Let $f$ be the flow at the end of the previous phase.
- $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- Each augmentation increases flow by $\Delta$. 

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